SOLUTIONS

Problem 1.

Find the critical points of the function

$$f(x, y) = 2x^3 - 3x^2y - 12x^2 - 3y^2$$

and determine their type i.e. local min/local max/saddle point. Are there any global min/max?

`Solution:` Partial derivatives

$$f_x = 6x^2 - 6xy - 24x, f_y = -3x^2 - 6y.$$

To find the critical points, we solve

$$f_x = 0 \implies x^2 - xy - 4x = 0 \implies x(x - y - 4) = 0 \implies x = 0 \text{ or } x - y - 4 = 0$$

$$f_y = 0 \implies x^2 + 2y = 0.$$

When $x = 0$ we find $y = 0$ from the second equation. In the second case, we solve the system below by substitution

$$x - y - 4 = 0, x^2 + 2y = 0 \implies x^2 + 2x - 8 = 0$$

$$\implies x = 2 \text{ or } x = -4 \implies y = -2 \text{ or } y = -8.$$

The three critical points are

$$(0, 0),\ (2, -2),\ (-4, -8).$$

To find the nature of the critical points, we apply the second derivative test. We have

$$A = f_{xx} = 12x - 6y - 24,\ B = f_{xy} = -6x,\ C = f_{yy} = -6.$$

At the point $(0, 0)$ we have

$$f_{xx} = -24, f_{xy} = 0, f_{yy} = -6 \implies AC - B^2 = (-24)(-6) - 0 > 0 \implies \boxed{(0,0) \text{is local max}}.$$

Similarly, we find

$$\boxed{(2, -2) \text{ is a saddle point}}$$

since

$$AC - B^2 = (12)(-6) - (-12)^2 =< 0$$

and

$$\boxed{(-4, -8) \text{ is saddle}}$$

since

$$AC - B^2 = (-24)(-6) - (24)^2 < 0.$$

The function has no global min since

$$\lim_{y \to \infty, x=0} f(x, y) = -\infty$$

and similarly there is no global maximum since

$$\lim_{x \to \infty, y=0} f(x, y) = \infty.$$

Problem 2.

Determine the global max and min of the function

$$f(x,y) = x^2 - 2x + 2y^2 - 2y + 2xy$$

over the compact region

$$-1 \le x \le 1, \; 0 \le y \le 2.$$

`Solution:` We look for the critical points in the interior:

$$\nabla f = (2x - 2 + 2y, 4y - 2 + 2x) = (0,0) \implies 2x - 2 + 2y = 4y - 2 + 2x = 0 \implies y = 0, x = 1.$$

However, the point $(1,0)$ is not in the interior so we discard it for now.

We check the boundary. There are four lines to be considered:

- the line $x = -1$:
  $$f(-1,y) = 3 + 2y^2 - 4y.$$
  The critical points of this function of y are found by setting the derivative to zero:
  $$\frac{\partial}{\partial y}(3 + 2y^2 - 4y) = 0 \implies 4y - 4 = 0 \implies y = 1 \text{ with } \boxed{f(-1,1) = 1}.$$
- the line $x = 1$:
  $$f(1,y) = 2y^2 - 1.$$
  Computing the derivative and setting it to 0 we find the critical point $y = 0$. The corresponding point $(1,0)$ is one of the corners, and we will consider it separately below.
- the line $y = 0$:
  $$f(x,0) = x^2 - 2x.$$
  Computing the derivative and setting it to 0 we find $2x - 2 = 0 \implies x = 1$. This gives the corner $(1,0)$ as before.
- the line $y = 2$:
  $$f(x,2) = x^2 + 2x + 4$$
  with critical point $x = -1$ which is again a corner.

Finally, we check the four corners

$$(-1,0), (1,0), (-1,2), (1,2).$$

The values of the function f are

$$\boxed{f(-1,0) = 3}, \boxed{f(1,0) = -1}, \boxed{f(-1,2) = 3}, \boxed{f(1,2) = 7}.$$

From the boxed values we select the lowest and the highest to find the global min and global max. We conclude that

global minimum occurs at $(1,0)$

global maximum occurs at $(1,2)$.

