Math 100a Fall 2015 Homework 4

Due Wednesday 10/21/2015 by 5pm in HW box in basement of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Read Section 3.3, primarily the material on direct products of groups. Then read Sections 3.4 and 3.5.

Assigned Problems (write up full solutions and hand in):

Section 3.2 #24

Section 3.3. #5, 6

Section 3.4 #2, 4, 6

Problems not from the text (also to be handed in)

A. Let G be a group and let a and b be elements of G which have finite order. For convenience, define $m = o(a) < \infty$ and $n = o(b) < \infty$. Suppose that a and b commute, that is, $ab = ba$. Let $k = o(ab)$ be the order of the element ab. The point of this problem is to study how the order of ab is related to the order of a and the order of b.

(a). Prove that k is finite and in fact that k divides $\operatorname{lcm}(m, n)$.

(b). Show that if $\gcd(m, n) = 1$, then $k = mn = \operatorname{lcm}(m, n)$.

(c). Give an example showing that k can be smaller than $\operatorname{lcm}(m, n)$ in general.

B. If two elements a and b of finite order do not commute, the result of the previous exercise fails completely; it is even possible for ab to have infinite order. In this exercise you see an example of this.

Let G be the group of all permutations of the infinite set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Remember that G is the set of all bijective functions $f : \mathbb{Z} \to \mathbb{Z}$ and that the operation in G is composition of functions, i.e. fg means $f \circ g$. Let $f, g \in G$ be the functions given by the formulas $f(x) = -x$ and $g(x) = 1 - x$ (you can take as given that these really are bijective functions and thus belong to G.) Prove that in the group G, $o(f) = 2$ and $o(g) = 2$, but that $o(fg) = \infty$.

C. In any group G, if H and K are subsets of G then we define $HK = \{hk | h \in H, k \in K\}$.

(a). Prove that if H and K are subgroups of G, then HK is a subgroup of G if and only if $HK = KH$. (do not use Proposition 3.3.2 in the text).

(b). Use the result of (a) to give an alternative proof of Proposition 3.3.2 in the text: namely, if H and K are subgroups of G and $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, then HK is a subgroup of G.

Optional problems (handing in not required)

(None this week)