Logistics

(1) Zoom lectures — MWF 3-3:50 PM.

(2) Office Hour — W 4-5:30 PM

(3) Psets — due Fridays, weekly

(4) Grades — 30% HWK

50% midterm

40% Final

(5) Midterm — take home, Feb 12

(6) Final — March 17, 3-6 PM

(7) Canvas/Gradescope/Website

math.ucsd.edu/~doprea/220w21.html

(8) Attendance
Part I: Sequences / Series / Products

1. Infinite products of holomorphic functions
   Weierstrass Problem

2. Sequences & series of meromorphic functions
   Mittag-Leffler Problem

3. Sequences of hol functions, Montl families

Part II: Geometric aspects / Conformal maps

4. Schwarz lemma, automorphisms of $\Delta$, $\mathbb{H}$, $\mathbb{H}^*$...

5. Riemann mapping theorem
Part III: Further topics (if time)

(6) Runge's theorem

(7) Schwarz Reflection

(8) Harmonic functions

(9) Hadamard factorization

(10) Little & Big Picard.

Some of these will only be covered in Math 220C.
Math 220A, Lecture 10: $f \neq 0$ entire has countably many zeroes that do not accumulate.

**Weierstrass Problem**

Given a sequence of distinct $\{a_n\}$, $a_n \to \infty$ and positive integers $\{m_n\}$, is there an entire function with zeroes only at $\{a_n\}$ with order exactly $\{m_n\}$?

**Weierstrass Problem**

Given $\{a_n\}, \{m_n\}$ as above, $\{a_{nj}\}_{0 \leq j < m_n}$ is there an entire function $f$ with $f^{(j)}(a_n) = A_{nj}$ for $0 \leq j < m_n$.
**Mittag-Leffler Problem**

Take \( \{ a_n \} \) as above.

We can always find a meromorphic function \( f \) in \( \mathbb{C} \) with poles only at \( a_n \), e.g., take \( g \) solving Weierstrass at \( \{ a_n \} \) and set \( f = 1/g \).

Mittag-Leffler asks if we can furthermore prescribe the Laurent principal parts.

Given \( \{ a_n \} \) distinct, \( a_n \to \infty \), and polynomials \( p_n \left( \frac{1}{z-a_n} \right) \) without constant terms, is there a meromorphic function in \( \mathbb{C} \) with poles only at \( a_n \) and Laurent expansion

\[
f = p_n \left( \frac{1}{z-a_n} \right) + \ldots \text{ near } a_n.
\]
Weierstrass – Poincaré Problem

Is any meromorphic function a quotient of two holomorphic functions?

Remark The three questions above can be asked & answered for all \( \mathbb{U} \subseteq \mathbb{C} \) open & connected.
Karl Weierstraß
1815 - 1897

Gösta Mittag-Leffler
1846 - 1927
We will also illustrate general theory e.g.

- factorization of sine, \( R \)-function
- Weierstrass problem.
- elliptic functions - Weierstrass \( \wp \)-function
- Mittag-Leffler
**Tools** — sequences, series, products of *holomorphic* & *meromorphic* functions.

**Last quarter**

sequences

series of holomorphic functions.

**This quarter**

Weierstrass requires *infinite products* of holomorphic functions. Intuitively, this makes sense. We could try to solve Weierstrass by setting \( f(z) = \prod_{n=1}^{\infty} (2 - a_n) \) but convergence is an issue.

Mittag-Leffler requires *infinite sums* of meromorphic functions.
Quick Review of the last lectures in Math 220A

Sequences \( \{ f_n \} \) holomorphic in \( u \in \mathbb{C} \)

Recall that the notion of convergence we considered was local uniform convergence \( \iff \) convergence on compact subsets, i.e.
\[
\lim_{n \to \infty} f_n = f \iff f_n \xrightarrow{\text{c}} f
\]

Weierstrass Convergence Theorem

Let \( f_n : U \rightarrow \mathbb{C} \) holomorphic, \( f_n \xrightarrow{\text{lu.}} f \). Then

1. \( f \) holomorphic
2. \( f^{(k)} \xrightarrow{\text{lu.}} f^{(k)} \)
$f_n : U \to \mathbb{C}$ holomorphic. Assume

$(\star) \forall K \subseteq U \text{ compact } \exists M_n(K), \|f_n\| \leq M_n(K)$.

over $K$, $\sum_{n=1}^{\infty} M_n(K) < \infty$.

$M$-test

$\Rightarrow f = \sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly on every $K$.

Weierstrass

$\Rightarrow f$ holomorphic and $f' = \sum_{n=1}^{\infty} f_n'$

Thm

This quarter

II infinite products $\prod_{n=1}^{\infty} f_n(z)$ Weierstrass

IV series of meromorphic functions $\sum_{n=1}^{\infty} f_n(z)$ Mittag-Leffler
Main Question: Given $f_k: D \rightarrow \mathbb{C}$ holomorphic, how do we define $f(z) = \prod_{k=1}^{\infty} f_k(z)$? Furthermore, is $f$ holomorphic?

Is $f$ holomorphic?

Is it true that $\text{Zero}(f) = \bigcup_{k} \text{Zero}(f_k)$?

Step back: Given $p_k \in \mathbb{C}$, how to define

$$P = \prod_{k=1}^{\infty} p_k?$$

Wrong answer: Form the partial products

$$P_n = \prod_{k=1}^{n} p_k$$ and define $P = \lim_{n \to \infty} P_n$. 

Infinite Products
If \( p_k = 0 \Rightarrow P = 0 \) no matter what the other \( p_k \)'s are. Thus one term would determine convergence of the product which is unfair.

We could have \( P = 0 \) even though \( p_k \neq 0 \) for \( k \neq 0 \) e.g.

\[
\prod_{k=1}^{n} \frac{1}{k} = 0
\]

Thus we have no control over the zeroes of a product of functions.

**Question** What kind of products will we consider?

**Definition**

\[
\prod_{k=1}^{n} p_k = P \text{ converges iff } \exists M \text{ such that } \\
\lim_{n \to \infty} \prod_{k=M}^{n} p_k \text{ exists and equals } \hat{P}. \text{ We then set} \\

P = p_1 \cdots p_{n-1} \hat{P}
\]
Remarks \[7\] the value of \( E \) is independent of \( m \) (check).

in the infinite products above only finitely many terms can be 0. (\( \mathring{p} \neq 0 \iff \mathring{p}_k = 0 \) for \( k \geq m \)).

With this definition we have control over the zeros. Indeed

\[
\mathring{p} = 0 \iff \mathring{p}_1 \cdots \mathring{p}_m \mathring{E} = 0 \quad (\mathring{p} \neq 0)
\]

\[
\iff \mathring{p}_1 = 0 \text{ or } \cdots \text{ or } \mathring{p}_{m-1} = 0
\]

\[
\iff \exists k \text{ with } \mathring{p}_k = 0.
\]

Thus this behaves in the same fashion as finite products.
Last time - Infinite products  Conway VII. 5

Given $p_k \in \mathbb{C}$, define $P = \prod_{k=n}^{\infty} p_k$ convergent product if $\exists N$ with

$$\lim_{n \to \infty} \prod_{k=n}^{N} p_k = P \neq 0$$

and set

$$P = p_1 \ldots p_{n-1} \quad \hat{P} = \text{value independent of } N.$$

Remarks

1. There exist finitely many zero terms

2. $P = 0 \iff \exists k \text{ with } p_k = 0$

3. $N = k$

$$P^n = \prod_{k=L}^{n} p_k \quad \to \quad \frac{\hat{P}}{P} = 1 \quad \text{as } n \to \infty.$$

Henceforth, we will assume $p_k = 1 + a_k$, $a_k \to 0$.
We seek to connect infinite products to infinite series.

Recall principal branch of logarithm \( z \neq 0, z \in \mathbb{C} \setminus \mathbb{R}_- \),

\[
\log (z) = \log r + i \theta \\
\theta \in (-\pi, \pi)
\]

\[ \text{Log (1+2), makes sense if \( z \text{ small since } 1+2 \notin \mathbb{R}_- \).} \]

Lemma \( \sum_{k=1}^{\infty} \log (1 + a_k) \) converges \( \iff \exists N > 0 \) such that \( \sum_{k=N}^{\infty} \log (1 + a_k) \) converges.

Proof: \( \text{Write} \)

\[
S_n = \sum_{k=1}^{n} \log (1 + a_k) \\
L_n = \prod_{k=1}^{n} (1 + a_k)
\]

\[ \Rightarrow \sum_{k=N}^{\infty} S_n = \prod_{k=N}^{\infty} \]

\[ \prod_{k=1}^{\infty} (1 + a_k) \]
If \( \lim s_n = s \), \( P_n = e^{s_n} \rightarrow e^s = \hat{P} \neq 0 \).

Assume \( P_n \rightarrow \hat{P} \). We wish to show \( s_n \rightarrow s \).

Pick \( \hat{P} \) such that \( \hat{P} \notin \mathbb{R}_{<0} \). We use the branch \( \log \alpha \).

\[
\log \alpha z = \log r + i\theta, \quad \theta \in (\alpha, \alpha + 2\pi)
\]

\[
e^{s_n} = P_n \quad \Rightarrow \quad s_n = \log \alpha P_n + 2\pi n l_n, \quad l_n \in \mathbb{Z}.
\]

We claim \( l_n = l_{n-1} \) if \( n \rightarrow \infty \) \( \Rightarrow \exists l, \quad l_n = l \).

\[
\log \alpha \rightarrow \log \alpha \hat{P} + 2\pi i l \quad = \quad s
\]

To prove the claim, consider

\[
S_n - S_{n-1} = \log \alpha P_n - \log \alpha P_{n-1} + 2\pi i (l_n - l_{n-1})
\]

\[
= \quad 0 \quad \text{as } n \rightarrow \infty
\]

Note \( S_n - S_{n-1} = \log (1 + a_n) \rightarrow \log 1 = 0 \)

\[
\log \alpha P_n - \log \alpha P_{n-1} \rightarrow \log \alpha \hat{P} - \log \alpha \hat{P} = 0
\]

This shows \( l_n - l_{n-1} \rightarrow 0 \) as \( n \rightarrow \infty \) \( \Rightarrow l_n = l_{n-1} \) if \( n \gg 0 \).
**Absolute convergence**

**Question:** How do we define absolutely convergent products $\prod_{k=1}^{\infty} p_k$?

**Wrong Answer:** $\prod_{k=1}^{\infty} |p_k|$ converges

But then for $p_k = (-1)^k$, $\prod_{k=1}^{\infty} (-1)^k$ converges absolutely, which is absurd.

**Def:** $\prod_{k=1}^{\infty} (1+a_k)$ converges absolutely iff $\exists N$ such that $\sum_{k=N}^{\infty} \log(1+a_k)$ converges absolutely.
Lemma TFAE

1. \( \prod_{k=1}^{\infty} (1 + a_k) \text{ converges absolutely} \)

2. \( \sum_{k=1}^{\infty} a_k \text{ converges absolutely} \)

3. \( \prod_{k=1}^{\infty} (1 + 10k) \text{ converges} \)

Proof Consider Taylor expansion in \( \Delta (a, s) \in \mathbb{C} \setminus (-\infty, 1] \)

\[
\log (1+2) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \ldots
\]

\[
\frac{\log (1+2)}{2} = 1 - \frac{2}{2} + \frac{2^2}{3} - \ldots
\]

\[
\lim_{2 \to 0} \frac{\log (1+2)}{2} = 1 \Rightarrow \exists \ p > 0 \text{ such that } \left| 2^1 \right| < p, 2 \neq 0.
\]

\[
\frac{1}{2} \leq \left| \frac{\log (1+2)}{2} \right| \leq \frac{3}{2}
\]

Important inequality \( \exists \ p \text{ s.t. } \left| 2^1 \right| < p \)

\[
\frac{1}{2} \left| 2^1 \right| < 1 \left| \log (1+2) \right| \leq \frac{3}{2} \left| 2^1 \right|.
\]
By defn, \[ \prod_{k=1}^{\infty} (1 + a_k) \] converges absolutely

\[ \iff \sum_{k=n}^{\infty} \log (1 + a_k) \] converges absolutely

\[ \iff \sum_{k=n}^{\infty} a_k \] converges absolutely (comparison test + important inequality)

Finally,

\[ \iff \sum_{k=n}^{\infty} |a_k| \] converges absolutely

\[ \iff \prod_{k=1}^{\infty} (1 + |a_k|) \] converges absolutely by \( \iff \)

for \( \tilde{a}_k = |a_k| \)

\[ \iff \prod_{k=1}^{\infty} (1 + |a_k|) \] converges

indeed, absolute convergence of the product is superfluous

\[ \sum_{k=n}^{\infty} \log (1 + |a_k|) = \sum_{k=n}^{\infty} \log (1 + |a_k|) \]
Remark (Rearrangements).