Problem 3.

Using Lagrange multipliers, optimize the function

$$f(x,y) = x^2 + (y+1)^2$$

subject to the constraint

$$2x^2 + (y-1)^2 \leq 18.$$

`Solution:` We check for the critical points in the interior

$$f_x = 2x, f_y = 2(y+1) \implies (0,-1) \text{ is a critical point .}$$

The second derivative test

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

shows this a local minimum with

$$\boxed{f(0,-1) = 0}.$$

We check the boundary

$$g(x,y) = 2x^2 + (y-1)^2 = 18$$

via Lagrange multipliers. We compute

$$\nabla f = (2x, 2(y+1)) = \lambda \nabla g = \lambda(4x, 2(y-1)).$$

Therefore

$$2x = 4x\lambda \implies x = 0 \text{ or } \lambda = \frac{1}{2}$$
$$2(y+1) = 2\lambda(y-1).$$

In the first case $x = 0$ we get

$$g(0,y) = (y-1)^2 = 18 \implies y = 1 + 3\sqrt{2}, 1 - 3\sqrt{2}$$

with values

$$\boxed{f(0, 1+3\sqrt{2}) = (2+3\sqrt{2})^2}, \boxed{f(0, 1-3\sqrt{2}) = (2-3\sqrt{2})^2}.$$

In the second case $\lambda = \frac{1}{2}$ we obtain from the second equation

$$2(y+1) = y - 1 \implies y = -3.$$

Now,

$$g(x,y) = 18 \implies x = \pm 1.$$

At $(\pm 1, -3)$, the function takes the value

$$\boxed{f(\pm 1, -3) = (\pm 1)^2 + (-3+1)^2 = 5.}$$

By comparing all boxed values, it is clear the $(0,-1)$ is the minimum, while $(0, 1+3\sqrt{2})$ is the maximum.

Problem 4.

Consider the function

$$w = e^{x^2 y}$$

where

$$x = u\sqrt{v}, \ y = \frac{1}{uv^2}.$$

Using the chain rule, compute the derivatives

$$\frac{\partial w}{\partial u}, \ \frac{\partial w}{\partial v}.$$

`Solution:` We have

$$\frac{\partial w}{\partial x} = 2xy\exp(x^2y) = 2u\sqrt{v}\frac{1}{uv^2}\exp\left(u^2v\cdot\frac{1}{uv^2}\right) = \frac{2}{v^{3/2}}\exp\left(\frac{u}{v}\right)$$

$$\frac{\partial w}{\partial y} = x^2\exp(x^2y) = u^2v\exp\left(\frac{u}{v}\right)$$

$$\frac{\partial x}{\partial u} = \sqrt{v},\ \ \frac{\partial x}{\partial v} = \frac{u}{2\sqrt{v}}$$

$$\frac{\partial y}{\partial u} = -\frac{1}{u^2v^2},\ \ \frac{\partial y}{\partial v} = -\frac{2}{uv^3}.$$

Thus

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{2}{v^{3/2}}\exp\left(\frac{u}{v}\right)\cdot\sqrt{v} - u^2v\exp\left(\frac{u}{v}\right)\cdot\frac{1}{u^2v^2} =$$

$$= \frac{2}{v}\exp\left(\frac{u}{v}\right) - \frac{1}{v}\exp\left(\frac{u}{v}\right) = \frac{1}{v}\exp\left(\frac{u}{v}\right).$$

Similarly,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = -\frac{u}{v^2}\exp\left(\frac{u}{v}\right).$$

Problem 5.

(i) *For what value of the parameter a, will the planes*

$$ax + 3y - 4z = 2,\ \ x - ay + 2z = 5$$

be perpendicular?

