ON VERLINDE SHEAVES AND STRANGE DUALITY OVER ELLIPTIC NOETHER-LEFSCHETZ DIVISORS

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Abstract. We extend results on generic strange duality for \( K3 \) surfaces by showing that the proposed isomorphism holds over an entire Noether-Lefschetz divisor in the moduli space of quasipolarized \( K3 \)s. We interpret the statement globally as an isomorphism of sheaves over this divisor, and also describe the global construction over the space of polarized \( K3 \)s.

1. Introduction.

1.1. Setup. For a fixed polarized complex \( K3 \) surface \((X, H)\), let \( v, w \in H^*(X, \mathbb{Z})\) be two primitive elements which are orthogonal in the sense that

\[
\int_X v \cup w = 0.
\]

Consider the moduli space \( \mathcal{M}_v \) of Gieseker \( H \)-stable sheaves \( E \) on \( X \) of Mukai vector \( v \):

\[
\text{ch}(E) \sqrt{\text{Todd}(X)} = v.
\]

The Mukai vector \( w \) induces a determinant line bundle \( \Theta_w \to \mathcal{M}_v \), constructed in [LP2][Li]. Specifically, if a universal family \( E \to \mathcal{M}_v \times X \) is available, we set

\[
\Theta_w = \text{det} R^p_! (E \otimes L \cdot q^* F)^{-1},
\]

for a complex \( F \to X \) of Mukai vector \( w \). Similarly we obtain the line bundle \( \Theta_v \to \mathcal{M}_w \).

If \( c_1(v \cdot w) \cdot H > 0 \), as explained in [Sc2], the set

\[
\Theta = \{(E, F) : H^0(E \otimes L \cdot F) \neq 0\} \hookrightarrow \mathcal{M}_v \times \mathcal{M}_w
\]

is the zero locus of a section of the line bundle

\[
\Theta_w \boxtimes \Theta_v \to \mathcal{M}_v \times \mathcal{M}_w,
\]

and induces a map

\[
D : H^0(\mathcal{M}_v, \Theta_w)^\vee \to H^0(\mathcal{M}_w, \Theta_v).
\]

According to Le Potier’s strange duality conjecture [LP1], \( D \) is expected to be an isomorphism.
1.2. Results. In [MOY] we established the conjecture for generic surfaces \((X,H)\) in the moduli space \(K_\ell\) of primitively quasipolarized \(K3\) surfaces of degree \(2\ell\), and for many pairs of Mukai vectors \((v,w)\) which satisfy
\[c_1(v) = c_1(w) = H.\]
The proof involves degeneration to the locus of elliptic \(K3\) surfaces with section and irreducible at worst nodal fibers.

In the present paper, we study the problem for elliptic \(K3\)s with arbitrary singular fibers. In other words, we consider the entire Noether–Lefschetz divisor
\[\mathcal{P}_1 \hookrightarrow K_\ell\]
consisting of pairs \((X,H)\) of elliptically fibered \(K3\)s which are quasipolarized by means of a numerical section \(H\). We show

**Theorem 1.** For any surface \((X,H)\) in \(\mathcal{P}_1\), fix two orthogonal Mukai vectors \(v\) and \(w\) of ranks \(r,s \geq 3\) with
\[c_1(v) = c_1(w) = H,\]
and satisfying further
\[\langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2.\]
Then the duality morphism \(D\) is an isomorphism.

In Section 2 we record basic properties of the Noether-Lefschetz divisor \(\mathcal{P}_1\). In Section 3, we prove the theorem above. In Section 4, the duality is stated globally as an isomorphism of sheaves, the Verlinde sheaves, over the entire divisor \(\mathcal{P}_1\). The Verlinde sheaves are also constructed more generally over the locus \(K_\ell^0 \hookrightarrow K_\ell\) of polarized \(K3\)s. It would be interesting to extend this construction to \(K_\ell\) in a suitable manner.

2. The Noether-Lefschetz divisor \(\mathcal{P}_1\). Let \((\mathcal{X}, \mathcal{H}) \to K_\ell\) be the moduli stack of quasipolarized \(K3\) surfaces \((X,H)\) of degree \(H^2 = 2\ell\) with \(\ell \neq 1\).

We consider the Noether-Lefschetz loci of quasipolarized elliptically fibered \(K3\) surfaces in \(K_\ell\). Specifically, for each \(k > 0\), we denote by \(\mathcal{P}_k\) the Noether-Lefschetz stack parametrizing triples \((X,H,F)\) consisting of quasipolarized \(K3\)’s of degree \(2\ell\), and divisor classes \(F\) over \(X\) satisfying
\[F^2 = 0, \ F \cdot H = k.\]
We claim that
\[\mathcal{P}_1 \hookrightarrow K_\ell\]
is a substack of \(K_\ell\) parametrizing exactly the quasipolarized \(K3\)s which can be elliptically fibered with section, and with the quasipolarization a numerical section. This
is expressed by the lemma below. The statement is standard, but a reference seemed difficult to find.

**Lemma 1.** Let \((X,H)\) be a quasipolarized \(K3\) surface of degree \(2\ell\) with \(\ell \neq 1\), and let \(F\) be a divisor class on \(X\) satisfying

\[
F^2 = 0,\quad F \cdot H = 1.
\]

Then

(i) \(F\) is effective and \(\mathcal{O}(F)\) is globally generated;
(ii) the induced map \(\pi : X \to \mathbb{P}^1\) is an elliptic fibration with section \(\sigma\), having \(F\) as the fiber class;
(iii) the quasipolarization equals \(H = \sigma + (\ell + 1)F\);
(iv) the class \(F\) satisfying the two numerical assumptions above is unique.

**Proof.** Note first that \(\chi(\mathcal{O}(F)) = 2\). Since \(-F \cdot H = -1\), and \(H\) is nef, \(-F\) cannot be effective, so

\[
h^2(\mathcal{O}(F)) = h^0(\mathcal{-F})) = 0, \quad \text{and} \quad h^0(\mathcal{O}(F)) \geq \chi(\mathcal{O}(F)) = 2.
\]

Thus \(F\) is effective.