Math 140A we learned that if \( \sum b_k \) is absolutely convergent then \( f: \mathbb{N} \to \mathbb{N} \) bijection then \( \sum_{k=1}^{\infty} b_{f(k)} \) converges to the same sum.

The same happens for absolutely convergent products

\( \prod f_k \) can be rearranged, \( b_k = \log(1 + a_k) \), \( f_k = 1 + a_k \).
Infinite Products of Holomorphic Functions

$f_k : U \rightarrow \mathbb{C}$ holomorphic, $u \in \mathbb{C}$

Assumption \[ \sum_{k=1}^{\infty} |f_k| \text{ converges locally uniformly} \]

Terminology \[ \sum_{k=1}^{\infty} f_k \text{ converges absolutely locally uniformly} \]

Define

\[(4) \quad F(z) = \prod_{k=1}^{\infty} \left( 1 + f_k(z) \right) \]

Remark

\[(*) \text{ converges absolutely } \forall z \in U \Rightarrow \text{ can rearrange the product.} \]
Proposition Under the above Assumption

the partial products of (*) converge locally uniformly.

F is holomorphic

\[ F(2) = 0 \iff \exists k \text{ with } 1 + f_k(2) = 0 \]

Proof will be given next time.

Examples

\[ \sum_{k=1}^{\infty} \frac{1}{11} \left( 1 - \frac{2^2}{k^2} \right) \]
defines an entire function

with zeroes only at the integers, nowhere else.

Indeed, apply the Proposition to \( f_k(2) = \frac{2^2}{k^2} \).

\[ \sum_{k=1}^{\infty} \frac{1}{11} (1 + 2 \cdot \frac{k}{2}) \]
is an entire function if \( |z| < 1 \)

with zeroes only at \( z = -\frac{k}{2} \).

Apply the Proposition to \( f_k(z) = 2 \cdot \frac{k}{z} \).
Math 220 B - Lecture 3

January 6, 2021
Assumption \[ \sum_{k=1}^{\infty} |f_k| \text{ converges locally uniformly.} \]

Then

\[ F(z) = \prod_{k=1}^{\infty} \left( 1 + f_k(z) \right). \]

converges absolutely for all \( z \in U \).

Remark II By Cauchy's criterion (Math 140B).

For \( K \subseteq U \) compact \( \forall \varepsilon > 0 \exists N_{K, \varepsilon} \text{ if } m, n > N_{K, \varepsilon} \)

\[ \Rightarrow \sum_{k=n}^{m} |f_k(z)| < \varepsilon \text{ for } z \in K \]

In practice, instead of Assumption above we might check:

\[ \sum_{k=n}^{m} \sup_{z \in K} |f_k(z)| < \varepsilon \Leftrightarrow \sum_{k=1}^{\infty} \sup_{z \in K} |f_k| < \infty. \]

"Normal convergence!"
This is simply the Weirestrass m-test, with \( M_k(k) = \sup_{x} \left| f_k(x) \right| \) so \( \| \Phi \| \leq B \).

**Proposition** Assume \( \sum |f_k| \) converges locally uniformly.

the partial products of (\( * \)) converge locally uniformly to \( F \).

**F is holomorphic**

\[ F(z_0) = 0 \iff \exists k \text{ with } 1 + f_k(z_0) = 0 \]

Furthermore,

\[ \text{ord}(F, z_0) = \sum_{k=1}^{\infty} \text{ord}(1 + f_k, z_0). \]

**Proof** Recall from last time

\[ \log(1 + z) \text{ is continuous in } \Delta(0,1) \]

**Important inequality**

\[ |\log(1 + z)| \leq \frac{3}{2} |z| \quad \text{if } |z| < 1 \]
Proof of \( \text{II} \). Let \( K \subseteq \mathbb{U} \) compact. By Remark \( \text{I} \)

\[ \exists N \text{ such that if } k \geq N \Rightarrow |f_k(z)| < \rho \text{ for } z \in K \]

\[ \Rightarrow \text{by important inequality} \]

\[ |\log (1 + f_k(z))| \leq \frac{3}{2} |f_k(z)| \text{ for } z \in K, k \geq N. \]

Since \( \sum_{k=N}^{\infty} |f_k| \) converges uniformly by assumption,

\[ \Rightarrow \sum_{k=N}^{\infty} \log (1 + f_k(z)) \text{ converges (absolutely), uniformly on } K. \]

Write \( G_n = \sum_{k=N}^{n} \log (1 + f_k(z)). \]

Note that \( G_n \) is continuous since \( \log (1 + w) \) is continuous for \( |w| < \rho \). Thus \( G \) is also continuous.

Since \( G_n \xrightarrow{\kappa} G \), by the claim \( \text{II} \) below

\[ \log (1 + f_k(z)) \xrightarrow{\kappa} e^c. \]

\[ \prod_{k=1}^{\infty} (1 + f_k(z)) \xrightarrow{\kappa} e^c (1 + f_1(z)) \cdots (1 + f_{\omega}(z)). \]
Uniform convergence after multiplication uses claim (b) below.

Thus
\[ F = e^c (1 + f_1) \ldots (1 + f_{n-1}) \text{ in } K, \] & the convergence is uniform, in \( K \), completing the proof.

\[ F \text{ holomorphic by (1) (local uniform convergence) & Weierstrass Convergence Theorem.} \]

Recall from last time that
\[ F(z_0) = 0 \iff \exists k \text{ with } 1 + f_k(z_0) = 0. \]

To prove the assertion about orders, consider (+)
in \( K = \overline{D}, \Delta \text{ neighborhood of } z_0 \)

\[ F(z_0) = e^c (1 + f_1(z_0)) \ldots (1 + f_{n-1}(z_0)) \]

\[ \Rightarrow \text{ord}(F, z_0) = \sum_{k=1}^{n-1} \text{ord}(1 + f_k, z_0) \]

\[ = \sum_{k=1}^{n-1} \text{ord}(1 + f_k, z_0). \]
using that $1 + f_k 
eq 0$ for $k \geq N$ (because $|f_k| < p < 1$ in $K$).

**Remark** Analyzing the proof, we see the argument only requires

$$\sum_k \frac{1}{\log (1 + f_k)}$$ converges locally uniformly.

The following standard claims were used in the proof:

**Claim 10** Let $u_n$ be continuous, $u_n \to u$. Then $e^{u_n} \to e^u$.

**Claim 11** If $u_n \to u, v_n \to v$ (both continuous). Then 

$$u_n v_n \to uv.$$

**Proof 10** Suffices to show $\sup_k |e^{u_n} - e^u| \to 0$. 
Compute

\[ \sup_k |\epsilon u_n - \epsilon u| = \sup_k |\epsilon u|. |\epsilon u_n - u| \]

\[ \leq \sup_k |\epsilon u|. \sup_k |\epsilon u_n - u| \]

\[ = M . \sup_k |\epsilon u_n - u| < \varepsilon M \text{ for } n \geq N. \]

**Why?**

By continuity, \( \exists \delta > 0: \quad |\epsilon u - 1| < \varepsilon \text{ if } |\epsilon| < \delta. \)

Since \( u_n \to u \Rightarrow \exists N \text{ with } |u_n - u| < \delta \text{ on } K \)

\[ \Rightarrow |\epsilon u_n - u| < \varepsilon \]

**Proof of (16)**

We show \( \sup_k |u_n v_n - u v| \to 0. \)

Indeed by triangle inequality

\[ \sup_k |u_n v_n - u v| \leq \sup_k (|u_n - u| |v_n - v|) + \sup_k |u| |v||v_n - v| \]

\[ + \sup_k |v| |u_n - u| \]

\[ \leq \sup_k |u_n - u|. \sup_k |v_n - v| + \sup_k |u|. \sup_k |v_n - v| + \sup_k |v|. \sup_k |u_n - u| \]

\[ \to 0 \text{ since } \sup_k |u_n - u| \to 0 \text{ and } \sup_k |v_n - v| \to 0. \]
Taking derivatives of products is messy. It is easier to take logarithmic derivatives.

If \( h \) is holomorphic \( \Rightarrow \frac{h'}{h} = \log \text{arithmic derivative} = \text{holomorphic away from Zero} \)

Addition formula

\[
\frac{h}{h} = f g \Rightarrow \frac{h'}{h} = \frac{f'}{f} + \frac{g'}{g}.
\]

\[
h' = f' g + f g' \Rightarrow \frac{h'}{h} = \frac{f' g + f g'}{f g} = \frac{f'}{f} + \frac{g'}{g}.
\]

Inductively

\[
h = f_1 \ldots f_n \Rightarrow \frac{h'}{h} = \frac{f_1'}{f_1} + \ldots + \frac{f_n'}{f_n}.
\]
We prove the same for infinite products.

1. \( g_k : u \rightarrow \mathbb{C} \) holomorphic, \( f_k = 1 + g_k \).

2. \( \sum_{k=1}^{\infty} g_k \) converges locally uniformly in \( u \).

Proposition

\[ f = \prod_{k=1}^{\infty} f_k \cdot \text{Away from } \text{Zero}(f) : \]

\[ \frac{f'_k}{f_k} = \sum_{k=1}^{\infty} \frac{f_k'}{f_k} \]

The RHS converges locally uniformly on \( U \setminus \text{Zero}(f) \).

Proof

Recall from (+) in the previous proof that

for \( k = \Delta \subseteq U \), \( \Delta \) neighborhood of an arbitrary point \( \mathcal{F} \) \( \mathcal{N} \) with

\[ F_n = \prod_{k=N}^{\infty} f_k \rightarrow \text{e.u.} \]

\( F = e^G \) on \( \Delta \)

\[ G = \sum_{k=N}^{\infty} \log (1 + g_k) \]
Note \( \mathbf{f} = f_1 \ldots f_N, \quad \prod_{k=N}^{\infty} f_k = f_N \ldots f_1 \), e.c.

**Finite case**

\[ \Rightarrow \quad \frac{\mathbf{f}'}{\mathbf{f}} = \frac{f_1'}{f_1} + \cdots + \frac{f_{N-1}'}{f_{N-1}} + \frac{(\mathbb{e} \mathbb{e}')}{\mathbb{e}}. \]

We need to show\[ \sum_{k=N}^{\infty} \frac{f_k'}{f_k} \text{ i.u. } (\mathbb{e} \mathbb{e}'). \]

To see this, by the **finite case** again

\[ \sum_{k=N}^{\infty} \frac{f_k'}{f_k} = \frac{F_n'}{F_n} \text{ i.u. } (\mathbb{e} \mathbb{e}'). \]

Note \( \frac{F_n'}{F_n} \text{ i.u. } \mathbb{e} \mathbb{e}^0 \) so by Weierstrass \( \frac{F_n'}{F_n} \text{ i.u. } (\mathbb{e} \mathbb{e}') \).

We finish using Claim 17 above (products) &

Claim 18 below.
Claim: If \( u_n \xrightarrow[k]{} u \), \( u_n \) continuous, \( u \) nowhere zero

\[ \lim_{k} \frac{1}{u_n} \rightarrow \frac{1}{u} \]
Math 220.8 - Lecture 4

January 11, 2021
0. Last time

\[ f_k : u \to \mathbb{C} \text{ holomorphic} \]

\[ \sum_{k = 1}^{\infty} |f_k| \text{ converges locally uniformly} \]

\[ h(z) = \frac{1}{\prod_{k = 1}^{\infty} (1 + f_k(z))} \text{ holomorphic} \]

\[ \frac{h'}{h} = \sum_{k = 1}^{\infty} \frac{f'_k}{1 + f_k} \]

The series on RHS converges absolutely locally uniformly on \( u \setminus \text{Zero}(h) \).

\text{Remark}

If \( \sum_{k = 1}^{\infty} \left| \frac{1}{\log(1 + f_k)} \right| \text{ converges locally uniformly} \]

the same conclusions hold.
Today we factorized sinc: Conway v11.6.

Euler, 1734

"De Summis Series Reciprocum"

\[ \Gamma \text{-function} \] Conway v11.7

Euler, Bernoulli, Gauss, Legendre, Weierstrass

These two topics are naturally connected
1. **Factorization of sine** (Euler, 1734)

**Theorem**

\[
\sin \pi x = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)
\]

**Idea**: Both sides have the same zeroes (with multiplicity)

**Question**: When do two entire functions have exactly the same zeroes?

**Lemma**: If \( f, g : \mathbb{C} \to \mathbb{C} \) entire, with the same zeroes and multiplicities, then \( f = g e^h \) for some \( h : \mathbb{C} \to \mathbb{C} \) entire.
Proof: Let $H = \frac{f}{g}$. $\Rightarrow$ $H$ entire without zeroes by hypothesis. We show $H = e^h$.