(ii) *Find a vector parallel to the line of intersection of the planes*

$$x - y + 2z = 2,\ \ 3x - y + 2z = 1.$$

(iii) *Find the plane through the origin parallel to*

$$z = 4x - 3y + 8.$$

(iv) *Find the angle between the vectors*

$$\mathbf{v} = (1, -1, 2),\ \ \mathbf{w} = (1, 3, 0).$$

(v) *A plane has equation*

$$z = 5x - 2y + 7.$$

For what values of a is the vector

$$(a, 1, \frac{1}{2})$$

normal to the plane?

`Solution:`

(i) The normal vectors to the two planes are

$$n_1 = (a, 3, -4),\ \ n_2 = (1, -a, 2).$$

The planes are perpendicular if n_1, n_2 are perpendicular. We compute the dot product

$$n_1\cdot n_2 = 0 \implies a\cdot 1 + 3\cdot(-a) + (-4)\cdot 2 = 0 \implies -2a - 8 = 0 \implies a = -4.$$

(ii) The vectors normal to the two planes are

$$n_1 = (1, -1, 2),\ n_2 = (3, -1, 2).$$

The line of intersection will be perpendicular to both n_1, n_2. But so is the cross product. Thus the line of intersection will be parallel to the cross product

$$n_1 \times n_2 = (1, -1, 2) \times (3, -1, 2) = (0, 4, 2).$$

(iii) The second plane must have the same normal vector hence the same coefficients for x, y, z. Since it passes through the origin, the equation is

$$z = 4x - 3y.$$

(iv) We compute the angle using the dot product

$$\cos\theta = \frac{v \cdot w}{||v|| \cdot ||w||} = \frac{-2}{\sqrt{6} \cdot \sqrt{10}} = -\frac{1}{\sqrt{15}}.$$

(v) The plane has the equation

$$5x - 2y - z = -7 \implies -\frac{5}{2}x + y + \frac{1}{2}z = \frac{7}{2}$$

hence a normal vector is $(-\frac{5}{2}, 1, \frac{1}{2})$. Comparing with the vector we are given, we see that

$$a = -\frac{5}{2}.$$

Problem 6.

(i) *Compute the second degree Taylor polynomial of the function*

$$f(x, y) = e^{x^2 - y}$$

around $(1, 1)$.

(ii) *Compute the second degree Taylor polynomial of the function*

$$f(x) = \sin(x^2)$$

around $x = \sqrt{\pi}$.

(iii) *The second degree Taylor polynomial of a certain function* $f(x, y)$ *around* $(0, 1)$ *equals*

$$1 - 4x^2 - 2(y - 1)^2 + 3x(y - 1).$$

Can the point $(0, 1)$ *be a local minimum for* f*? How about a local maximum?*

`Solution:`

(i) After computing all derivatives and substituting, we find the answer

$$1 + 2(x - 1) - (y - 1) + 3(x - 1)^2 + \frac{1}{2}(y - 1)^2 - 2(x - 1)(y - 1).$$

(ii) We have $f(\sqrt{\pi}) = 0$. The first derivative is

$$f_x = 2x\cos x^2 \implies f_x(\sqrt{\pi}) = 2\sqrt{\pi}\cos\pi = -2\sqrt{\pi}.$$

The second derivative is

$$f_{xx} = 2\cos x^2 - 2x\sin x^2 \implies f_{xx}(\sqrt{\pi}) = -2.$$

The Taylor polynomial is

$$-2\sqrt{\pi}(x - \sqrt{\pi}) - (x - \sqrt{\pi})^2 = -x^2 + \pi.$$

(iii) From the Taylor polynomial we find

$$f_x(0,1) = f_y(0,1) = 0$$

so $(0,1)$ is a critical point. We can find the second derivatives

$$\frac{1}{2}f_{xx}(0,1) = -4,\ \frac{1}{2}f_{yy}(0,1) = -2,\ f_{xy}(0,1) = 3.$$

By the second derivative test

$$AC - B^2 = (-8)(-4) - 3^2 > 0,\quad A = -8 < 0 \implies (0,1) \text{ is a local maximum.}$$

Problem 7.