We treat separately the two possibilities that \(\mathcal{O}(F)\) be nef or not. First, if \(\mathcal{O}(F)\) is nef, by the theorem of Piatetski-Shapiro and Shafarevich [PS] there exists an elliptic fibration

\[
\pi : X \to \mathbb{P}^1
\]

such that \(F = mf\), where \(f\) is the class of a fiber. In fact,

\[
F \cdot H = 1 \implies m = 1, \quad F = f, \quad H \cdot f = 1.
\]

We next show that the fibration has a section. It is easy to check that the class

\[
\Sigma = H - (\ell + 1)f
\]

has self-intersection \(-2\). Since \(\chi(\mathcal{O}(\Sigma)) = 1\), \(\Sigma\) is either effective or anti-effective. In fact, \(\Sigma\) is effective, since \(\Sigma \cdot H > 0\). Let \(C\) be a curve in the linear series \(\mathcal{O}(\Sigma)\). Now, for any component \(R\) of a fiber we have \(R \cdot f = 0\) by Zariski’s lemma, cf. III.8.2 [BPV]. Since \(C \cdot f = 1\), \(C\) must have a component which intersects each fiber with multiplicity 1. The other components of \(C\) must be supported on components of the fibers. The transversal component gives a section \(\sigma\) of the elliptic fibration \(\pi\).

We now argue that \(H = \sigma + (\ell + 1)f\). From the above discussion, we already know that

\[
H = \sigma + mf + \sum m_iR_i
\]
where $R_i$ are components of fibers and $m = \ell + 1$. In fact, by absorbing other fiber classes into the constant $m$, we may assume $R_i$ are supported on fibers with two components or more. We have the following possibilities:

(i) fibers of type $I_n$, consisting in a polygon of rational curves $C_1, \ldots, C_n$;
(ii) fibers of type $III$, consisting of 2 rational curves $C_1, C_2$ meeting tangentially;
(iii) fibers of type $IV$ consisting of 3 concurrent rational curves $C_1, C_2, C_3$;
(iv) fibers of type $I_n^*$ which can be written as

$$C_1 + C_2 + C_3 + C_4 + 2(D_1 + \ldots + D_n)$$

where

$$C_1 \cdot D_1 = C_2 \cdot D_1 = C_3 \cdot D_n = C_4 \cdot D_n = 1$$

and $D_i \cdot D_{i+1} = 1$ for $1 \leq i \leq n - 1$;
(v) fibers of type $II^*, III^*, IV^*$ corresponding to the graphs $E_6, E_7, E_8$.

Consider a fiber of type (i) and its contribution $\sum m_i C_i$ to the divisor $H$. We claim this contribution is a multiple of the fiber. Indeed, label the components so that $C_1$ intersects the section $\sigma$. Since $H \cdot C_i \geq 0$ for all $i$, we obtain the inequalities

$$-2m_1 + m_2 + m_n \geq -1, -2m_2 + m_1 + m_3 \geq 0, \ldots, -2m_n + m_1 + m_{n-1} \geq 0.$$ 

If $-2m_1 + m_2 + m_n \geq 0$, then after adding the above inequalities, we conclude that we must have equality throughout. Thus $m_1 = \ldots = m_n = m$ which shows that $\sum m_i C_i = mf$ as claimed. The case

$$-2m_1 + m_2 + m_n = -1$$

is impossible. Indeed, since

$$\sum_{k \neq 1} (-2m_k + m_{k-1} + m_{k+1}) = -( -2m_1 + m_2 + m_n ) = 1$$

we conclude that for some index $k_0$

$$-2m_k + m_{k-1} + m_{k+1} = \begin{cases} 1 & \text{if } k = k_0 \\ 0 & \text{if } k \neq 1, k_0. \end{cases}$$

This system is easily seen not to have any solutions. The remaining fiber types (ii)-(v) are entirely similar, and we will not verify them explicitly. In all cases, we find that $\sum m_i C_i$ must contribute a multiple of the fiber, hence

$$H = \sigma + mf$$

for some integer $m$. In fact, $m = \ell + 1$ by computing $H^2 = 2\ell$. This completes the proof when $O(F)$ is nef.
We assume now that \( O(F) \) is not nef and we will reach a contradiction. Then there exists an irreducible curve \( \Gamma_1 \) such that 
\[
F \cdot \Gamma_1 < 0.
\]
The curve \( \Gamma_1 \) is a component of an effective curve of class \( F \) and furthermore \( \Gamma_1^2 < 0 \).
Thus \( \Gamma_1 \) is a smooth rational curve on \( X \). Let \( H' \) be an ample class, and set \( F_0 = F \).
The reflection of \( F \) along \( \Gamma_1 \) then yields an effective class, cf. proof of Theorem 2.2 in [S]:
\[
F_1 = F_0 + (F_0 \cdot \Gamma_1)\Gamma_1
\]
which has the property that
\[
F_1^2 = F_0^2 = 0, \quad F_1 \cdot H' < F_0 \cdot H'.
\]
If \( F_1 \) is not nef, then we continue the process reflecting along a smooth rational curve \( \Gamma_2 \).
The process will eventually stop since \( F_i \cdot H' \) is a decreasing sequence of non-negative integers. At the end, we find a nef line bundle \( O(F_k) \) of zero self-intersection, where
\[
F_k = F + (F_0 \cdot \Gamma_1)\Gamma_1 + (F_1 \cdot \Gamma_2)\Gamma_2 + \ldots + (F_{k-1} \cdot \Gamma_k)\Gamma_k.
\]
Therefore \( F_k = mf \), where \( m \geq 0 \) by nefness. In particular,
\[
F = mf + \sum n_i \Gamma_i
\]
where \( n_i = -F_{i-1} \cdot \Gamma_i > 0 \). Using \( F \cdot H = 1 \) we conclude
\[
m(H \cdot f) + \sum n_i (H \cdot \Gamma_i) = 1.
\]
Since \( H \) is nef, the intersection numbers above are nonnegative. If \( H \cdot f = 0 \), since \( H^2 > 0 \), by the Hodge index theorem we find \( f^2 \leq 0 \). Since equality occurs, \( f \) must be numerically trivial which is not the case since it intersects \( H' \) nontrivially. Therefore
\[
H \cdot f = 1, \quad m = 1, \quad H \cdot \Gamma_i = 0 \text{ for all } i.
\]
The argument given in the nef case then shows that the elliptic fibration \( \pi \) has a section \( \sigma \), and
\[
H = \sigma + (\ell + 1)f.
\]
We conclude
\[
H \cdot \Gamma_i = \sigma \cdot \Gamma_i + (\ell + 1)f \cdot \Gamma_i = 0.
\]
Thus either \( \sigma \cdot \Gamma_i \leq 0 \) or \( f \cdot \Gamma_i \leq 0 \). This means \( \Gamma_i \) is contained in \( \sigma \) or in the fiber \( f \).
The first case cannot occur since then
\[
\Gamma_i = \sigma \text{ and } \sigma \cdot \Gamma_i + (\ell + 1)f \cdot \Gamma_i = 0 \text{ shows } \ell = 1.
\]
which is not allowed. Thus $\Gamma_i$ is a component of the fiber of $f$. However, in this case $f \cdot \Gamma_i = 0$ by Zariski’s lemma. Since