The function $\frac{H'}{H}$ is entire so it admits primitive $h$. Then

$$\Rightarrow \frac{H'}{H} = h'$$

$$(H e^{-h})' = H' e^{-h} - H e^{-h} h' = e^{-h} (H' - H h') = 0$$

$$\Rightarrow H e^{-h} = c \neq 0 \Rightarrow H = c e^{h} = e^{\log c + h}.$$ 

Remark: The same holds for $f, g: u \to \mathbb{C}$, $u$ simply connected.
Proof of the sine factorization

(1) convergence:

Note that \( \sum_{k=1}^{\infty} \left| \frac{2^2}{k^2} \right| \) converges locally uniformly \( \Rightarrow \sum_{k=1}^{\infty} \left( 1 - \frac{2^2}{k^2} \right) \) converges.

(2) location of zeroes:

Both sides \( \sin \pi^2 \) & \( \prod_{k=1}^{\infty} \left( 1 - \frac{2^2}{k^2} \right) \) have simple zeroes at the integers & nowhere else.

(3) completing the proof

By the Lemma, \( F \) is entire

\[
\sin \pi^2 = e^h \quad \prod_{k=1}^{\infty} \left( 1 - \frac{2^2}{k^2} \right)
\]

We show \( h = 0 \). Compute logarithmic derivative

\[
\frac{\pi \cos \pi^2}{\sin \pi^2} = \left( e^h \right)' = -\frac{\pi}{e^h} + \frac{\pi^2}{e^h} + \sum_{k=1}^{\infty} \frac{-2^2}{k^2} + \frac{1}{1 - \frac{2^2}{k^2}}
\]

\[
\pi \cot \pi^2 = \pi' + \frac{1}{e^h} + \sum_{k=1}^{\infty} \frac{2^2}{k^2 - \pi^2}
\]
Recall Math 220, HWk 6:

6. Let \( a \in \mathbb{R} \setminus \mathbb{Z} \). Let \( \gamma_n \) be the boundary of the rectangle with corners \( n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n - \frac{1}{2} - ni, n + \frac{1}{2} - ni \). Evaluate

\[
\int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz
\]

via the residue theorem. Making \( n \to \infty \), show that

\[
\pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}.
\]

Thus \( h' \equiv 0 \Rightarrow h \text{ constant}. \) We show \( h \equiv 0 \).

From

\[
\sin \frac{\pi 2}{\pi 2} = e^{h} \frac{1}{|i|} \left( 1 - \frac{2^2}{k^2} \right), \text{ make } 2 \to 0
\]

\[
\int_{1}^{e^{h(0)}} 1 \iff h(0) = 0 \Rightarrow h \equiv 0.
\]

This completes the proof.
\[ \sin \frac{\pi}{2} = \frac{\pi}{2} \sum_{k=1}^{\infty} \left( 1 - \frac{1}{4k^2} \right) = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k+1)} \]

\[ \Rightarrow \frac{\pi}{2} = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{(2k)!}{(2k-2)!} \frac{2k+1}{2k} \]

\[ \frac{\pi}{2} = \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdots \quad \text{Wallis, 1655} \]

\[ 2 = \left( \frac{4}{3} \right) \frac{2\pi}{\pi} = \frac{e^{-\pi} - e^{\pi}}{2\pi} \]

\[ \cos \frac{\pi}{2} = \frac{\sin \frac{2\pi}{2}}{2 \sin \frac{\pi}{2}} = \frac{2\pi}{2} \sum_{k=1}^{\infty} \left( 1 - \frac{4k^2}{\pi^2} \right) = \frac{2\pi}{2} \sum_{k=1}^{\infty} \left( 1 - \frac{4k^2}{\pi^2} \right) \]

Splitting into \( k \) even and odd:

\[ \cos \frac{\pi}{2} = \frac{1}{2} \sum_{k=1}^{\infty} \left( 1 - \frac{4k^2}{\pi^2} \right) \]
2. \( \Gamma - \text{function} \) — probability, statistics, combinatorics, ...

"\( \prod \) is the function that must be introduced in analysis." (Gauss to Bessel, 1811)

\[
\Gamma(x) = 1 \cdot 2 \cdot 3 \ldots \cdot x = \Gamma(x+1)
\]

"The theory of analytic factorials does not seem to have the importance some mathematicians used to attribute to it."

\[ G(2) = \Gamma \left( \sum_{n=1}^{\infty} \frac{x^n}{n^n (1 + \frac{x}{n})} e^{-x} \right) \]

Remark: The convergence (absolutely & locally uniformly) of the product is HWK 1, #4. There, you show

\[
\sum_{n=1}^{\infty} \left\lfloor \log \left( \frac{1 + \frac{x}{n}}{\frac{x}{n}} \right) e^{-\frac{x}{n}} \right\rfloor \text{ converges locally uniformly.}
\]
Properties of the function $G$

\[ G(z) G(-z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{n^2} \right) e^{2 \pi i } \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) e^{-2 \pi i } \]

\[ = \prod_{n=1}^{\infty} \left( 1 + \frac{z^2}{n^2} \right) \left( 1 - \frac{z^2}{n^2} \right) e^{2 \pi i / n} e^{-2 \pi i / n} \]

\[ = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = \frac{\sin \pi z}{\pi z} \text{ by Euler.} \]

\[ G(z+1) = 2 G(z) e^{-\gamma} \text{ where } \gamma \text{ is Euler constant.} \]

\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right). \]

We will prove this next time.

**Definition**

\[ \Gamma(z) = \frac{e^{-\gamma z^2} z}{\Gamma(z)} \]

**Remark** \( G \) has zeroes at \(-1, -2, \ldots -n, \ldots\)

\[ \Rightarrow \Gamma \text{ meromorphic in } \mathbb{C} \text{ with zeroes at } -1, -2, \ldots -n, \ldots \]
Math 220B — Lecture 5

January 13, 2021
The $\Gamma$-function (Conway VII. 7)

Definition \( G(z) = \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \).

Properties of the function \( G \):

1. \( G(z) G(-z) = \frac{\sin \pi z}{\pi z} \) (Euler)

2. \( G(z-1) = \frac{1}{z} G(z) e^{\gamma} \) for some constant \( \gamma \).

We inspect zeroes of both sides.

Zeroes of \( G \): \(-1, -2, \ldots, -n, \ldots\)

Zeroes of \( G(z-1) = \frac{1}{z} G(z) e^{\gamma} \):

\( Giez) \): \( 0, -1, -2, \ldots, -n, \ldots \) \( G(z-1) \): \( 0, -1, -2, \ldots, -n \) have the same zeroes

\( \Rightarrow \) \( G(z-1) = \frac{1}{z} G(z) e^{\gamma} \) for some function \( \gamma(z) \).

We need \( \gamma(z) = \text{constant} \). We verify \( \gamma' = 0 \).

Take logarithmic derivatives

\( \frac{G'(z-1)}{G(z-1)} = \frac{1}{z} + \frac{G'(z)}{G(z)} + \gamma' \) (\(*\))
\[ G(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{-\frac{x}{n}}. \]

Since logarithmic derivative turns products into sums

\[ \frac{G'(x)}{G(x)} = \sum_{n=1}^{\infty} \left(\frac{1}{x} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2+n} - \frac{1}{n}\right) \]

\[ \Rightarrow \frac{G'(x)}{G(x)} = \sum_{n=1}^{\infty} \left(\frac{1}{x} - \frac{1}{n}\right) = \left(\frac{1}{x} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{2+n} - \frac{1}{n}\right) \]

\[ = \left(\frac{1}{x} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{2+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \]

\[ = \left(\frac{1}{x} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{2+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty} \frac{1}{n+1} \]

\[ = \frac{1}{x} + \frac{G'(x)}{G(x)} \]

This implies \( \gamma'(x) = 0 \) and \( \gamma(x) = c = \text{constant} \).
The above constant is

\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right) \]

\text{Euler constant}

Indeed, \( G(0) = 1 \) by definition of the function \( G \).

By \[ c(x) = 2 \quad \Rightarrow \quad c(1) = e^{-\gamma}, \]

Using the definition

\[ c(1) = \prod_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right) e^{-\frac{1}{k}} = \]

\[ = \lim_{n \to \infty} \prod_{k=1}^{n} \left( 1 + \frac{1}{k} \right) e^{-\frac{1}{k}} = \]

\[ = \lim_{n \to \infty} \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots \frac{n+1}{n} \cdot e^{-\frac{1}{n} - \frac{2}{n} - \ldots - \frac{n}{n}} = e^{-\gamma} \]

\[ \Rightarrow \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right) \]

\[ = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1) \right) \]
**Definition** \( \Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \frac{1}{\zeta(z)} = \Gamma - \text{function} \)

**Properties of \( \Gamma \)**

17. \( \Gamma(1) = \frac{e^{-\gamma}}{\zeta(1)} = 1 \) using \( \zeta(1) = e^{-\gamma} \) from above.

18. \( \Gamma(2z+1) = 2 \Gamma(z) \) "\( \Gamma \) behaves like a factorial"

In particular, by induction

\[ \Gamma(n) = (n-1)! \quad \text{for } n > 0, n \in \mathbb{Z}. \]

This follows by direct computation,

\[ \Gamma(2z+1) = \frac{e^{-\gamma z}}{(2z+1)} \cdot \frac{1}{\zeta(2z+1)} = \frac{e^{-\gamma z}}{2} \cdot \frac{1}{\zeta(2)} \cdot z = 2 \Gamma(z) \]

\[ \Leftrightarrow \quad \zeta(2) = (2z+1) \zeta(2z+1) = \gamma \text{ which is true}. \]

(See above)
In particular, \(\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \).

To prove this, we use [116] + Euler:

\[
\Gamma (2) = \Gamma (1 - 2) = \Gamma (2) (-2) \Gamma (-2) =
\]

\[
= \frac{\pi}{\sin \pi z} = \frac{\pi}{\sin \pi (-2)} = \frac{1}{2 \pi (2) \Gamma (-2)} = \frac{\pi}{\sin \pi \frac{1}{2}} \quad \text{(see above)}
\]

We use the definition

\[
\Gamma (2) = \lim_{n \to \infty} \frac{n^2}{2 (n+1) \ldots (2+n)} \quad \text{(Gauss' definition)}
\]

\[
\Gamma (2) = \frac{\pi}{\sin \pi \frac{1}{2}} \cdot \frac{1}{\Gamma (2)} = \frac{\pi}{\sin \pi z} \cdot \frac{1}{\Gamma (2)} = \frac{\pi}{2 \pi (2) \Gamma (-2)}
\]

\[
= \lim_{n \to \infty} \frac{\pi}{2} \cdot \frac{1}{\Gamma (2)} \left( \frac{1}{2 + \frac{1}{n}} \right)^{-1} \left( \frac{1}{2 + \frac{1}{n+1}} \right)^{-1} \ldots \left( \frac{1}{2 + \frac{1}{2}} \right)^{-1} \ldots = \frac{\pi}{2} \Gamma (2)
\]

\[
= \lim_{n \to \infty} \frac{n!}{2 (2+1) \ldots (2+n)} \cdot \frac{2 (1 + \frac{1}{2} + \ldots + \frac{1}{n})}{\Gamma (2)} \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n} \right) \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} \right)
\]

\[
= \lim_{n \to \infty} \frac{n!}{2 (2+1) \ldots (2+n)}
\]
Exercise (Conway VII. 7. 3)

Legendre duplication formula.

Use Gauss’ definition to check

\[ \sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) \]
Note that \( r(z) = \frac{e^{-\frac{z}{2}}}{c(z)} \) is meromorphic with poles at 0, -1, -2, ... since \( c \) has zeroes at -1, -2, ... .

What are the residues?

\[
\text{Res} (r, -n) = \lim_{z \to -n} (z + n) r(z) =
\]

\[
= \lim_{z \to -n} \frac{c(z)}{c(z + 1) \cdots (z + n)}
\]

\[
= \frac{\Gamma(n)}{(-n) \cdots (-1)} = \frac{1}{(-1)^n n!} = \frac{(-1)^n}{n!}
\]
Remark. It can be shown
\[ r(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \] for \( Re(z) > 0 \).