(i) *The temperature $T(x,y)$ in a long thin plane at the point (x,y) satisfies Laplace's equation*

$$T_{xx} + T_{yy} = 0.$$

Does the function

$$T(x,y) = \ln(x^2 + y^2)$$

satisfy Laplace's equation?

(ii) *For the function*

$$f(x,y) = \sin(x^2 + y^2)\ln(x^4y^4 + 1)\tan(xy)$$

is it true that

$$f_{xyxyy} = f_{yyxyx}?$$

`Solution:`

(i) We compute

$$T_x = \frac{2x}{x^2+y^2} \implies T_{xx} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}$$
$$T_y = \frac{2y}{x^2+y^2} \implies T_{yy} = \frac{2x^2 - 2y^2}{(x^2+y^2)^2}.$$

Therefore,

$$T_{xx} + T_{yy} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2} + \frac{2x^2 - 2y^2}{(x^2+y^2)^2} = 0.$$

(ii) The two derivatives are equal as the order in which derivatives are computed is unimportant.

Problem 8.

Consider the function $f(x,y) = \frac{x^2}{y^4}$.

(i) *Carefully draw the level curve passing through $(1,-1)$. On this graph, draw the gradient of the function at $(1,-1)$.*

(ii) *Compute the directional derivative of f at $(1,-1)$ in the direction $\mathbf{u} = \left(\frac{4}{5}, \frac{3}{5}\right)$. Use this calculation to estimate*

$$f((1,-1) + .01\mathbf{u}).$$

(iii) *Find the unit direction $\mathbf{v}$ of steepest descent for the function f at $(1,-1)$.*

(iv) *Find the two unit directions $\mathbf{w}$ for which the derivative $f_{\mathbf{w}} = 0$.*

`Solution:`

(i) The level is $f(1,1) = 1$. The level curve is

$$f(x,y) = f(1,1) = 1 \implies x^2 = y^4 \implies x = \pm y^2.$$

The level curve is a union of two parabolas through the origin. The gradient

$$\nabla f = \left(\frac{2x}{y^4}, \frac{-4x^2}{y^5}\right) \implies \nabla f(1,-1) = (2,4)$$

is normal to the parabolas.

(ii) We compute

$$f_{\mathbf{u}} = \nabla f \cdot \mathbf{u} = (2,4) \cdot \left(\frac{4}{5}, \frac{3}{5}\right) = 4.$$

For the approximation, we have $f(1,-1) = 1$ and

$$f((1,-1) + .01\mathbf{u}) \approx f(1,-1) + .01 f_{\mathbf{u}} = 1 + .01 \cdot 4 = 1.04.$$

(iii) The direction of steepest decrease is opposite to the gradient. We need to divide by the length to get a unit vector:

$$\mathbf{v} = -\frac{\nabla f}{||\nabla f||} = -\frac{(2,4)}{\sqrt{2^2+4^2}} = \left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right).$$

(iv) Write

$$\mathbf{w} = (w_1, w_2).$$

We have

$$f_{\mathbf{w}} = \nabla f \cdot \mathbf{w} = (2,4) \cdot \mathbf{w} = 2w_1 + 4w_2 = 0 \implies w_1 = -2w_2.$$

Since $\mathbf{w}$ has unit length

$$w_1^2 + w_2^2 = 1 \implies (-2w_2)^2 + w_2^2 = 1 \implies w_2 = \pm\frac{1}{\sqrt{5}}.$$

Therefore

$$\mathbf{w} = \pm\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

Problem 9.