$$F = f + \sum n_i \Gamma_i$$

has zero self intersection, we find

$$\left(\sum n_i \Gamma_i\right)^2 = 0,$$

where $\Gamma_i$ are components of the fiber. This yields $\sum n_i \Gamma_i = nf$ for some integer $n$, again by Zariski’s lemma. Thus $F = (n + 1)f$, and since $F \cdot H = 1$ then $F$ is the fiber class.

Finally, we establish the uniqueness of $F$ as claimed in (iv). If $F'$ is another class with

$$F'^2 = 0, \ F' \cdot H = 1$$

then we can write

$$F' = a\sigma + R$$

where $R$ is supported on components of fibers. We have $R \cdot f = 0$ and

$$F' \cdot H = (a\sigma + R) \cdot (\sigma + (\ell + 1)f) = 1 \implies R \cdot \sigma = 1 - a(\ell - 1).$$

In addition

$$F'^2 = 0 \implies -2a^2 + 2a(R \cdot \sigma) + R^2 = 0.$$ 

This yields

$$R^2 = -2a + 2a^2(\ell + 1).$$

By Zariski’s lemma, $R^2 \leq 0$, which implies $a = 0$. Furthermore, we obtain $R^2 = 0$, showing that $R = mf$, again by Zariski’s lemma. Moreover, $R \cdot \sigma = 1$ hence $m = 1$. Therefore $F' = f$, proving uniqueness. \(\square\)

3. **Strange duality along $P_1$.** We now show Theorem 1 of the Introduction. For $(X, H) \in P_1$, we consider the orthogonal Mukai vectors

$$(2) \quad v = r + H + a[pt], \ w = s + H + b[pt]$$

with $r, s \geq 3$, satisfying further

$$(3) \quad \langle v, v \rangle + \langle w, w \rangle \geq 2(r + s)^2.$$ 

We form the moduli spaces of stable sheaves $M_v$ and $M_w$ together with the corresponding theta line bundles. Stability of the sheaves in $M_v$ and $M_w$ is with respect to a polarization which is suitable in the sense of Friedman. For such polarizations, and sheaves of fiber degree 1, stability on the surface is equivalent to stability of the restriction to a generic fiber, cf. Theorem 5, chapter 6 of [F]. \(^1\) Both moduli spaces are smooth and projective.

\(^1\)As shown in the appendix of [MOY], this choice of polarization is in fact irrelevant under the stronger assumptions that

$$\langle v, v \rangle \geq 2(r - 1)(r^2 + 1), \ \langle w, w \rangle \geq 2(s - 1)(s^2 + 1).$$
Under these conditions, in [MOY], the strange duality map

\[ D : H^0(\mathcal{M}_v, \Theta_w)^\vee \to H^0(\mathcal{M}_w, \Theta_v) \]

was proven to be an isomorphism over the open sublocus of \( \mathcal{P}_1 \) consisting of surfaces with Picard rank 2.

We now assume that \( X \) has Picard rank larger than 2. The elliptic fibration has finitely many reducible fibers. Fourier-Mukai functors were studied in this setting in [HMS]. Specifically, let

\[ \pi : X \to \mathbb{P}^1 \]

be any quasipolarized elliptically fibered \( K3 \) surface with section class \( \sigma \) and fiber class \( f \). Consider the product \( Y = X \times_{\mathbb{P}^1} X \) with projections \( p \) and \( q \) to the two factors, and let

\[ \Delta \subset X \times_{\mathbb{P}^1} X \]

be the diagonal. The \( \pi \)-relative Fourier-Mukai functor

\[ S : \mathcal{D}(X) \longrightarrow \mathcal{D}(X) \]

with kernel

\[ \mathcal{P} = \mathcal{I}_{\Delta} \otimes \mathcal{O}(p^*\sigma + q^*\sigma) \]

is an equivalence of bounded derived categories of coherent sheaves by Proposition 2.16 of [HMS]. As \( (X, H) \) is in \( \mathcal{P}_1 \), by Lemma 1

\[ c_1(v) = c_1(w) = \sigma + (\ell + 1)f. \]

Along the lines of [B], we shall prove shortly that the Fourier-Mukai transform \( S \) induces a birational morphism, regular in codimension 1, between the moduli spaces \( \mathcal{M}_v \) and \( \mathcal{M}_w \) on the one hand, and the Hilbert schemes of \( d_v \) respectively \( d_w \) points on \( X \) on the other:

\[ \Psi_v : \mathcal{M}_v \longrightarrow X^{[d_v]}, \quad \Psi_w : \mathcal{M}_w \longrightarrow X^{[d_w]} .\]

Assuming this for the moment, we explain how to complete the proof of Theorem 1, much as in [MOY]. We determine first the exact numerics of the transformation \( S \) by a cohomological Fourier-Mukai calculation. Let \( V \in \mathcal{D}(X) \) be any complex of rank \( r \), Euler characteristic \( \chi \), and first Chern class

\[ c_1(V) = k\sigma + mf, \]

