Math 100a Fall 2015 Homework 3

Due Wednesday 10/14/2015 by 5pm in HW box in basement of AP&M

Reading

All references are to Beachy and Blair, 3rd edition.

Read Section 3.2 and begin to read Section 3.3.

Assigned Problems (write up full solutions and hand in):

Section 3.2 #1(b)(d), 8, 11, 14, 19(a)(b), 20, 21, 22

Problems not from the text (also to be handed in)

A. Let G be the group $\mathbb{Z}$ of integers under the operation of $+$. For any $m \geq 1$, the set $m\mathbb{Z} = \{mq | q \in \mathbb{Z}\}$ of multiples of m is a subgroup of $\mathbb{Z}$ (see Example 3.2.1 in the text).

Prove that *every* subgroup of $\mathbb{Z}$ is either equal to $m\mathbb{Z}$ for some $m \geq 1$ or else is the trivial subgroup $\{0\}$.

(Hint: If H is a subgroup of G and $H \neq \{0\}$, prove that H contains a positive number and let m be the smallest positive element of H. Then show that $H = m\mathbb{Z}$ for this m. Begin by picking $n \in H$ and writing $n = qm + r$ using the division algorithm.)

B. Let $G = \mathbb{Z}_{13}^{\times}$, the units group of integers mod 13 under multiplication. Thus G has 12 elements. Find all the cyclic subgroups of G. In particular, show that G is itself cyclic and that G has a unique cyclic subgroup containing d elements for each d that is a divisor of 12.

Optional problem (handing in not required)

C. Let G be a cyclic group of order n, with generator a. Thus we can write $G = \langle a \rangle = \{e, a, a^2, \dots, a^{n-1}\}$ and $a^n = e$. In this problem, you will find all subgroups of G. As a consequence, you will prove that *every subgroup of G is cyclic, equal to $\langle a^d \rangle$ for some positive divisor d of n.*

(a). Show that if $d \geq 1$ and $d|n$, then $\langle a^d \rangle$, in other words the cyclic subgroup of G generated by a^d, is a subgroup of G with exactly n/d elements.

(b). Let H be any subgroup whatsoever of G (don't assume H is cyclic.) Define the set of integers $S = \{m \in \mathbb{Z} | a^m \in H\}$. Show that S has positive elements and let d be the smallest positive integer in S. Prove that $d|n$. (Hint: first use the division algorithm to write $n = qd + r$.)

(c). Again let H be any subgroup of G and let d be the integer found in part (b). Prove that $H = \langle a^d \rangle$. (Hint: suppose that $a^c \in H$. Use the division algorithm to write $c = qd + r$.)