Math 220B Final Exam Review

To review, we list below the Main Topics covered in this class (this is not a comprehensive list):

7. Riemann Mapping Theorem.
8. Schwarz Reflection Principle.

Additional Practice Problems

Please review the course material and the homework problems. In case you need more practice problems, a list is below. There’s no need to solve them all before the final; they’re here just in case you think you need more practice.

1. (Qualifying Exam 2019.) Show that there are no bijective holomorphic maps $f : \{0 < |z| < 1\} \to \{1 < |z| < 2\}$.

2. (Conway, Chapter VI.2.6, page 133.) Assume $f$ is holomorphic in a region containing $\Delta = \Delta(0, 1)$, and that $|f(z)| = 1$ if $|z| = 1$. Assume that $f$ has a simple zero at $z = \frac{1}{4}(1 + i)$ and a double zero at $z = \frac{1}{2}$. Can $f(0) = 1/2$?

3. (Conway, Chapter VI.2.8, page 133.) Is there a holomorphic function $f : \Delta \to \Delta$ such that $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$? Is it unique?

4. (Qualifying Exam 2017.) Let $f : h^+ \to \mathbb{C}$ such that $f(i) = 0$ and $|f(z)| < 1$ for all $z \in h^+$. What is the maximum value of $|f(2i)|$?

5. Let $f$ be a holomorphic map between the strip $S = \{-1 < \text{Re } z < 1\}$ and the unit disc $\Delta(0, 1)$ such that $f(0) = 0$. What is the maximum value of $|f'(0)|$?

6. Let $A = \{2 < |z| < 3\}$ and $f(z) = \frac{e^z}{z}$. Can the function $f$ be approximated locally uniformly in $A$ by polynomials? Can it be approximated locally uniformly in $A$ by rational functions with poles only at 1 and 4? By rational functions with poles only at 4?
7. (Conway, Chapter VII.6.1, page 166.) Show that

\[ \cos \pi z = \prod_{n=1}^{\infty} \left( 1 - \frac{4z^2}{(2n-1)^2} \right). \]

8. Show that \( f(z) = -\cos z \) determines a biholomorphism between the half infinite strip \( \{ z = x + iy : 0 < x < \pi, y > 0 \} \) and the upper half plane.

9. Let \( \mathcal{F} \) be the family of holomorphic functions on \( \Delta(0,1) \) such that

\[ |f'(z)|(1 - |z|^2) + |f(0)| \leq 1. \]

Is \( \mathcal{F} \) normal?

10. (Qualifying Exam 2017.) Construct a meromorphic function with simple poles at \( z = n \) and residues equal to \( n/\sqrt{n} \) for \( n = 1, 2, 3 \ldots \)

11. (Qualifying Exam 2008.) Prove that there exist a sequence \( R_n \) of rational functions whose finite poles are only at \( 3/2 \) such that

\[ \lim_{n \to \infty} R_n(z) = 1 \quad \text{for} \quad |z| = 1, \quad \lim_{n \to \infty} R_n(z) = 2 \quad \text{for} \quad 2 \leq |z| \leq 3. \]

12. (Qualifying Exam 2020.) Let \( G \neq \mathbb{C} \) be a connected set such that \( \hat{\mathbb{C}} \setminus G \) is connected. Show that if \( f : G \to G \) is holomorphic and admits 2 fixed points then \( f \) is the identity.

13. (Conway, Chapter VIII.3.5, page 209.) Assume that \( a_n \) is an infinite sequence such that \( |a_n| \to \infty \) and let \( A_n \in \mathbb{C} \). Show that there exists an entire function \( f \) such that \( f(a_n) = A_n \).

In fact, prove the stronger statement that fixing \( a_n, m_n, \) and values \( A_{nk} \) for \( 0 \leq k \leq m_n \), one can construct an entire function \( f \) such that

\[ f^{(k)}(a_n) = A_{nk} \]

for all \( 0 \leq k \leq m_n \).

14. (Qualifying Exam 2009.) Assume that \( \alpha_n \) is an infinite sequence such that \( |\alpha_n| \to \infty \) and let \( \beta_n \) be complex numbers. Show that there exists an entire function such that \( f(\alpha_n) = \beta_n \) with multiplicity 2, that is \( f - \beta_n \) has a zero of order 2 at \( \alpha_n \).

15. (Qualifying Exam 2016.) Let \( a_k = 1 - \frac{1}{k^2} \) for \( k \geq 1 \). Let \( f_n(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - a_k} \).

(i) Show that \( f_k \) converges to a holomorphic function \( f : \Delta(0,1) \to \Delta(0,1) \).

(ii) Show that there does not exist an open set \( U \subset \mathbb{C} \) and a holomorphic function \( g : U \to \mathbb{C} \) such that \( \overline{\Delta}(0,1) \subset U \) and \( f(z) = g(z) \) for \( z \in \Delta \).