Indeed, in this case, the different moduli spaces are birational in codimension 1.
for integers $k$ and $m$. Recalling $p$ and $q$ are the projections from $Y = X \times_{\mathbb{P}^1} X$, we have

$$
\det S(V) = \det Rq_*(p \otimes p^* V) = \det Rq_*(I_\Delta \otimes p^* V(\sigma) \otimes q^* O(\sigma)) = \det Rq_*(I_\Delta \otimes p^* V(\sigma)) \otimes O(\sigma)^{\chi(V|p)} = \det Rq_*(p^* V(\sigma)) \otimes \det Rq_*(O_\Delta \otimes p^* V(\sigma))^{-1} \otimes O(k\sigma) = \det Rq_*(p^* V(\sigma)) \otimes \det V(\sigma)^{-1} \otimes O(k\sigma) = \det Rq_*(p^* V(\sigma)) \otimes O(-r\sigma - mf).
$$

To calculate the first term, it is more convenient to work on the product

$$
j : Y \hookrightarrow X \times X.
$$

Let $\bar{p}, \bar{q}$ denote the two projections from $X \times X$, and let $\text{pr} = \pi \times \pi : X \times X \to \mathbb{P}^1 \times \mathbb{P}^1$. Observing that

$$
j_* O_Y = \text{pr}^* O_{\Delta/\mathbb{P}^1 \times \mathbb{P}^1} = \text{pr}^* (O_{\mathbb{P}^1 \times \mathbb{P}^1} - O_{\mathbb{P}^1}(-1) \boxtimes O_{\mathbb{P}^1}(-1)) = O_{X \times X} - \bar{p}^* O(-f) \otimes \bar{q}^* O(-f),
$$

we calculate

$$
\det Rq_*(p^* V(\sigma)) = \det R\bar{q}_* (\bar{p}^* V(\sigma) \otimes j_* O_Y) = \det R\bar{q}_* (\bar{p}^* V(\sigma)) \otimes \det R\bar{q}_* (\bar{p}^* V(\sigma) \otimes \bar{p}^* O(-f) \otimes \bar{q}^* O(-f))^{-1} = \det(R\bar{q}_* (\bar{p}^* V(\sigma - f))) \otimes O(-f))^{-1} = O(-f)^{-\chi(V(\sigma - f))} = O((\chi - 2r + m - 3k)f).
$$

To summarize, we obtained

$$
\det S(V) = O(-r\sigma + (\chi - 2r - 3k)f).
$$

Now let $E$ and $F$ be stable sheaves whose Mukai vectors $v$ and $w$ are given by (2). By the preceding calculation

$$
\det S(E^\vee) = O(-r\sigma + (a - r + 3)f), \quad \det S(F) = O(-s\sigma + (b - s - 3)f).
$$

Assuming the birational isomorphism with the Hilbert scheme, for generic $E$ and $F$ we therefore have that

$$
S(E^\vee) = I_Z \otimes O(r\sigma - (a - r + 3)f)[-1],
$$

$$
S(F) = I_W^\vee \otimes O(-s\sigma + (b - s - 3)f),
$$

where $Z$ and $W$ are zero dimensional subschemes of lengths $d_v$ and $d_w$ respectively. In fact, we will only explain the first equality below; the second can be deduced from the first by Grothendieck duality as in Proposition 2 of [MOY].
We finally calculate
\[ H^0(E \otimes L F) = \text{Hom}_{D(X)}(E^\vee, F) = \text{Hom}_{D(X)}(S(E^\vee), S(F)) = \text{Ext}^1(I_Z \otimes L, I_W^\vee) = \text{Ext}^1(I_{W^\vee}^\vee, I_Z \otimes L)^\vee = H^1(I_W \otimes L^\vee). \]

On the third line, using (4) and (5), we have set
\[ L = O((r + s)\sigma + (r + s - a - b)f). \]

The orthogonality condition
\[ H^2 = -rb - sa \]
for the Mukai vectors \( v \) and \( w \) together with the bound (3) on the dimensions \( d_v \) and \( d_w \) ensure that \(-a - b > r + s\), so the line bundle \( L \) is big and nef, without higher cohomology on \( X \).

Thus, under the birational map
\[ \Psi_v \times \Psi_w : \mathcal{M}_v \times \mathcal{M}_w \rightarrow X^{[d_v]} \times X^{[d_w]} \]
the two theta divisors
\[ \Theta = \{(E, F) : \mathbb{H}^0(E \otimes L F) \neq 0\} \subset \mathcal{M}_v \times \mathcal{M}_w, \]
and
\[ \theta_L = \{(I_Z, I_W) : \mathbb{H}^0(I_Z \otimes L I_W \otimes L) \neq 0\} \subset X^{[d_v]} \times X^{[d_w]} \]
coincide. The line bundles \( \Theta_w, \Theta_v \) on the two higher-rank moduli spaces and \( L^{[d_v]}, L^{[d_w]} \) on the two Hilbert schemes are also identified. As explained in Section 3 of [MO], for line bundles \( L \) without higher cohomology, \( \theta_L \) is known to induce an isomorphism
\[ H^0(X^{[d_v]}, L^{[d_v]}) \rightarrow H^0(X^{[d_w]}, L^{[d_w]}). \]
Therefore, under the identifications above, \( \Theta \) also induces the isomorphism of equation (1):
\[ D : H^0(\mathcal{M}_v, \Theta_w)^\vee \rightarrow H^0(\mathcal{M}_w, \Theta_v). \]

We turn now to the proof that \( \Psi_v \) is an isomorphism in codimension 1, which was given for a surface \( \pi : X \rightarrow \mathbb{P}^1 \) with irreducible fibers in [BH], [MOY]. We thus take up the case when the fibration has at least one reducible fiber. We shall explain that the inverse
\[ \Psi_v^{-1} : X^{[d_v]} \rightarrow \mathcal{M}_v \]
is a regular embedding defined on a subscheme \( U \subset X^{[d_v]} \) with \( \text{codim} (X^{[d_v]} \setminus U) \geq 2 \). The same is then true about \( \Psi_v \) on \( \mathcal{M}_v \). Indeed, if this were not the case, as the two moduli spaces are holomorphic symplectic, \( \Psi_v \) would at least admit by [H], Section 2.2,
an extension $\Psi_v$ to a regular embedding defined away from codimension 2 on $\mathfrak{M}_v$. Thus $\Psi_v$ would extend over a divisorial locus $D \subset \mathfrak{M}_v$ where the original map $\Psi_v$ is assumed undefined. But then $\Psi_v(D) \subset X^{[d_v]} \setminus U$, a contradiction as the latter has codimension 2 in $X^{[d_v]}$.