Step 1. Convergence of RHS. Fix \( z \), \( Re(z) > 0 \).

\[ \left| \int_0^1 e^{-t} t^{z-1} \, dt \right| \leq \int_0^1 \left| e^{-t} t^{z-1} \right| \, dt \] since \( |e^{-t}| \leq 1 \)
\[ \leq \int_0^1 t^{Re(z)-1} \, dt \]
\[ = \left. \frac{t^{Re(z)}}{Re(z)} \right|_{t=0}^{t=1} = \frac{1}{Re(z)} \text{ using } Re(z) > 0. \]

Pick \( A \) with \( t^{z-1} \leq e^{1/2} \) when \( t > A \)

\[ \left| \int_A^\infty e^{-t} t^{z-1} \, dt \right| \leq \int_A^\infty \left| e^{-t} t^{z-1} \right| \, dt \leq \int_A^\infty e^{-t} t^{1/2} \, dt \]
\[ = \int_A^\infty e^{-t/2} \, dt \]
\[ = 2 e^{-A/2} < \infty. \]

\[ \int_A^1 e^{-t} t^{z-1} \, dt < \infty \text{ by continuity of } e^{-t} t^{z-1} \text{ in } t. \]
**Step 2** Using integration by parts, one easily shows

\[ \int_0^n (1 - \frac{t}{n})^n t^{n-1} \, dt = \frac{n^2}{2} \frac{n!}{(2+n)\ldots(2+n)} \]

*Exercise* - check the details.

**Step 3** Make \( n \to \infty \). From real analysis

\((1 - \frac{t}{n})^n \to e^{-t}\) as \( n \to \infty \). This will also be explained below. We will argue that

\[ (1) \quad \int_0^n (1 - \frac{t}{n})^n t^{n-1} \, dt \to \int_0^\infty e^{-t} t^{n-1} \, dt \]

\( \ll \text{Step 1} \)

\[ \frac{n^2}{2} \frac{n!}{(2+n)\ldots(2+n)} \to \Gamma(n) \text{ by Gauss' formula} \]

This shows \( \Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt \).
Rigorous justification of convergence in (3)

Claim: \( 0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{e^{-t} t^2}{n} \) if \( 0 \leq t \leq n \).

Assuming the claim, we prove Step 3. Compute

\[
\int_0^n e^{-t} t^{2-1} dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{2-1} dt =
\]

\[
= \int_0^n \left(e^{-t} - \left(1 - \frac{t}{n}\right)^n\right) t^{2-1} dt + \int_n^\infty e^{-t} t^{2-1} dt \rightarrow 0
\]

We claim both terms converge to 0 as \( n \rightarrow \infty \).

Term II: \( \int_n^\infty e^{-t} t^{2-1} dt \rightarrow 0 \) as \( n \rightarrow \infty \) because \( \int_0^\infty e^{-t} t^{2-1} dt \) converges by Step 1.
Proof of claim (only in the notes)

(a) first inequality.

Use \(1 - y \leq e^{-y}\) for \(y \geq 0\).

Take \(y = \frac{t}{n}\) \(\Rightarrow 1 - \frac{t}{n} \leq e^{-\frac{t}{n}} \Rightarrow \left(1 - \frac{t}{n}\right)^n \leq e^{-t} \leq e^{-t} \leq e^{-y}\).
(b) second inequality.

The inequality to prove is

\[ e^{-t} - \left(1 - \frac{t}{n}\right)^n \geq \frac{t^2 e^{-t}}{n} \iff \]

\[ \iff 1 - e^t \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n}. \quad (*) \]

Use \( e^y \geq 1 + y \) for \( y \geq 0 \) proven just as above. Take \( y = \frac{t}{n} \)

Since \( e^t = (e^{t/n})^n \geq \left(1 + \frac{t}{n}\right)^n \), to show \((*)\) we show

\[ 1 - \left(1 + \frac{t}{n}\right)^n \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n} \iff \]

\[ \iff \left(1 - \frac{t^2}{n^2}\right) \leq \left(1 - \frac{t^2}{n^2}\right)^n. \quad \text{This is true.} \]

Indeed, use \((1 - y)^n \geq 1 - ny\) for \( y = \frac{t^2}{n^2} \).

The last inequality can be proved by induction on \( n \).
1. The Weierstrass Problem

Given \( \{a_n\} \) distinct, \( a_n \to \infty \).

\( \{m_n\} \) positive integers

Find entire functions \( f \) with zeroes only at \( a_n \) of order \( m_n \).

Remark: This also makes sense for arbitrary regions \( U \subseteq \mathbb{C} \).

Main Theorem

The Weierstrass problem is always solvable in \( \mathbb{C} \).

Henceforth, \( \{a_n\} \) will be an infinite sequence. The finite case is easy.
Corollary Every meromorphic function in \( E \) is quotient of two entire functions.

Proof Let \( h \) be meromorphic. Let \( E \) be the collection of poles of \( h \) listed with multiplicity. Let \( g \) be the solution to the Weierstrass problem for \( E \).

(The set \( E \) has no limit point in \( E \). By Remark [[[ the hypothesis of Weierstrass is satisfied.]]) Then \( f \) is entire. & \( h = \frac{f}{g} \).
Remarks

Any two solutions $f_1$ & $f_2$

\[ f_1 = e^t f_2 , \quad t \in \mathbb{R} \]

If $\{a_n\}$ has no limit point in $\mathbb{R}$ then $a_n \to \infty$.

Indeed, if not, \exists \, \epsilon > 0 \text{ such that } \forall N \in \mathbb{Z}, \quad |a_n| \leq \epsilon.

This means $\exists$ subsequence of $\{a_n\}$ bounded by $\epsilon$. Since $\bar{A}(\epsilon)$ is compact, this will have a convergent subsequence, with limit $a \in \mathbb{R}$.

Repetitions & zeros known.

We will agree from now on that $\{a_n\}$ may contain repetitions. That is, by relabelling we can repeat each zero as many times as their multiplicity.

We assume $a_n \to \pm \infty$. If $0$ is a zero for $f$, we will add it via multiplication by $z^m$ at the end.
2. Solution to the Weierstrass Problem

Naive Attempt: We could try \( \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) \)

*Issue*: Convergence!

Idea: Try \( f(z) = \prod_{n=1}^{\infty} f_n(z) \) where

\( f_n \) has zero at \( a_n \). E.g. \( f_n(z) = \left( 1 - \frac{z}{a_n} \right)^{b_n} \)

Hope \( f(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right)^{b_n} \) converges.

For example,

\[
\prod_{n=1}^{\infty} \left( 1 + \frac{s}{n^2} \right) \text{ does not converge}
\]

\[
\prod_{n=1}^{\infty} \left( 1 + \frac{s}{n^2} \right) e^{-\frac{1}{n}} = \vartheta(2) \text{ does converge.}
\]
Weierstrass elementary / primary factors

Define

\[ E_p(z) = \begin{cases} 
1 - z & \text{if } p = 0 \\
(1 - z) \exp \left( z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right) & \text{if } p > 0.
\end{cases} \]

\( \Rightarrow \ E_p \) is entire.

Remark

Zero of \( E_p(z) \) is at \( z = 1 \).

\( \Rightarrow \ E_p \left( \frac{z}{a} \right) \) has a simple zero at \( z = a \).

We look for an answer of the form

\[ f(z) = \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{a_n} \right) \]

for suitable \( p_n > 0 \).

Issue:

Can we pick \( p_n \) such that (4) converges absolutely and locally uniformly?

Recall:

\[ \sum_{n=1}^{\infty} f_n \]

converges locally uniformly \( \Rightarrow \prod_{n=1}^{\infty} (1 + f_n) \)

converges absolutely locally uniformly.

We wish to use this for \( f_n = E_{p_n} \left( \frac{z}{a_n} \right) - 1 \).
Growth of the elementary factors

Lemma \[ |1 - E_p(x)| \leq 12|p|^{1/2} \] if \( |x| \leq 2 \).

Proof The proof will be given next time.

Lemma Given \( a_n \to \infty \), \( a_n \neq 0 \), \( \exists p_n \) natural numbers (not unique) such that

\[ \forall r > 0 \implies \sum_{n=1}^{\infty} \left| \frac{r}{a_n} \right|^{p_n+1} < \infty. \]

Proof For instance, take \( p_n = n-1 \). Let \( r > 0 \).

Since \( a_n \to \infty \), \( \exists N \) such that \( |a_n| > \frac{r}{2} \) if \( n \geq N \).

\[ \implies \left| \frac{r}{a_n} \right| \leq \frac{1}{2} \implies \left( \frac{r}{a_n} \right)^n \leq \frac{1}{2^n}. \]

Since \( \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \), \( \sum_{n=1}^{\infty} \left( \frac{r}{a_n} \right)^n < \infty \). by comparison test.
Weierstrass Factorization

**Thm** Let \( a_n \to \infty, \ a_n \neq 0 \). Pick \( p_n \) as in the previous lemma:

\[
y > 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty.
\]

Then

\[
\prod_{n=1}^{\infty} \left[ \frac{z}{a_n} \right]^{p_n} \quad \text{converges absolutely and locally uniformly}
\]

to an entire function with zeroes at \( a_n \) and no other zeroes.

**Proof** Let \( f_n = \prod_{n=1}^{\infty} \left[ \frac{z}{a_n} \right]^{p_n} - 2 \). Pick \( K \) compact, \( K \subseteq \Delta (0, r) \).

for some \( r \). We will argue that \( \prod_{n=1}^{\infty} \left[ \frac{z}{a_n} \right]^{p_n} \) converges locally uniformly.

It suffices to show \( \sum_{n=1}^{\infty} |f_n| \) converges uniformly on \( \Delta (0, r) \).

Note for \( \Delta (0, r) \): 1st Lemma

\[
|f_n(z)| = \left| \prod_{n=1}^{\infty} \left[ \frac{z}{a_n} \right]^{p_n} - 2 \right| \leq \left| \left[ \frac{z}{a_n} \right]^{p_n} \right| \leq \left( \frac{r}{|a_n|} \right)^{p_n+1}.
\]

This requires \( \left| \frac{z}{a_n} \right| \leq 1 \), which is true for \( n \in \mathbb{N} \) since \( a_n \to \infty \).

Since \( \sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty \), Weierstrass M-test \( \sum_{n=1}^{\infty} |f_n| \) converges uniformly in \( \Delta (0, r) \), as needed.
The statement about zeroes follows from Lecture 3 & the fact that $E_{p_n}(\frac{2}{a_n})$ vanishes only at $z = a_n$.

**Corollary** Any (not identically 0) entire function can be written as

$$f(z) = \sum e^{\beta_n} \prod_{n=1}^\infty E_{p_n}(\frac{z}{a_n}), \beta = \text{entire}.$$  

for a non-unique choice of $p_n$ & $\beta$.

**Remark** For the same function $f$, several $p_n$'s may work. Changing $p_n$ into $\tilde{p}_n$ can be absorbed in the exponential.
Proof: WLOG we may assume \( f(0) \neq 0 \). Else if \( \text{ord}_a f(0) = m \) we add the factor \( z^m \).

Let \( \{a_i\} \) be the zeroes of \( f \) listed with multiplicity.

Both \( f \) and \( \Pi \in \mathbb{E} \left( \frac{z}{a_i} \right) \) solve the Weierstrass problem.

Apply Remark \( \square \) to conclude.

Remark: Weierstrass' theorem allows us to define functions which were not even hinted at before.

Poincare: "Weierstrass' most important contribution to the theory of complex variables is the discovery of primary factors."
Example

\[ Q(x) = \prod_{k=1}^{\infty} \left(1 + x^{2k}\right) = \prod_{k=1}^{\infty} E \left(-\frac{x^{2k}}{4k^2}\right) \]

Note \( p_k = 0 \) & \( \sum_{k=0}^{\infty} \left(\frac{x}{k}\right)^{p_k+1} < \infty \), so the hypothesis of Weierstrass factorization holds.

\[ G(x) = \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2}\right) e^{-\frac{x^2}{k^2}} = \prod_{k=1}^{\infty} E \left(-\frac{x^2}{k^2}\right) \]

Note \( p_k = 1 \) & \( \sum_{k=0}^{\infty} \left(\frac{x}{k}\right)^{p_k+1} < \infty \).

\[ \sin \pi x = \pi x \prod_{k=0}^{\infty} \left(1 - \frac{x^2}{k^2}\right) = \pi \prod_{k=1}^{\infty} \left(1 + \frac{x^2}{k^2}\right) e^{\frac{x^2}{k^2}} \left(1 - \frac{x^2}{k^2}\right) \]

\[ = \pi \prod_{k=1}^{\infty} E \left(\frac{x^2}{k^2}\right) \]

Do we get any new examples we didn't know?

Yes, see hawk for the Weierstrass \( \tau \)-function.
Last time

We defined the elementary primary factors

\[ E_p(2) = \begin{cases} \frac{1}{p^2} & , p = 0 \\ \left(1 - \frac{1}{p}\right) \exp \left(2 + \frac{2^2}{2} + \ldots + \frac{2^p}{p^2} \right) & , p > 0 \end{cases} \]

We saw that given \( a_n \to \infty, a_n \neq 0 \).

\[ f(2) = \sum_{n=1}^{\infty} E_p \left( \frac{a_n}{a_n} \right) \]

are entire with zeroes at \( a_n \).