Consider the function

$$f(x,y) = \sqrt{\ln(e^{2x}y^3)}.$$

(i) *Write down the tangent plane to the graph of f at $(2,1)$.*

(ii) *Find the approximate value of the number*

$$\sqrt{\ln(e^{4.1}(1.02)^3)}.$$

`Solution:`

(i) Using the chain rule, we compute

$$f_x = \frac{1}{2}\frac{\frac{2e^{2x}}{e^{2x}}}{\sqrt{\ln(e^{2x}y^3)}} = \frac{1}{\sqrt{\ln(e^{2x}y^3)}} \implies f_x(2,1) = \frac{1}{\sqrt{\ln e^4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

Similarly,

$$f_y = \frac{1}{2}\frac{\frac{3y^2e^{2x}}{y^3e^{2x}}}{\sqrt{\ln(e^{2x}y^3)}} = \frac{3}{2y}\frac{1}{\sqrt{\ln(e^{2x}y^3)}} \implies f_y(2,1) = \frac{1}{2}\frac{3}{\sqrt{\ln e^4}} = \frac{3}{2}\cdot\frac{1}{\sqrt{4}} = \frac{3}{4}.$$

We compute

$$f(2,1) = \sqrt{\ln e^4} = \sqrt{4} = 2.$$

The tangent plane is

$$z - 2 = \frac{1}{2}(x-2) + \frac{3}{4}(y-1) \implies z = \frac{1}{2}x + \frac{3}{4}y + \frac{1}{4}.$$

(ii) The number we are approximating is

$$f(2.05, 1.02) \approx \frac{1}{2}\cdot 2.05 + \frac{3}{4}\cdot 1.02 + \frac{1}{4} = 2.04.$$

Problem 10.

Suppose that

$$z = e^{3x+2y}, \;\; y = \ln(3u-w), \;\; x = u + 2v.$$

Calculate

$$\frac{\partial z}{\partial v}, \frac{\partial z}{\partial w}.$$

`Solution:`

By the chain rule

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} = 3e^{3x+2y}\cdot 2 = 6e^{3x}e^{2y} = 6e^{3u+6v}e^{2\ln(3u-w)} = 6e^{3u+6v}(3u-w)^2.$$

Similarly,

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial w} = 2e^{3x+2y}\cdot\frac{-1}{3u-w} = 2e^{3u+6v}e^{2\ln(3u-w)}\frac{-1}{3u-w}$$
$$= 2e^{3u+6v}(3u-w)^2\cdot\frac{-1}{3u-w} = -2e^{3u+6v}(3u-w).$$

Problem 11.

(i) *Find z such that*

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots = 3.$$

(ii) *Calculate the series*

$$\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \ldots + \frac{2^{99}}{3^{100}}.$$

`Solution:`

(i) This is a geometric series with step $\frac{1}{z}$. Its sum equals

$$\frac{1}{1-\frac{1}{z}} = 3 \implies 1 - \frac{1}{z} = \frac{1}{3} \implies z = \frac{3}{2}.$$

(ii) This is a finite geometric series with 100 terms and initial term $1/3$ and step $2/3$. The sum equals

$$\frac{1}{3} \cdot \frac{1 - \left(\frac{2}{3}\right)^{100}}{1 - \frac{2}{3}} = 1 - \left(\frac{2}{3}\right)^{100}.$$

Problem 12.

The probability density function for the outcome x of a certain experiment is

$$p(x) = Ce^{-x}, \text{ for } x \geq 0.$$

(i) *What is the value of the constant C?*
(ii) *What is the cumulative distribution function?*
(iii) *What is the median of the experiment?*
(iv) *What is the mean of the experiment?*
(v) *What is the probability that the outcome of the experiment is bigger than 1?*

`Solution:`

(i) The pdf must integrate to 1 hence

$$\int_0^\infty p(x)dx = 1 \implies C \int_0^\infty e^{-x} = 1 \implies -C \cdot e^{-x}|_{x=0}^\infty = 1 \implies -C(0-1) = 1 \implies C = 1.$$

(ii) The cdf is obtained by integrating the pdf:

$$P(x) = \int_0^x p(t)dt = \int_0^x e^{-t}\, dt = -e^{-t}|_{t=0}^{t=x} = 1 - e^{-x}.$$