We are thus left to analyze the domain of $\Psi^{-1}_v$. The inverse is a Fourier-Mukai transform whose kernel is a complex $\mathcal{Q}[1]$ over $X \times \mathbb{P}^1 X$. We write $T$ for the Fourier-Mukai transform with kernel $\mathcal{Q}$ so that $S \circ T = [-1], T \circ S = [-1]$.

We claim that for generic $Z$, the sheaf
\[
M = I_Z \otimes \mathcal{O}(r \sigma - (a - r + 3)f)
\]
is $WIT_0$ for the kernel $\mathcal{Q}$. Its transform is then a stable torsion free sheaf in $\mathfrak{M}_v$, cf. Section 7 of [B]. To prove the claim, we adapt arguments of [B], as follows. On general grounds, cf. Lemma 6.1 in [B], there is a short exact sequence
\[
0 \to A \to M \to B \to 0
\]
where $A$ is $T$-$WIT_0$, while $B$ is $T$-$WIT_1$. We prove that $B = 0$, following Lemma 6.4 in [B]. Assuming otherwise, we have $T(B) \neq 0$, and therefore there exists $x \in X$ and a non-zero morphism
\[
T^1(B) \to \mathcal{C}_x.
\]
Note however that
\[
\mathcal{C}_x = T^1(I_x(o)),
\]
where $I_x$ is the ideal sheaf of the point $x$ in its fiber, and $o$ denotes the intersection of the fiber through $x$ with the section. In fact, $I_x(o) = S^0(\mathcal{C}_x)$, by Lemma 6.3.7 of [C]. By Parseval, we now obtain a non-zero morphism
\[
M \to B \to I_x(o).
\]
This morphism must factor through the restriction of $M$ to the fiber $C$ through $x$, yielding a non-zero map
\[
I_Z|_C \otimes \mathcal{O}(ro) \to I_x(o).
\]
Thus it suffices to show
\[
\text{Hom}_C(I_Z|_C \otimes \mathcal{O}((r - 1)o), I_x) = 0.
\]
We prove this is the case for $r \geq 3$ and subschemes $Z$ such that

(i) $Z$ intersects any smooth fiber in at most two points;
(ii) $Z$ intersects any singular fiber in at most one point which is not a node or a cusp (if the fiber is irreducible) or does not lie on at least two irreducible components. This locus has complement of codimension 2 in the Hilbert scheme of $X$.

When $C$ is a smooth fiber, $\zeta = Z \cap C$ has length at most equal to 2, by (i). Then

$$I_Z|_C = I_{\zeta/C} \oplus T$$

where $T$ is a torsion sheaf supported at $\zeta$. This can be seen by restricting the ideal sequence of $Z$ to the curve $C$. In fact, the same statement also holds when $C$ is singular, as $Z$ is subject to (ii). When $C$ is smooth, it suffices therefore to prove

$$\text{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0 \iff H^0(\mathcal{O}_C(-(r-1)o + \zeta - x)) = 0.$$ 

Since for $r \geq 3$ the degree is negative, the conclusion follows. When $C$ is a singular fiber, the scheme $\zeta = Z \cap C$ has length 1. We show

$$\text{Hom}_C(I_{\zeta/C}((r-1)o), \mathcal{O}_C) = 0$$

which gives $\text{Hom}_C(I_{\zeta/C}((r-1)o), I_x) = 0$.

Indeed, by duality, this is the same as proving

$$H^1(I_{\zeta/C}((r-1)o)) = 0.$$ 

Here we used that the dualizing sheaf of $C$ is trivial. Assume first $\zeta \neq o$. From the exact sequence

$$0 \rightarrow I_{\zeta/C}(o) \rightarrow I_{\zeta/C}((r-1)o) \rightarrow \mathbb{C}_o^{r-2} \rightarrow 0$$

we see it suffices to show

$$H^1(I_{\zeta/C}(o)) = 0.$$ 

Next, from the exact sequence

$$0 \rightarrow \mathcal{O}(-o) \rightarrow \mathcal{O} \rightarrow \mathbb{C}_o \rightarrow 0$$

we conclude

$$H^0(\mathcal{O}(-o)) = 0, \ H^1(\mathcal{O}(-o)) = \mathbb{C} \implies H^0(\mathcal{O}(o)) = \mathbb{C}, \ H^1(\mathcal{O}(o)) = 0.$$ 

The exact sequence

$$0 \rightarrow I_{\zeta/C}(o) \rightarrow \mathcal{O}_C(o) \rightarrow \mathbb{C}_\zeta \rightarrow 0$$

and the fact that

$$H^0(\mathcal{O}_C(o)) \rightarrow \mathbb{C}_\zeta$$

is an isomorphism for $\zeta \neq o$ yield $H^1(I_{\zeta/C}(o)) = 0$, as claimed. The vanishing of higher cohomology also holds for $\zeta = o$ since $H^1(\mathcal{O}((r-2)o)) = 0$. This completes the proof. \hfill $\Box$

4. **The Verlinde sheaves.** We will reinterpret Theorem 1 as giving an isomorphism of sheaves defined over the divisor $\mathcal{P}_1$ in the moduli space of quasipolarized $K3$s.
4.1. **Construction.** For a fixed integer \( n \), we may consider over \( \mathcal{K}_\ell \) the relative Hilbert scheme of \( n \) points

\[
\pi : \mathcal{X}^{[n]} \to \mathcal{K}_\ell,
\]

viewed as the relative moduli stack of rank 1 torsion free sheaves of trivial determinant and second Chern number \(-n\).