The \( p_n \)'s are chosen so that

\[ \forall r > 0, \quad \sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty \]
Remark 12: Convergence requires the estimate

\[ |1 - E_p(z)| \leq 12|p|^{-1} \] if \( |z| \leq 2 \).

Analogly: The factorization

\[ f(z) = Z \text{ e } \frac{2}{1} \prod_{n=1}^{N} E_{p_n} \left( \frac{z}{a_n} \right) \]

is reminiscent of the factorization of integers into primes.

\[ \text{ primes } \leftarrow E_p \]
\[ \text{ units } \leftarrow \zeta \]

Difference: Not canonical/ uniqueness of \( p_n \)'s.

We can however ask questions with arithmetic flavor.

Wedderburn: Can we write \( 1 = Af + Bg \)

when \( f, g \) have no common zeroes?
Remarks: We have freedom in the choice of \( p_m \).

Question: Is there a canonical choice?

Assume \( \exists \, k \in \mathbb{Z}_{\geq 0} \) with \( \sum_{n=1}^{\infty} \frac{1}{10^n} \, t^n < \infty \).

If such \( k \) exists, pick the smallest one. This is called the genus of the canonical product

\[
\prod_{n=1}^{\infty} E \left( \frac{2}{a_n} \right)
\]

Example:

(1) \( Q(2) = \prod_{k=1}^{\infty} (1 + e^{\frac{2}{k}}) = \prod_{k=1}^{\infty} E \left( -e^{\frac{2}{k}} \right) \)

-genus 0

(2) \( G(2) = \prod_{k=1}^{\infty} \left(1 + \frac{2}{k^2} \right) e^{-\frac{2}{k}} = \prod_{k=1}^{\infty} E \left( -\frac{2}{k} \right) \)

-genus 1

(3) \( \tau = 2 \prod_{n \in \mathbb{N^+}} E \left( \frac{\pi^2}{2n^2} \right) \) genus 2. (HWK)
Remark

The genus controls the growth of zeroes via the expression

$$\sum_{n=1}^{\infty} \frac{1}{\log n^{z+1}}.$$

Remarkably, genus controls the growth of entire functions (Hadamard factorization theorem). This will be covered in Math 220c.
Proof of the estimate

\[ |1 - E_p(z)| \leq |z|^{p+1} \quad \text{for } |z| \leq 1. \]

where \( E_p(z) = (1-z) \). \( u = z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \)

Write \( E_p(z) = \sum_{k=0}^{\infty} a_k z^k. \)

By definition \( E_p(0) = 1 \implies a_0 = 1. \)

\[ E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k. \]

\( \square \) Claim \( \quad a_1 = a_2 = \cdots = a_k = 0 \)

\( \square \) Claim \( \quad a_k \text{ real } \quad \text{and } \quad a_k \leq 0 \quad \forall k \geq p+1. \)

\( \square \) Claim \( \quad \sum_{k=p+1}^{\infty} a_k = -1. \)

Assuming the claim, we compute

\[ |E_p(z) - 1| = \left| \sum_{k=1}^{\infty} a_k z^k \right| = \left| \sum_{k=p+1}^{\infty} a_k z^k \right| \]

\[ = |z|^{p+1} \left| \sum_{k=p+1}^{\infty} a_k z^{k-p-1} \right| \]
\[ \sum_{k=p+1}^{n} \frac{1}{a_k} \left| x \right|^{k-1} \leq 1, \quad 1 \leq \left| x \right| \]

\[ \sum_{k=p+1}^{n} \frac{1}{a_k} \leq 1, \quad 1 \leq \left| x \right| \]

Proof of the claim.

\[ E_p\left(z\right) = (1 - z) e^u, \quad u = z + \frac{2^1}{x} + \ldots + \frac{2^p}{x} \]

Note \( u' = 1 + \frac{2^1}{x} + \ldots + \frac{2^p}{x} \Rightarrow (1 - z) u' = 1 - 2^p. \)

Compute

\[ E_p'\left(z\right) = \left((1 - z) e^u\right)' = -e^u + (1 - z) u' e^u \]

\[ = -e^u + (1 - 2^p) e^u \]

\[ = -e^u \cdot 2^p \quad (1) \]

Since

\[ E_p\left(z\right) = 1 + \sum_{k=0}^{\infty} a_k \frac{2^k}{x} \Rightarrow E_p'\left(z\right) = \sum_{k=0}^{\infty} a_k \frac{k-1}{x} \quad (2) \]
The terms in (1) have powers of $2^p$.

Comparing with (2) we see $a_k = 0 \neq 1 \leq k \leq p$.

Also for $k \geq p+1$,

$$a_k = -\frac{1}{k} \cdot \text{Coefficient of } z^{k-p-1} \text{ in } e^u.$$

Since

$$e^u = e \cdot e^{\frac{z}{2}} \cdot ... \cdot e^{\frac{z}{p}} \quad \text{& using the expansion of the exponential, we see that}$$

$$\text{Coefficient of } z^{k-p-1} \text{ in } e^u \quad \text{is equal to } a_k \leq 0.$$

Set $\alpha = 1$:

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{n} a_k \Rightarrow \sum_{k=p+1}^{n} a_k = -1.$$
Further remarks - Looking forward (not needed)

A divisor is a formal sum

\[ D = \sum_{p \in \mathbb{C}} n_p \frac{1}{p} \quad \text{where} \quad n_p \in \mathbb{Z} \]

We require that this sum be locally finite.

A divisor is non-negative (effective) if \( n_p \geq 0 \) for all \( p \).

Example \( D = 3 [a] + 5 [b] \) is a divisor.

Any entire function gives rise to a divisor

Indeed,

\[ \text{div}(f) = \sum_{p \text{ zero for } f} \text{ord}(f, p) \frac{1}{p} \]

Example

\[ f = (z - a)(z - b) = 3 [a] + 5 [b] \]
The Weierstrass Problem can be rephrased:

Every effective divisor is the divisor of an entire function:

\[ D \geq 0, \quad D = \text{div}(f). \]

For a meromorphic function \( f \):

\[ \text{div}(f) = \sum_p \text{ord}(f, p) [p] \]

where \( p \) is a zero or pole.

**Question**: Is every divisor the divisor of a meromorphic function?

**Yes**: For a general divisor \( D \) we can separate:

\[ D = D_+ - D_- , \quad D_+, D_- \text{ non-negative}. \]

Write \( D_+ = \text{div} f_+ \) and \( D_- = \text{div} f_- \) and set \( f = f_+/f_- \).
Then \( \text{div}(f) = \text{div}(f_+) - \text{div}(f_-) \) (check)

\[ = D_+ - D_- = D. \]

These questions naturally lead to sheaf cohomology.

(Math 220c).

Next time the Weierstrass problem in \( u \leq \sigma \).

This is a bit more involved.
Weierstrass Problem for arbitrary regions

Question: Given \( U \subseteq \mathbb{C} \), \( \{a_n\} \subseteq U \) without limit point in \( U \), find \( f \) holomorphic in \( U \) with zeroes only at \( \{a_n\} \).

The sequence \( \{a_n\} \) may contain repetitions according to multiplicities of the zeroes.

Main Theorem: The Weierstrass Problem can be solved in \( U \).
Remark: It is not true any two solutions $f_1, f_2$ satisfy $f_1 = e^{f_2}$.

Counterexample: $u = \mathbb{C}^\times$, $f_1 = 1$, $f_2 = 2$. $h$ would have to be a logarithm, which is undefined in $\mathbb{C}^\times$.

Any meromorphic function in $u$ is quotient of two holomorphic functions.

The same proof for $u = \mathbb{C}$ works for all $u$. 
**How to prove** Weierstrass for $u$?

We could again try

$$f(x) = \prod_{n=1}^{\infty} p_n \left( \frac{x}{a_n} \right).$$

Convergence used $a_n \to \infty$.

Indeed, if we wish to have

$$\sum_{n=1}^{\infty} \left( \frac{r}{1-a_n} \right)^{n+1} < \infty \quad \text{we'd need} \quad \frac{r}{1-a_n} \to 0 \quad \Rightarrow \quad a_n \to \infty.$$ 

Since $a_n \in \mathbb{R}$, this may not be the case. e.g. if $u$ is bounded. **How to deal with bounded regions for instance?**
New ideas

(1) Use biholomorphisms to change the region U e.g. via $z \to \frac{1}{z}$. If U were bounded, the new region would be unbounded.

(2) Think of $U \subseteq \mathbb{C}$ as $U \subseteq \mathbb{C}$ and prescribe values at oo as well.

New idea

Even for unbounded regions, we can try new functions:

$$f(z) = \sum_{n=1}^{\infty} \frac{E \left[ \frac{a_n - b_n}{z - b_n} \right]}{z - a_n}$$

for good choices of $b_n$.

This also has zeroes at $z = a_n$ since $E \left[ \frac{1}{z} \right] = 0$. 
Weierstrass Problem in \( u \subseteq \mathbb{C} \)

**Step (1)** Assume \( \exists R > 0 \) in neighborhood of \( \infty \).

\[ \{ |z| > R \} \subseteq u \]

\[ |a_n| \leq R \quad \forall n. \]

**Construct** \( f \) holomorphic in \( u \) such that

\[ f \text{ has zeroes at } a_n \]

\[ \lim_{z \to \infty} f(z) = 1. \]

**Step (2)** General case. Use easy trick.

Use \( z \to \frac{1}{z} \) to reduce to Step 1.
Topological Fact used in the Proof (Rudin)

\[ K \cap F = \emptyset, \text{ } K \neq \emptyset \text{ compact, } F \neq \emptyset \text{ closed.} \]

\[ d = \operatorname{dist}(K, F) = \inf \{ |k - f| : k \in K, f \in F \} > 0 \]

**Proof**

Assume \( d = 0 \). Then \( \exists k_n \in K, f_n \in F \) with

\[ |k_n - f_n| \to 0 \]

Passing to a subsequence, assume \( k_n \to k \in K \).

It follows that \( f_n \to k \) as well.

Since \( F \) closed, \( k \in F \). Thus \( k \in K \cap F = \emptyset \). Contradiction.
Step \( 1 \): \( \exists R > 0, \{1 \leq |z| \leq R\} \subseteq U \) and \( |a_n| \leq R \).

\( \text{Claim: } f \) has at most one zero at \( a_n \).

\( \lim_{z \to a_n} f(z) = 1 \)

Note \( K = \mathbb{C} \setminus U \subseteq \{1 \leq |z| \leq R\} \Rightarrow K \) bounded and closed

\( \Rightarrow K \) compact.

Since \( |a_n - 2| \) is continuous, \( \exists b_n \in K \) with

\[ |a_n - b_n| = \min_{z \in K} |a_n - z| \]

Write \( S_n = |a_n - b_n| > 0 \) since \( an \in U \), \( b_n \notin U \).

Claim: \( S_n \to 0 \).

**Proof:** Assume otherwise. Then \( \exists \varepsilon > 0 N \exists n \geq N \) with

\[ |S_n| \geq \varepsilon \]

Passing to a subsequence we may assume \( |S_n| \geq \varepsilon \).
Note \( \{a_n\} \subseteq \overline{B}(0,R) = \text{compact}. \) Passing to a subsequence we may assume \( a_n \to a. \) Since \( \{a_n\} \) has no limit point in \( U \to a \in K. \) Then by the definition of \( b_n: \)

\[
|a_n - a| \geq |a_n - b_a| = \delta_n > \epsilon.
\]

This contradicts \( a_n \to a. \) Thus \( \delta_n \to 0. \)
Claim. $f(z) = \sum_{n=1}^{\infty} \frac{a_n - b_n}{2 - b_n}$ converges absolutely and locally uniformly in $U$, and vanishes only at $a_n$.

Proof. It suffices to show

$$\sum_{n=1}^{\infty} \left| \frac{a_n - b_n}{2 - b_n} - 1 \right|$$

converges absolutely and locally uniformly in $U$. To this end, let $K' \subseteq U$ compact.

Let $\delta = d(K, K') > 0$ since $K \cap K' = \emptyset$.

For $z \in K'$, $|z - b_n| > \delta$.

Recall

$$|1 - E_0(\omega)| \leq |\omega|^{-1} + |\omega|^{-1}.$$ 

Thus

$$\left| 1 - E_n \left( \frac{a_n - b_n}{2 - b_n} \right) \right| \leq \left| \frac{a_n - b_n}{2 - b_n} \right| \leq \frac{1}{|z|^N}, \quad z \in K', \quad n \geq N.$$ 

We conclude by Weierstrass M-test since $\sum_{n=1}^{\infty} \frac{1}{|z|^N} < \infty$. 