(iii) To find the median, we set the pdf to $1/2$:

$$P(T) = \frac{1}{2} \implies 1 - e^{-T} = \frac{1}{2} \implies e^{-T} = \frac{1}{2} \implies T = \ln 2.$$

(iv) The mean is computed by the integral

$$\text{mean} = \int_0^\infty xp(x)\, dx = \int_0^\infty xe^{-x} = 1.$$

The last integral was found by integration by parts

$$\int_0^\infty xe^{-x} = -xe^{-x}|_{x=0}^\infty + \int_0^\infty e^{-x}dx = 0 + \int_0^\infty e^{-x}dx = -e^{-x}|_{x=0}^\infty = 1.$$

(v) The probability the outcome is at most 1 is $P(1) = 1 - e^{-1}$. The probability that $x \geq 1$ is

$$1 - P(1) = e^{-1}.$$

Problem 13.

Consider the function $f(x, y) = 5 - (x+1)^2 - y^2$.

(i) *Draw the cross section corresponding to $x = 1$.*
(ii) *Draw the contour diagram of f showing at least three levels.*
(iii) *Draw the graph of f.*
(iv) *What is the equation of the tangent plane to the graph of f at $(1, 0, 1)$?*

`Solution:`

(i) The cross section is the parabola $z = 1 - y^2$.

(ii) The level curve for level c is

$$f(x,y) = c \implies (x+1)^2 + y^2 = 5 - c.$$

The level curves are circles of centers $(-1,0)$ and radius $\sqrt{5-c}$. For instance, picking levels $c=1, c=2, c=4$ we get circles of radii $2, \sqrt{3}, 1$ of center $(-1,0)$.

(iii) The graph of f is a downward pointing paraboloid with vertex at $(-1,0,5)$.

(iv) The partial derivatives

$$f_x = -2(x+1) \implies f_x(1,0) = -4$$

$$f_y = -2y \implies f_y(1,0) = 0.$$

The tangent plane is

$$z - 1 = -4(x-1) + 0(y-0) \implies z = -4x + 5.$$

Problem 14.

Find the point on the plane

$$2x + 3y + 4z = 29$$

that is closest to the origin. You may want to minimize the square of the distance to the origin.

`Solution:` We minimize the square of the distance to the origin

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraint

$$g(x,y,z) = 2x + 3y + 4z = 29.$$

We use Lagrange multipliers

$$\nabla f = (2x, 2y, 2z), \nabla g = (2,3,4).$$

Then

$$\nabla f = \lambda \nabla g \implies 2x = 2\lambda, 2y = 3\lambda, 2z = 4\lambda \implies x = \lambda, y = \frac{3\lambda}{2}, z = 2\lambda.$$

Since

$$2x + 3y + 4z = 2\lambda + \frac{9\lambda}{2} + 8\lambda = \frac{29\lambda}{2} = 29 \implies \lambda = 2 \implies x = 2, y = 3, z = 4.$$

The closest point is $(2,3,4)$.

Problem 15.

Find the critical points of the function $f(x,y) = 2x^3 + 6xy + 3y^2$ and describe their nature.

`Solution:`

We set the first derivatives to zero:

$$f_x = 6x^2 + 6y = 0 \implies x^2 + y = 0$$

$$f_y = 6x + 6y = 0 \implies x + y = 0.$$

We solve

$$y = -x^2 = -x \implies x^2 = x \implies x = 0 \text{ or } x = 1.$$

The critical points are $(0,0)$ and $(1,-1)$.

We compute the second derivatives

$$A = f_{xx} = 12x, B = f_{xy} = 6, C = f_{yy} = 6.$$

For the critical point $(0,0)$ we have

$$AC - B^2 = 0 \cdot 6 - 6^2 < 0 \implies (0,0) \text{ is saddle point.}$$

For the critical point $(1,-1)$ we have

$$AC - B^2 = 12 \cdot 6 - 6^2 > 0, A > 0 \implies (1,-1) \text{ is local minimum.}$$