More generally, to consider spaces of higher rank sheaves as the \( K3 \) surface varies in moduli, we restrict attention to the open substack

\[
\mathcal{K}^0_\ell \hookrightarrow \mathcal{K}_\ell
\]

where the line bundle \( \mathcal{H} \) over the universal surface

\[
\pi : \mathcal{X} \to \mathcal{K}_\ell
\]

is ample. We construct

\[
M[v] \to \mathcal{K}^0_\ell,
\]

the moduli space of \( \mathcal{H} \)-semistable sheaves with rank \( r \), determinant \( d\mathcal{H} \) and Euler characteristic \( a - r \) over the fibers of \( \pi : \mathcal{X}^0 \to \mathcal{K}^0_\ell \).

The construction of the theta bundles over \( M[v] \) is subtler. To start, let

\[
\pi : \mathcal{X}^0_1 \to \mathcal{K}^0_{\ell,1}
\]

be the universal family over the moduli stack \( \mathcal{K}^0_{\ell,1} \) of polarized \( K3 \)s with a marked point. It has a canonical section

\[
\sigma : \mathcal{K}^0_{\ell,1} \to \mathcal{X}^0_1.
\]

Let

\[
\mathcal{V} = (r - d)\mathcal{O} + d\mathcal{H} + \alpha \mathcal{O}_\sigma,
\]

\[
\mathcal{W} = (s - e)\mathcal{O} + e\mathcal{H} + \beta \mathcal{O}_\sigma,
\]

be classes in the \( K \)-theory of \( \mathcal{X}^0_1 \). Over a fixed marked polarized \( K3 \) surface \((X,H,p)\), they have the Mukai vectors

\[
v = r + dH + a[pt], \quad w = s + eH + b[pt],
\]

for

\[
\alpha = a - r - \frac{dH^2}{2},
\]

\[
\beta = b - s - \frac{eH^2}{2}.
\]

We further denote as

\[
\pi_v : M[v]_1 \to \mathcal{K}^0_{\ell,1}
\]
the relative moduli space of semistable sheaves of type \( v \) over the fibers of \( \pi : X_1^\circ \to K_{t,1}^\circ \).

The class \( \mathcal{W} \) induces standardly a determinant line bundle
\[
\Theta_v \to M[v]_1,
\]
via descent from
\[
Q \to M[v]_1,
\]
where \( Q \) is an open subscheme of a suitable quot scheme. Explicitly, over \( Q \), we have
\[
\Theta_v = \det R_p ! (\mathcal{E} \otimes q^* \mathcal{W})^{-1}
\]
for the universal quotient sheaf \( \mathcal{E} \to Q \times K_{t,1}^\circ X_1^\circ \). The fiber of the forgetful map
\[
M[v]_1 \to M[v]
\]
over a point \( (X, H, E \to X) \in M[v] \) is the surface \( X \). To describe the restriction of \( \Theta_v \) to this fiber, we let \( \Delta \subset X \times X \) be the diagonal and denote by \( p, q \) the projections from \( X \times X \) to the two factors. Then
\[
\Theta_v|_X = \det R_{p!} (q^* E \otimes ((s - e) \mathcal{O} \oplus q^* (eH) \oplus \beta \mathcal{O}_\Delta))^{-1} = \det E^{-\beta} = H^{-\beta d}.
\]

We conclude that the product line bundle
\[
(7) \quad \Theta_v \otimes \pi_v^* \mathcal{H}^{\beta d} \text{ on } M[v]_1
\]
restricts trivially to the fibers of the map
\[
M[v]_1 \to M[v]
\]
forgetting the marking. By the seesaw lemma, the product (7) is in fact the pullback to \( M[v]_1 \) of a line bundle \( \Theta_v \to M[v] \):
\[
\Theta_v \otimes \pi_v^* \mathcal{H}^{\beta d} = \text{pr}^* \Theta_v.
\]
While the determinant line bundle \( \Theta_v \) is uniquely defined for a fixed K3 surface, over the relative moduli space \( M[v] \), \( \Theta_v \) depends on choice of \( \mathcal{H} \), and therefore can be canonically defined only up to tensoring by line bundles pulled back from \( K_{t}^\circ \).

Remark. The same construction gives the theta line bundle on the relative moduli space \( SU_g(r) \to M_g \) of semistable rank \( r \) bundles with trivial determinant over smooth curves of genus \( g \). They are naturally defined on the basechanged moduli space
\[
SU_{g,1}(r) = SU_g(r) \times_{M_g} M_{g,1} \to M_{g,1},
\]
relative to the \( K \)-theory class
\[
\mathcal{O} + (g - 1) \mathcal{O}_\sigma
\]
on the universal curve \( \mathcal{C} \to M_{g,1} \), and are then seen to be pulled back under the forgetful map
\[
SU_{g,1}(r) \to SU_g(r).
\]
Pushing forward the $k$-tensor powers of the theta line bundles to $M_g$, we obtain the Verlinde bundles
\[ V_{r,k} \to M_g. \]
Their first Chern classes remain unknown in general.