Proof of \( \text{III} \)

\[ \lim_{x \to 0} f(x) = 1. \]

Equivocally

\[ \lim_{x \to 0} f \left( \frac{1}{x} \right) = 1. \]

We compute

\[ g(x) = f \left( \frac{1}{x} \right) = \prod_{n=1}^{\infty} E_n \left( \frac{a_n - b_n}{\frac{1}{2} - b_n} \right) = \frac{2 (a_n - b_n)}{1 - 2 b_n}. \]

(\text{x})

We show the product (\text{x}) converges absolutely and locally uniformly in \( \Delta \left( 0, \frac{1}{R} \right) \). The limit will be holomorphic at \( x = 0 \) hence continuous. Then

\[ \lim_{x \to 0} g(x) = g(0) = 1. \implies \lim_{x \to 0} f \left( \frac{1}{x} \right) = 1. \]
To show convergence, let \( \bar{D} (0, p) \subseteq D (0, \frac{1}{R}) \). \( \Rightarrow p R < 1 \).

We have for \( z \in \bar{D} (0, p) \): 

\[
\left| \frac{z (a_n - b_n)}{1 - z b_n} \right| \leq \frac{p S_n}{1 - 2 b_n} \leq \frac{p S_n}{1 - 2 R b_n} \leq \frac{1}{2}.
\]

for \( n \geq N \), since \( S_n \to 0 \).

Then

\[
\left| 1 - E_n \left( \frac{z (a_n - b_n)}{1 - z b_n} \right) \right| \leq \left| \frac{2 (a_n - b_n)}{1 - z b_n} \right| ^{n+1} \leq \frac{1}{2^{n+1}} \Rightarrow
\]

\( \Rightarrow \) Weierstrass M-test

\[
\sum_{n=1}^{\infty} \left| 1 - E_n \left( \frac{z (a_n - b_n)}{1 - z b_n} \right) \right| \text{ converges absolutely and locally uniformly in } \bar{D} (0, \frac{1}{R}).
\]
\textbf{Case (2) General case}

\textit{wlog } \forall \in U \& a_n \neq 0

Indeed we may take \( a \in U \), \( a \neq a_n \) \( \forall n \). Let

\[ U^{\text{new}} = \left\{ u - a, u \in U \right\}, \quad a_n^{\text{new}} = a_n - a, \]

\[ \Rightarrow 0 \in U^{\text{new}}, \quad a_n^{\text{new}} \neq 0. \quad \text{Let } f^{\text{new}} \text{ solves Weierstrass for } \]

\[ (U^{\text{new}}, \{a_n^{\text{new}}\}) \text{ let } f(2) = f^{\text{new}}(2 - a), \text{ solves Weierstrass for } \]

\[ (U, \{a_n\}). \]

\textit{Trick to reduce to Case 1.}

Define \( \widetilde{U} = \left\{ \frac{1}{x} : x \in U \cup \{0\} \right\} \). This is open by the open mapping theorem for \( U \cup \{0\}, x \mapsto \frac{1}{x} \).
Let $\tilde{a}_n = \frac{1}{a_n} \in \tilde{u}$

Claim $(\tilde{u}, \{\tilde{a}_n\})$ satisfies Step 1.

Let $\tilde{f}$ be the solution to Weierstrass for $(\tilde{u}, \{\tilde{a}_n\})$

Let $f(z) = \tilde{f}(\frac{1}{z})$ is holomorphic in $u \setminus \{0\}$.

Since $\lim_{z \to 0} f(z) = 1 \Rightarrow \lim_{z \to 0} f(z)^2 = 1$. Thus $0$ is removable singularity and $f$ extends to $u$. Its zeroes are only at $a_n$.

Proof of the claim

Since $0 \in u \Rightarrow \exists \varepsilon$ with $B(0, \varepsilon) \subseteq u.$

$\Rightarrow \{z \in \mathbb{C} : |z| < \frac{1}{\varepsilon}\} \subseteq \tilde{u}.$

Since $0 \in u \Rightarrow \{a_n\}^\infty_{n=0}$ do not have $0$ as limit point

$\Rightarrow \exists \varepsilon' \text{ with } |a_n| \geq \varepsilon' \Rightarrow |\tilde{a}_n| \leq \frac{1}{\varepsilon'}.$

Let $R = \max \{\frac{1}{2}, \frac{1}{\varepsilon'}\} \Rightarrow |a_n| \leq R & \{|z| \leq R\} \subseteq u.$
Exercise

Follow the above proof for $u = e$. What function $f$ does the proof produce?
The Mittag-Leffler Problem

Conway VIII. 3 simplified.

Weierstrass Problem

Given \( \{a_n\} \) distinct, \( a_n \to \infty \).

Let \( \{m_n\} \) positive integers

Find entire functions \( f \) with zeroes only at \( a_n \) of order \( m_n \).

Answer

We can always solve the Weierstrass Problem. And we even have a factorization of the solution.

Remark

The function \( 1/f \) is meromorphic & its poles are only at \( a_n \) & their order equals \( m_n \).

The Mittag-Leffler Problem asks a sharper question.
The Mittag-Leffler (ML) Problem for $a$

Given \( \{a_n\} \) distinct, \( a_n \to a \).

Laurent principal parts (singular parts)

\[ g_n(z) = \frac{A_{nm_n}}{(z-a_n)^m_n} + \frac{A_{nm_n-1}}{(z-a_n)^{m_n-1}} + \ldots + \frac{A_{n1}}{(z-a_n)} \]

Main Theorem. We can always find meromorphic function $f$ with poles only at $a_n$ & Laurent principal parts $g_n$ near $a_n$.

Remark. If $f_1, f_2$ are two solutions $\Rightarrow f_1 - f_2 =$ entire since the singular parts at $a_n$ cancel out.

\[ f_1 = f_2 + h \]

Remark. This makes sense for $u \leq a$. 

SUR LA RÉPRÉSENTATION ANALYTIQUE
DES
FONCTIONS MONOGÈNES UNIFORMES
D'UNE VARIABLE INDÉPENDANTE

PAR
G. MITTAG-LEFFLER
A STOCKHOLM.

Les recherches dont je vais exposer ici l'ensemble, ont été publiées auparavant, quant à leurs traits les plus essentiels, dans le Bulletin (Öfver- sigt) des travaux de l'Académie royale des sciences de Suède, ainsi que dans les Comptes-rendus hebdomadaires de l'Académie des sciences à Paris. Leur but est de faire parvenir, dans un certain sens, la théorie des fonctions analytiques uniformes d'une variable, à ce degré d'achèvement auquel la théorie des fonctions rationnelles est arrivée depuis longtemps.

Soit \( x \) une grandeur variable complexe à variabilité illimitée, et \( x' \) un point donné fini\(^{(1)}\) dans le domaine de la variable \( x \). Soit enfin \( R \) une quantité positive donnée. Je dis que l'ensemble des points \( x \) remplissant la condition \( |x - x'| < R \), constitue le voisinage ou l'entourage ou les environs du point \( x \)\(^{(2)}\) correspondant à \( R \). Chacun de ces points est dit appartenir au voisinage ou à l'entourage ou aux environs \( R \), ou être

\(^{(1)}\) C'est-à-dire représentant une valeur donnée finie.

Acta Math 4 (1884)
2. (Generalized Weierstraß problem. Monday, January 25.) Let \( \{a_n\} \) be distinct complex numbers with \( a_n \to \infty \). Fix complex numbers \( \{A_n\} \). Show that there exists an entire function \( f \) such that
\[
f(a_n) = A_n.
\]
Further Connections

In HWK3, Problem 3 we will see that we can derive Mittag-Leffler for simple poles from Weierstrass factorization.
Discussion of the proof

Given \( \{a_n\} \), \( a_n \to \infty \), \( L_n = \) Laurent principal parts

we try \( f = \sum_{n=1}^{\infty} L_n \) as solution to Mittag-Leffler

issue. As usual, this may not converge

New idea

Pick \( h_n \) entire functions & argue

\[
f = \sum_{n=1}^{\infty} \left( L_n - h_n \right) \text{ converges}
\]

Since \( h_n \) are entire, we are not changing the Laurent

principal parts

Compare this to Weierstrass

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{2}{a_n} \right) \text{ vs. } \prod_{n=1}^{\infty} \left( 1 - \frac{2}{a_n} \right) e^{b_n}
\]

may not converge could converge.
**Terminology**

\[ \sum_{n=1}^{\infty} (L_n - h_n) = \text{Mittag-Leffler series} \]

\[ h_n = \text{convergence enhancing corrections} \]

The \( h_n \)'s are not unique!

**Remark**  
\[ \text{WLOG } a_n \neq 0, \forall n. \]

The contributions of the poles at 0 are added at the end:

\[ \frac{A_m}{2^m} + \ldots + \frac{A_1}{2^1} + \text{Solution with } a_n \neq 0. \]
Proof: The proof is part of the theorem. Conway VIII.3.

Fix $r_n \to \infty$, $c_n < |a_n|$

\[ \sum_{n=0}^{\infty} c_n < \infty \]

e.g. $c_n = \frac{1}{2^n}$, $c_n = \frac{1}{n^2}$, ...

Consider $g_n(z) = \frac{A_{nmn}}{(2-a_n)^m} + \frac{A_{nmn-1}}{(2-a_n)^{m-1}} + \cdots + \frac{A_n}{2-a_n}$

Since $a_n \neq 0$, $g_n$ is holomorphic at $z = 0$ in $\Delta(0, 1/a_n)$

We can Taylor expand $g_n$ in $\Delta(0, 1/a_n)$ around 0.

Since $\Delta(0, r_n) \subseteq \Delta(0, 1/a_n)$, the Taylor series of $g_n$ converges uniformly in $\Delta(0, r_n)$. We can pick a Taylor polynomial $h_n$ such that

$|g_n - h_n| < c_n$ in $\Delta(0, r_n)$. 
Let \( f = \sum_{k=1}^{\infty} (g_k - h_k) \)

We show

Claim: \( f \) meromorphic with poles only at \( a_k \) & principal parts \( g_k \) near \( a_k \) \( \Rightarrow \) \( f \) solves Mittag-Leffler.

Proof: Let \( r > 0 \).

Since \( r \stackrel{k}{\longrightarrow} \infty, \Rightarrow r_k > r \) if \( k \geq N \). Then

\[ |g_k - h_k| < c_k \text{ in } \Delta(0, r) \subseteq \Delta(0, r_k) \text{ if } k \geq N. \]

By Weierstrass m-test \( \sum_{k=N}^{\infty} (g_k - h_k) \) converges uniformly in \( \Delta(0, r) \). Note that since \( |g_k| > r_k > r \)

\[ \Rightarrow g_k - h_k \text{ holomorphic in } \Delta(0, r). \text{ Thus the sum } \]

the pole \( a_k \) is not in \( \Delta(0, r) \)

\[ \sum_{k=N}^{\infty} (g_k - h_k) \text{ is holomorphic in } \Delta(0, r). \]
The sum \( \sum_{k=1}^{N-1} (g_k - h_k) \) is meromorphic as a finite sum of meromorphic functions in \( \Delta(0, r) \). The poles are only at those \( g_j \)'s with \( |g_j| < r \) and the Laurent principal parts are \( g_j \). This is because \( h_k \) are polynomials, so they do not contribute to the Laurent principal parts.

Thus \( f = \sum_{k=1}^{N-1} (g_k - h_k) + \sum_{k=N}^{\infty} (g_k - h_k) \) is meromorphic with poles at \( |g_j| < r \) for all \( \Delta(0, r) \).

Varying \( z \) we get the claim & finish the proof.
Summary of the proof

**Step 1** Pick \( r_n \to \infty, \quad 1/a_n > r_n \)

\[ c_n, \quad \sum c_n < \infty \]

**Step 2** Taylor expand \( g_n \) near \( 0 \)

Pick Taylor polynomial \( h_n \) with

\[ |g_n - h_n| < c_n \quad \text{in} \quad \Delta (0, r_n) \]

**Step 3** \( f = \sum_{n=1}^{\infty} (g_n - h_n) \)
Examples (will be repeated next time)

**Poles at** \(-n \in \mathbb{Z}\), principal parts \(\frac{1}{2+n}\).

For \(n \neq 0\), we expand \(\frac{1}{2+n}\) at \(z = 0\).

\[
2_n = \frac{1}{2+n} = \frac{1}{n} \cdot \frac{1}{1 + \frac{x}{n}} = \frac{1}{n} \left(1 - \frac{x}{n} + \frac{x^2}{n^2} - \cdots\right)
\]

\[
= \frac{1}{n} - \frac{x}{n^2} + \frac{x^2}{n^3} - \cdots
\]

Let \(h_n = \frac{1}{n}\) \(\Rightarrow\) \(g_n - h_n = \frac{1}{2+n} - \frac{1}{n} = \frac{x}{n(2+n)}\).