4.2. Global strange duality. Over $K_\ell^\circ$ we define now the Verlinde complexes
\[ W = R\pi_{v*}\Theta_w, \quad V = R\pi_{w*}\Theta_v. \]
Consider the fiber product
\[ \pi : M[v] \times_{K_\ell^\circ} M[w] \to K_\ell^\circ, \]
edowed with the canonical Brill-Noether locus,
\[ \Theta = \{(X,H,E,F) \text{ so that } \mathbb{H}^0(X,E \otimes^L F) \neq 0 \} \subset M[v] \times_{K_\ell^\circ} M[w]. \]
One expects $\Theta$ to be a divisor. This was established in [MOY] when $v$ and $w$ satisfy
\[ c_1(v) = c_1(w) = H. \]
The corresponding line bundle, also denoted for simplicity as $\Theta$, is in any case always defined on the product space, and splits by the seesaw lemma as
\[ \Theta \simeq \Theta_w \boxtimes \Theta_v. \]
The above equation is correct up to a line bundle twist
\[ \mathcal{T} \to K_\ell^\circ \]
which will be found explicitly below, and which for now we absorb into any one of the theta bundles. The two line bundles $\Theta_w$ and $\Theta_v$ are ambiguous up to reverse twistings by a line bundle from $K_\ell^\circ$,
\[ (\Theta_v, \Theta_w) \sim (\Theta_v \otimes \pi_w^*\mathcal{L}, \Theta_w \otimes \pi_v^*\mathcal{L}^{-1}), \text{ for } \mathcal{L} \in \text{Pic } K_\ell^\circ, \]
while $\Theta$ is canonical. Pushing forward the canonical theta line bundle via $\pi$, we get
\[ R\pi_*\Theta \simeq W \otimes^L V, \]
and the above ambiguity carries over to the Verlinde complexes $W$ and $V$. The divisor
\[ D : W^\vee \to V. \]
In [MOY], also having assumed that
\[ \chi(v), \chi(w) \leq 0, \]
we showed that over a Zariski open subset of $K_\ell^\circ$, the higher cohomology sheaves vanish while $H^0(D)$ induces an isomorphism between the zeroth cohomology sheaves.
Remark. Even though not necessary for our argument, let us determine the twist $\mathcal{T} \to K^{\circ}_{\ell}$ in the decomposition
\[(12) \quad \Theta = \Theta_{w} \otimes \Theta_{v} \otimes \text{pr}^{*}\mathcal{T}\]
over $M[v] \times_{K^{\circ}_{\ell}} M[w]$, where pr is the projection to $K^{\circ}_{\ell}$. Above, we absorbed this twist into the Verlinde complexes, for the ease of exposition.

First, we may pass to the moduli stack $\mathcal{M}[v]$ and $\mathcal{M}[w]$ of all sheaves over $X$, without changing the above equations. We let $E \to \mathcal{M}[v] \times_{K^{\circ}_{\ell}} X_{1}^{\circ}$, $F \to \mathcal{M}[w] \times_{K^{\circ}_{\ell}} X_{1}^{\circ}$ be the universal families of sheaves, and further set, on the same product spaces,
\[E = E - \text{pr}^{*}_{2}V, \quad F = F - \text{pr}^{*}_{2}W.\]
Considering now the triple product
\[\mathcal{M}[v] \times_{K^{\circ}_{\ell}} \mathcal{M}[w] \times_{K^{\circ}_{\ell}} X_{1}^{\circ},\]
we calculate
\[\Theta \otimes \Theta_{v}^{-1} \otimes \Theta_{w}^{-1}\]
as the pushforward
\[(\det \mathcal{R}_{p_{12}*} (p_{13}^{*}E \otimes L p_{23}^{*}F - p_{13}^{*}E \otimes L p_{23}^{*}W - p_{23}^{*}F \otimes L p_{3}^{*}V))^{-1} \otimes \text{pr}^{*}\mathcal{H}^{-d_{\beta} - e_{\alpha}},\]
where $\mathcal{H} \to K^{\circ}_{\ell,1}$ is viewed on $\mathcal{M}[v] \times_{K^{\circ}_{\ell}} \mathcal{M}[w] \times_{K^{\circ}_{\ell}} X_{1}^{\circ}$ via pullback by the natural projection
\[\text{pr} : \mathcal{M}[v] \times_{K^{\circ}_{\ell}} \mathcal{M}[w] \times_{K^{\circ}_{\ell}} X_{1}^{\circ} \to K^{\circ}_{\ell,1}.\]
We apply Grothendieck-Riemann-Roch to compute
\[\text{ch} \mathcal{R}_{p_{12}*} (p_{13}^{*}E \otimes L p_{23}^{*}F) .\]
By construction, $\text{ch} \bar{E}$ and $\text{ch} \bar{F}$ restrict trivially over the fibers of
\[p_{12} : \mathcal{M}[v] \times_{K^{\circ}_{\ell}} \mathcal{M}[w] \times_{K^{\circ}_{\ell}} X_{1}^{\circ} \to \mathcal{M}[v] \times_{K^{\circ}_{\ell}} \mathcal{M}[w].\]
The Chern character of the pushforward above is thus supported in codimension 2 or higher, and therefore gives
\[\det \mathcal{R}_{p_{12}*} (p_{13}^{*}E \otimes L p_{23}^{*}F) = \mathcal{O}.\]
Recalling the morphism \( \pi : X^c_i \to K_{l,1}^c \) which describes the universal surface, we find that

\[
\Theta \otimes \Theta^{-1}_v \otimes \Theta^{-1}_w = \det R_p 1_2 \left[ p_3^* (\mathcal{V} \otimes L \mathcal{W}) \right] \otimes \text{pr}^* \mathcal{H}^{-\delta - \epsilon a}
\]

\[
= \text{pr}^* \left( \det R \pi_*(\mathcal{V} \otimes L \mathcal{W}) \otimes \mathcal{H}^{-\delta - \epsilon a} \right)
\]

\[
= \text{pr}^* \left( \left( \pi^* \mathcal{H} \right)^e \left( (r-d) \mathcal{O} + d \mathcal{H} + \alpha \mathcal{O}_{\sigma} \right) \otimes \left( (s-e) \mathcal{O} + e \mathcal{H} + \beta \mathcal{O}_{\sigma} \right) \right) \otimes \mathcal{H}^{-\delta - \epsilon a}
\]

Here, we wrote

\[
\lambda = (\det R \pi_* \mathcal{O}_X)^{-1} \to K_{l,1}^c
\]

for the Hodge bundle. This yields the following

**Proposition 1.** The twist \( \mathcal{T} \) defined by equation (12) is given by

\[
\mathcal{T} = \lambda^{-(r-d)(s-e)} \otimes (\det \pi_* \mathcal{H})^{e(r-d)+d(s-e)} \otimes (\det \pi_* \mathcal{H}^2)^{de}
\]

**4.3. Extensions of the Verlinde sheaves and desiderata.** We now turn our attention to the locus of elliptic \( K^3 \) with section, where the Verlinde sheaves and the isomorphism \( D \) can be extended from \( \mathcal{P} \cap K_{l,1}^c \) to all of \( \mathcal{P}_1 \) by the results of Section 3, as we now explain.

The universal data over \( \mathcal{P}_1 \) consists of the triple

\[
(\mathcal{X}, \mathcal{H}, \mathcal{F}) \to \mathcal{P}_1,
\]

where \( \mathcal{F} \) denotes the universal fiber class of the elliptic fibration. We consider the line bundle

\[
\mathcal{L} = \mathcal{H}^{r+s} \otimes \mathcal{O}(\mathcal{F})^{-(r+s)\ell - a - b},
\]

which restricts over each \( (X, H, F) \) to

\[
L = \mathcal{O}((r + s)\sigma + (r + s - a - b)f).
\]

In the product of Hilbert schemes we have the universal theta divisor

\[
\theta = \{(X, Z, W) : H^0(X, I_Z \otimes^L I_W \otimes \mathcal{L}|_X) \neq 0 \} \subset \mathcal{X}^{[d_v]} \times_{\mathcal{P}_1} \mathcal{X}^{[d_w]}.
\]

To write the corresponding line bundle, we denote by

\[
\mathcal{Z} \subset \mathcal{X}^{[d_v]} \times_{\mathcal{K}_l} \mathcal{X}, \quad \mathcal{W} \subset \mathcal{X}^{[d_w]} \times_{\mathcal{K}_l} \mathcal{X},
\]

the universal subschemes, and set standardly

\[
\mathcal{L}^{[d_v]} = \det R_p \left( \mathcal{O}_Z \otimes q^* \mathcal{L} \right), \quad \mathcal{L}^{[d_w]} = \det R_p \left( \mathcal{O}_W \otimes q^* \mathcal{L} \right).
\]

From the product

\[
\mathcal{X}^{[d_v]} \times_{\mathcal{K}_l} \mathcal{X}^{[d_w]} \times_{\mathcal{K}_l} \mathcal{X},
\]


we calculate

\[ \theta = \det \left( R_{p_{12}^*} \left( p_{13}^* \mathcal{I}_Z \otimes L p_{23}^* \mathcal{I}_W \otimes p_3^* \mathcal{L} \right) \right)^{-1} \]

\[ = \det \left( R_{p_{12}^*} \left( p_{13}^* (\mathcal{O} - \mathcal{O}_Z) \otimes L p_{23}^* (\mathcal{O} - \mathcal{O}_W) \otimes p_3^* \mathcal{L} \right) \right)^{-1} \]

\[ = \mathcal{L}^{[d_v]} \boxtimes \mathcal{L}^{[d_w]} \otimes \pi^*(\det \pi_* \mathcal{L})^{-1} \otimes \det R_{p_{12}^*} \left( p_{13}^* \mathcal{O}_Z \otimes L p_{23}^* \mathcal{O}_W \otimes p_3^* \mathcal{L} \right) \]

\[ = \mathcal{L}^{[d_v]} \boxtimes \mathcal{L}^{[d_w]} \otimes \pi^*(\det \pi_* \mathcal{L})^{-1}. \]

On the third line, the last bundle is the determinant of a complex of sheaves supported on the codimension 2 locus of intersecting subschemes in \( X^{[d_v]} \times_{K} X^{[d_w]} \) – thus it is trivial. Lemma 5.1 of [EGL] implies that

\[ \pi_* \mathcal{L}^{[d_v]} = \Lambda^{[d_v]} \pi_* \mathcal{L}, \quad \pi_* \mathcal{L}^{[d_w]} = \Lambda^{[d_w]} \pi_* \mathcal{L}. \]

The higher direct images of the line bundles \( \mathcal{L}^{[d_v]}, \mathcal{L}^{[d_w]} \) vanish by Theorem 5.2.1 of [Sc1]. We therefore finally have

\[ \pi_* \theta \simeq \Lambda^{d_v} (\pi_* \mathcal{L}) \otimes \Lambda^{d_w} (\pi_* \mathcal{L}) \otimes (\det \pi_* \mathcal{L})^{-1} \cong W' \otimes V'. \]

We set

\[ W' = \pi_* \mathcal{L}^{[d_v]}, \quad V' = \pi_* \mathcal{L}^{[d_w]} \otimes (\det \pi_* \mathcal{L})^\vee. \]

As before these sheaves are only defined up to reverse twistings by a line bundle from \( \mathcal{P}_1 \). The divisor \( \theta \) induces the duality isomorphism

\[ D' : W'^\vee \rightarrow V' \]

over \( \mathcal{P}_1 \), which is a global version of (6).

Section 3 shows that the universal relative Fourier-Mukai transform induces a birational map

\[ X^{[d_v]} \times_{\mathcal{P}_1^0} X^{[d_w]} \rightarrow M[v] \times_{\mathcal{P}_1^0} M[w] \]

regular in codimension 1 over each fiber, such that the divisors \( \theta \) and \( \Theta \) are precisely matched. Because of regularity in codimension 1, the pushforward sheaves \( \pi_* \theta \) and \( R^0 \pi_* \Theta \) coincide. Therefore

\[ W' \otimes V' \cong \mathcal{H}^0(W) \otimes \mathcal{H}^0(V) \]

over \( \mathcal{P}_1^0 \). We can furthermore align the line bundle twists inherent in the definition of \( W, V, W', V' \) so that

\[ \mathcal{H}^0(D) = D' \]

over this locus. We thus extended the Verlinde sheaves from \( \mathcal{P}_1^0 \hookrightarrow \mathcal{P}_1 \).

The resolution of the following query will however be of much greater interest.
Question 1. Is it possible to extend $W, V$ from

\[ K_\ell^0 \hookrightarrow K_\ell \]

in such a fashion that

\[ c_1(W) = -c_1(V) \]?

Combined with the results of [MOY], this would establish the strange duality conjecture over the entire locus where there is no higher cohomology, since the Baily-Borel compactification of $K_\ell$ has one dimensional boundary. It would be interesting to investigate whether $D$ is in fact a quasi-isomorphism between the complexes $W^\vee$ and $V$.

Regarding the canonical line bundle $\Theta$, it is also natural to wonder

**Question 2.** Is the Chern character $\text{ch}(R\pi_*\Theta)$ in the ring generated by the Hodge class $\lambda = -c_1(R^2\pi_*\mathcal{O}_{X^\vee})$ studied in [GK]?

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**References**


ON VERLINDE SHEAVES AND STRANGE DUALITY


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