Let \(r_n = \sqrt{n!} \quad \text{if} \quad 1 \leq r_n \quad \text{then}\)

\[
\Rightarrow |g_n - h_n| = \frac{|x|}{n \cdot 2+n/1} \leq \frac{\sqrt{n}}{n \cdot (n-r_n)} = c_n \quad i f \quad n > 0
\]

Note \(\lim_{n \to \infty} \frac{c_n}{n^{-3/2}} = 1\) \& \(\sum_{n} n^{-3/2} < \infty\). Thus \(\sum_{n} c_n < \infty\).

A similar argument works for \(n < \infty\).
\[ f = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2+n} - \frac{1}{n} \right) + \frac{1}{2} \]

is the solution we need to add to the Mittag-Leffler Problem. This at the end for \( n = 0 \).

Remark

Note that the \( n \) \& \(-n \) terms can be collected

\[ f = \sum_{n=1}^{\infty} \left( \frac{1}{2+n} + \frac{1}{2-n} \right) + \frac{1}{2} \]

\[ = \sum_{n=1}^{\infty} \frac{2^2}{2^2-n^2} + \frac{1}{2} = \frac{\pi}{\csc \frac{\pi}{2}} \] by

HWK 6 in Math 220A.
Last time - Mittag-Leffler Problem

Given

- $a_n \to \infty$ distinct and
- Laurent principal parts $g_n$

find $f$ meromorphic with poles at $a_n$ & principal parts $g_n$ at $a_n$

Construction

Step 1
Expand $g_n$ into Taylor series at 0.

Step 2
Pick $h_n$ a Taylor polynomial & check

$|2g_n - h_n| < c_n$ in $\Delta(0, r_n)$ with $\sum c_n < \infty$.

and $r_n < |a_n|$, $r_n \to \infty$

Step 3
Solution

$f = \sum_{n=1}^{\infty} (g_n - h_n) + \text{add Laurent principal part at } 0$. 
Today - 4 historically important examples

- we group them in pairs of two
Example 11 \((-n, \frac{1}{2+n})\), \(n \in \mathbb{Z}\).

\[
\frac{1}{2+n} \quad \frac{1}{2+1} \quad \frac{1}{2} \quad \frac{1}{2-1} \quad \frac{1}{2-2}
\]

\[
\ldots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \ldots
\]

**Step 1**  
Taylor expand:

\[
g_n = \frac{1}{2+n} = \frac{1}{n} \cdot \frac{1}{1+\frac{2}{n}} = \frac{1}{n} \left(1 - \frac{2}{n} + \frac{2^2}{n^2} - \ldots\right)
\]

\[= \frac{1}{n} - \frac{2}{n^2} + \frac{2^2}{n^3} - \ldots\]

\[-h_n = \frac{1}{n}, \quad n \neq 0\]

**Step 2**

\[L = \int r_n = \frac{1}{n^{1/2}}. \quad \text{If } |z| \leq r_n:\]

\[
|g_n - h_n| = \left|\frac{1}{2+n} - \frac{1}{n}\right| = \frac{|2|}{(n)(n+2)} \leq \frac{r_n}{(n)(n+2)} = c_n.
\]

Since \(\lim_{n \to \infty} \frac{c_n}{(n)^{1/2}} < \infty\) and \(\sum \frac{1}{(n)^{1/2}} < \infty\), \(\Rightarrow \sum c_n < \infty\).
Steps

Mittag-Leffler solution

\[ f = \sum_{n \neq 0} \left( \frac{1}{x+n} - \frac{1}{n} \right) + \frac{1}{2}. \]

Collecting the terms for \( n \) \& \(-n\) we find

\[ f = \sum_{n \neq 0} \left( \frac{1}{x+n} + \frac{1}{x-n} \right) + \frac{1}{2} \]

\[ = \sum_{n \neq 0} \frac{2x}{x^2 - n^2} + \frac{1}{2} = \frac{\pi}{\text{csc} \ \pi x} \]

Poles at \(-n \in \mathbb{Z}\), principal parts \(\frac{1}{(2+n)^2}\).

\[
\left( -n, \frac{1}{(2+n)^2} \right)
\]

\[
\begin{align*}
    \frac{1}{(2+1)^2} & \quad \frac{1}{2^2} & \quad \frac{1}{(2-1)^2} & \quad \frac{1}{(2-2)^2} \\
    \ldots & \quad -2 & \quad -1 & \quad 0 & \quad 1 & \quad 2 & \quad \ldots
\end{align*}
\]

**Step 1**

\[ g_n = \frac{1}{(2+n)^2} \]

\[ h_n = 0 \]

**Step 2**

\[ r_n = \frac{1}{2} \cdot |n|^{1/2} \cdot 1 \geq |n| \leq r_n \]

\[
|g_n - h_n| = \left| \frac{1}{(2+n)^2} \right| \leq \frac{1}{(|n|-r_n)^2} = c_n.
\]

\[
\lim_{n \to \infty} \frac{c_n}{|n|^{-2}} = 1 \quad \Rightarrow \quad \sum_{n \neq 0} \frac{1}{n^2} < \infty \quad \Rightarrow \quad \sum_{n \neq 0} c_n < \infty.
\]
**Step 3** Mittag-Leffler function

\[ f^n = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^2}. \]

We have seen \( f = \frac{\pi^2}{\sin^2 \pi z} \) in Math 220A, HWK 6, #7.

6. Let \( a \in \mathbb{R} \setminus \mathbb{Z} \). Let \( \gamma_n \) be the boundary of the rectangle with corners \( n + \frac{1}{2} + ni, -n - \frac{1}{2} + ni, -n + \frac{1}{2} - ni, n + \frac{1}{2} - ni \). Evaluate

\[ \int_{\gamma_n} \frac{\pi \cot \pi z}{z^2 - a^2} \, dz \]

via the residue theorem. Making \( n \to \infty \), show that

\[ \pi \cot \pi a = \frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{1}{a^2 - n^2}. \]

7. Let \( a \in \mathbb{R} \setminus \mathbb{Z} \). Let \( \gamma_n \) be the boundary of the rectangle with corners \( \pm \left( n + \frac{1}{2} \right) \pm ni \).

Evaluate

\[ \int_{\gamma_n} \frac{\pi \cot \pi z}{(z + a)^2} \, dz \]

via the residue theorem, and use this to show that

\[ \sum_{n=-\infty}^{\infty} \frac{1}{(a + n)^2} = \frac{\pi^2}{\sin^2(\pi a)}. \]
Remark  Compare \[16\] & \[17\]

\[
\left( -n, \frac{1}{2+n} \right) \leftrightarrow \left( -n, \frac{1}{(2+n)^2} \right)
\]

\[
\pi \cot \frac{\pi^2}{2} \leftrightarrow \frac{\pi^2}{\sin^2 \frac{\pi^2}{2}}
\]

These are related by differentiation (up to a sign).
For the next examples, we replace

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \]

by the lattice

\[ \Lambda = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 = \left\{ m \omega_1 + n \omega_2 : m, n \in \mathbb{Z} \right\}, \quad \frac{\omega_1}{\omega_2} \notin \mathbb{R} \]
Main Difference

\[-2 \quad -1 \quad 0 \quad 1 \quad 2\]

\[\sum_{n \neq 0} \frac{1}{\ln^n} \text{ converges if } \alpha = 2\]
\[\text{if } \alpha > 1.\]

For the lattice,

\[\sum_{\lambda \in \Lambda} \frac{1}{|\lambda|^\alpha} \text{ converges if } \alpha = 3 \quad (H W K \ 2)\]
\[\text{if } \alpha > 2.\]
Poles at $\lambda \in \Lambda$, principal parts \( \frac{1}{\lambda - \Delta} \).

\[
(f, \frac{1}{\lambda - \Delta})_{\lambda \in \Lambda}.
\]

**Step 1** \( \lambda \neq 0 \)

Taylor expand

\[
2_\lambda = \frac{1}{\lambda - \Delta} = \frac{1}{\lambda} \cdot \frac{-1}{1 - \frac{\Delta}{\lambda}}
\]

\[
= \frac{-1}{\lambda} \left(1 + \frac{\Delta}{\lambda} + \frac{\Delta^2}{\lambda^2} + \ldots\right)
\]

\[
= - \frac{1}{\lambda} - \frac{\Delta}{\lambda^2} - \frac{\Delta^2}{\lambda^3} - \ldots
\]

\[
h_\lambda = - \frac{1}{\lambda} - \frac{\Delta}{\lambda^2}
\]

**Step 2** Let \( r_\lambda = \min \left(\frac{1}{2} \Delta_1, \Delta_1^{\frac{3}{4}}\right) \).

If \( |\Delta| \leq r_\lambda \) then

\[
\left| 2_\lambda - h_\lambda \right| = \left| \sum_{k=2}^{\infty} \frac{\Delta^k}{\lambda^{k+1}} \right|
\]

\[
= \frac{|\Delta|^2}{12/3} \sum_{k=0}^{10} \left| \frac{\Delta^k}{\lambda^{k+1}} \right| \leq \frac{r_\lambda^2}{12/3} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} =
\]
\[ = 2 \cdot \frac{r^2}{|z|^3} \leq 2 \cdot \frac{1}{|z|^{3/2}} = c_2. \]

Since \[ \sum_{\lambda \neq 0} \frac{1}{|\lambda|^{3/2}} < \infty, \] we get \[ \sum_{\lambda \neq 0} c_2 < \infty. \]

**Step 3** Mittag-Leffler solution

\[ 3 = \frac{1}{2} + \sum_{\lambda \neq 0} \left( \frac{1}{2-2} + \frac{1}{\lambda} + \frac{2}{\lambda^2} \right) \]

Weierstraß \( z \)-function (HWX 5, #3)
Pole at $\lambda \in \Lambda$, principal parts $\frac{1}{(z - \lambda)^2}$:

\[
\left( \lambda, \frac{1}{(z - \lambda)^2} \right)_{\lambda \in \Lambda}
\]

**Step 1**

$\lambda \neq 0$

\[
\lambda^2 = \frac{1}{(z - \lambda)^2} = \frac{1}{\lambda^2} \cdot \frac{1}{\left(1 - \frac{z}{\lambda}\right)^2} = \frac{1}{\lambda^2} \left(1 + \frac{2z}{\lambda} + \frac{3z^2}{\lambda^2} + \cdots \right)
\]

\[= \frac{1}{\lambda^2} + \frac{2z}{\lambda^3} + \frac{3z^2}{\lambda^4} + \cdots
\]

\[
\frac{1}{(1 - \omega)^n} = 1 + 2\omega + 3\omega^2 + \cdots
\]

\[
\gamma_n = \frac{1}{\lambda^2}
\]

**Step 2**

$\gamma_n = \min \left\{ \left( \frac{12}{\lambda^2}, \frac{12}{\lambda^2} \right) \right\}$

\[
\left| \frac{z}{\lambda} - \lambda \right| = \left| \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right| = \left| \frac{z^2 - 2z\lambda}{\lambda^2 (z - \lambda)^2} \right|
\]

\[
\lambda \leq \frac{12}{\lambda^2}
\]

\[
\frac{r_n^2 + 2r_n (1/\lambda)}{1/\lambda^2 (1/\lambda - r_n)^2} \leq \frac{4. \frac{r_n^2 + 2r_n (1/\lambda)}{1/\lambda^2}}{1/\lambda^2} = c_n.
\]
Note

$$\lim_{\lambda \to \infty} \frac{c_\lambda}{\lambda^{1/4}} < \infty \Rightarrow \sum c_\lambda \sim \sum \frac{1}{\lambda^{1/4}} < \infty.$$ 

Step 3

The Mittag-Leffler solution

$$f(x) = \frac{1}{x^2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{(2-x)^2} - \frac{1}{x^2} \right) = \text{Weierstrass p function.}$$

Homework 3, #4.
Compare \[\text{[III]} \& \text{[IV]}\]

\[
\begin{align*}
&\left(\frac{1}{x - 2}, \frac{1}{(x - 2)^2}\right) & \overset{\text{derivative}}{\leftrightarrow} & \left(\frac{1}{x - 2}, \frac{1}{(x - 2)^2}\right) \\
&\xi & \overset{\text{derivative}}{\leftrightarrow} & \eta = -\xi'
\end{align*}
\]

\[
\begin{align*}
&\left(-n, \frac{1}{x + n}\right) & \overset{\text{derivative}}{\leftrightarrow} & \left(-n, \frac{1}{(x + n)^2}\right) \\
&\pi \cot \frac{\pi}{2} & \overset{\text{derivative}}{\leftrightarrow} & \frac{\pi^2}{\sin^2 \frac{\pi}{2}} = -\left(\pi \cot \frac{\pi}{2}\right)'
\end{align*}
\]

Remark. The Mittag-Leffler Problem makes sense for all \(u \leq c\). We will not give the details here (but see Conway VIII. 3).
Math 2208 - Lecture 11

January 29, 2021
Next few lectures - Normal Families

Montel

Terminology

Arzela

Ascoli

Why climb the mountain - Motivation

Sequences of complex numbers

\{ a_n \} bounded \implies \exists \text{ convergent subsequence}

Indeed, if \( |a_n| \leq M \Rightarrow a_n \in \overline{\Delta}(0, M) \). The closed disc \( \overline{\Delta}(0, M) \) is compact.
We wish to make similar statements for sequences of functions (continuous or holomorphic).

**Dream Statement**

Given a "bounded" sequence of functions, there exists a "convergent" subsequence.

**Question A** What could "---" mean?

**Question B** Is this connected to compactness?

Answer is "yes" but it has no consequences for the current lecture.
Remark: Dream statement makes sense in

- **Real analysis** (continuous functions): Arzela–Ascoli

- **Complex analysis** (holomorphic functions): Montel

We will investigate both.

**Question A**

fn : u → a "convergent" could mean

- pointwise
- weak
- uniform
- strong
- local uniform
- OK for us
- uniform convergence on compact sets
I. "bounded" could mean

pointwise bounded \( \rightarrow \) weak

\[ \forall x \in U \exists M(x) \text{ with } |f_n(x)| < M(x) \text{ for all } n \]

uniformly bounded \( \rightarrow \) strong

\[ \exists M \forall x \in U \quad |f_n(x)| < M \text{ for all } n \]

locally uniformly bounded \( \rightarrow \) ok for us

\[ \forall x \exists D_x \subseteq U \text{ neighborhood of } x \text{, such that the } \]

restrictions \( f_n|_{D_x} \) are uniformly bounded.

Ok for us

uniformly bounded on compact sets

\[ \forall K \exists M(K), \quad |f_n(x)| \leq M(K) \text{ for all } x \in K \text{ for all } n \]
Remark: We have \( \exists ! v \Leftrightarrow \exists ! \alpha \) that is, locally uniformly bounded \( \Rightarrow \) uniformly bounded on each compact.

Why? \( \Leftarrow \) If \( x \in u \), let \( K = \overline{\Delta_x} \) be a compact neighborhood of \( x \).

\[ \Rightarrow \text{For all } x \in u, \exists \Delta_x \text{ where } f_n/\Delta_x \text{ are bounded by } M_x. \]

Then \( K \subseteq U \Delta_x \Rightarrow K \subseteq U \Delta_y \), and let \( m \in K \):

\[ M = \max (M_{x_1}, \ldots, M_{x_n}) > 0. \]

This is a bound for all \( f_n \)'s over \( K \).
Example

1. \( f_n(x) = \sin nx \) uniformly bounded by 1 in \( \mathbb{R} \).

2. \( f_n(x) = 2^n \) in \( \Delta(0,1) \) uniformly bounded by 1.

3. \( f_n(x) = n2^n \) locally uniformly bounded in \( \Delta(0,1) \) but not uniformly bounded.

Proof

For \( 0 \leq r < 1 \): \( |f_n(x)| \leq nr^n \) in \( \Delta(0,r) \). Since

\[
\lim_{n \to \infty} nr^n = 0 \implies \{ nr^n \} \text{ is bounded by } M = 1
\]

\[\implies |f_n(x)| \leq M \text{ in } \Delta(0,r).\]

Each \( K \subseteq \Delta(0,1) \) compact, \( K \subseteq \overline{\Delta(0,r)} \) for \( r < 1 \)

\[\implies \{ f_n \} \text{ uniformly bounded (locally on compacts)}.\]

Since \( f_n \left( \frac{1}{\sqrt{2}} \right) = \frac{n}{2} \to \infty \).

\[\implies \{ f_n \} \text{ not uniformly bounded.}\]
**Dream Statement Revisited**

\[ f_n : u \to \mathbb{C} \text{ locally uniformly bounded} \]

\[ \implies f_n \text{ admits a locally convergent subsequence} \]

---

**Question** Could this be true?

---

**Example** No.

Let \( u = \mathbb{R} \). The sequence

\[ f_n (x) = \sin nx \]

is uniformly bounded, but we can't get a convergent subsequence not even pointwise.
Question c1  Could this be true in complex analysis i.e. holomorphic functions?  YES

Question c2  What is the correct statement in real analysis i.e. continuous functions?

Answer to c1

Main Theorem (Montel)

\[ f_n : U \to \mathbb{C} \text{ holomorphic and locally uniformly bounded} \]

\[ \Rightarrow f_n \text{ admits a locally uniformly convergent subsequence.} \]
More generally - Families

A family of continuous or holomorphic functions.

Required for applications (Riemann - mapping & Picard's theorems)

Any sequence determines \( F = \{ f_1, f_2, ... f_n, ... \} = \text{family}. \)

\[
\tilde{F} = \{ f: \Delta (0,1) \rightarrow \mathbb{C} \text{ holomorphic} \} = \{ f(z) = \sum_{k=1}^{\infty} a_k z^k, |a_k| \leq |a_k| \}
\]

\[
\tilde{F} = \{ f: \Delta (0,1) \rightarrow \mathbb{C} \text{ holomorphic, } f(0) = 1, \text{Re} f > 0 \}
\]
**Def**

$F$ is normal if all sequences in $F$ admit a locally uniformly convergent subsequence.

**Remark**

The limit does not have to be in $F$.

---

**Example**

1. $F$ normal family of holomorphic functions

$\Rightarrow$ $F'$ is normal where $F' = \{ f' : f \in F \}$

**Proof**

Definition + Weierstrass Convergence

Let $\{ f_n' \} \subseteq F'$. be a sequence with $f_n \in F$.

Pick a subsequence $f_{n_k} \rightarrow f$. By Weierstrass,

$f_{n_k} \rightarrow f'$ showing $F'$ is normal.
Remark

We can define $F$ uniformly bounded, locally uniformly bounded etc just as before.

Examples

17 $F = \{ f: \Delta(0, r) \to \mathbb{C} \text{ holomorphic, } f = \sum_{k=0}^{\infty} a_k z^k, \ l_a \leq |f| \}$

locally uniformly bounded.

Indeed, since all compacts $K \subseteq \Delta(0, r)$ suffices to work over $\Delta(0, r)$. Then

$$|f(z)| \leq \sum_{k=0}^{\infty} |a_k| |z|^k \leq \sum_{k=0}^{\infty} k |r|^k = \frac{r}{(1-r)^2} + 121r$$

and $f \in F$

$\implies F$ locally uniformly bounded.
F family of holomorphic functions in U

\( F \) locally uniformly bounded. \( \Rightarrow \)

\( F' \) locally uniformly bounded.

**Proof** Cauchy’s estimates.

Take \( z \in U. \Rightarrow \exists \Delta(z, r) \subseteq U \) such that \( \forall f \in F: \)

\[
|f| \leq M \quad \text{over} \quad \Delta(z, r).
\]

We bound \( |f'| \) over \( \Delta(z, \frac{r}{2}) \).

Let \( a \in \Delta(z, \frac{r}{2}). \) By Cauchy’s estimate

\[
|f'(a)| \leq \frac{\sup |f| \text{over} \overline{\Delta(a, \frac{r}{2})}}{\frac{r}{2}} \leq \frac{M}{\frac{r}{2}}.
\]

where we used \( \overline{\Delta(a, \frac{r}{2})} \subseteq \Delta(z, r) \).

\( \frac{r}{2} \)

We have seen that \( \overline{F} = \bigcup_{n} f_{n} \) is uniformly bounded but \( \overline{F'} = \bigcup_{n} n^{2} f_{n} \) is not uniformly bounded.
Mental Rephrased (Dream Statement)

\( F \) family of holomorphic functions in \( u \in \mathbb{C} \)

\( F \) locally uniformly bounded \( \iff \) \( F \) normal.

Remark Both sides are well behaved under taking derivatives as we noted.
Paul Montel (1876 - 1975) studied normal families of functions. He proved the above theorem in his thesis in 1907. In 1927 he published a monograph on normal families.

Students: Cartan, Dieudonné
1. Last time

Point #1 All notions we use are local e.g.

local boundedness, local uniform convergence, local equicontinuity (today)

Point #2 Work with families

A family of continuous or holomorphic functions in U
Definitions

$L_{locally\ uniform}$

$F_{normal} \iff$ every sequence in $F$ has convergent subsequence

$L_{locally}$ $\iff$ $\forall x \exists \Delta_x \subseteq U$, $F_{\Delta_x}$ uniformly bounded

$F$ locally bounded $\iff$ i.e. $\exists M \in \mathbb{R} + \forall f \in F$, $|f| \leq M$ in $\Delta_x$.

Montel's Theorem $F$ family of holomorphic functions in $U$.

$F_{normal} \iff F$ locally bounded

This fails in real analysis.

$F = \{ \sin n \pi \}$ locally bounded in $\mathbb{R}$ & not normal.

(We can't even arrange pointwise convergence.)
Question c.i.e. What is the correct statement in real analysis i.e. continuous functions?

Remark

This requires the notion of equicontinuity.

Then will be several versions.
II. Notions of Equi-continuity

-strongest

If **equicontinuous on** $U$

$v > 0 \exists \delta > 0 \forall x, y \in U \forall f \in \mathcal{F} : |f(x) - f(y)| < \varepsilon$

Main Point If $\mathcal{F} = \{ f \}$ this says $f$ uniformly continuous.

In general, this says all $f \in \mathcal{F}$ are uniformly continuous, "uniformly".

that is, the same $\delta$ in the definition of uniform continuity works for all $f \in \mathcal{F}$, uniformly.
Fix $m > 0$. The family

$$\mathcal{F} = \{ f : (0,1) \to \mathbb{R}, \; \text{s.t.} \; |f(x) - f(y)| \leq m |x-y| \}$$

is equicontinuous.

Suffices to take $\delta = \frac{\varepsilon}{m}$ and note

$$|x-y| < \delta \implies |f(x) - f(y)| \leq m |x-y| < \varepsilon \quad \forall f \in \mathcal{F}.$$

For $\mathcal{F} = \{ f = \sum_{k=0}^{2021} a_k x^k, \; |a_k| \leq 1 \}$ equicontinuous on $[-1,1]$.

$$\left| \frac{f(x) - f(y)}{x-y} \right| = \left| \sum_{k=0}^{2021} a_k (x^k - y^k) \right|$$

$$\leq \sum_{k=0}^{2021} |a_k| |x-y|^k$$

$$\leq \sum_{k=0}^{2021} 1 \cdot (1 + \ldots + 1) = \sum_{k=0}^{2021} k = m \quad \text{use part (i)}$$

For $\mathcal{F} = \{ f_n \}$; $f_n(x) = nx$ not equicontinuous in $[0,1]$.

See also the Proposition at the end of lecture.
Variations

- **Equicontinuous**

- **Equicontinuous at each point (Conway)**

  \[ \forall x \in U \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \in B(x, \delta) \Rightarrow |f(y) - f(x)| < \varepsilon \quad \forall f \in F \]

  When \( F = \{ f \} \) this says \( f \) is continuous at each point.

- **Locally Equicontinuous**

  \[ \forall x \in U \quad \exists \delta > 0 \quad \forall y \in B(x, \delta) \quad f \text{ is equicontinuous} \]

- **Equicontinuous on all compacts (Rudin, Ahlfors, us)**

  \[ \forall K \subseteq U \text{ compact, } \frac{F}{K} \text{ equicontinuous} \]
are equivalent.

Just use $K = \overline{\Delta_x}$ where $\Delta_x$ is a bounded neighborhood of $*$ in $\mathcal{U}$.

clear from definitions

requires a compactness argument

($\text{HWK 4, \#6 or Conway VII.1}$).
Question C: Characterization of normality?

Theorem (Angela - Ascoli)

$F$ family of continuous functions

$F$ normal $\iff$ $F$ is locally equicontinuous & locally bounded.

Theorem (Montel)

$F$ family of holomorphic functions.

$F$ normal $\iff$ $F$ locally bounded.

Question D: Why is local equicontinuity needed in real analysis?

Question E: Why is local equicontinuity not needed in complex analysis?
**Answer to E**

**Proposition**

F is family of holomorphic functions.

\[ \overline{F} \text{ is locally bounded} \implies \overline{F} \text{ is locally equicontinuous.} \]

**Proof**

Fix \( a \in U \).

\[ \exists \overline{\Delta}(a, 2r) \text{ such that} \]

\[ \overline{F}/\overline{\Delta}(a, r) \text{ is bounded by } M. \]

**Claim**

\[ \overline{F}/\overline{\Delta}(a, r) \text{ is equicontinuous.} \]

Let \( z, w \in \overline{\Delta}(a, r) \). Take \( f \in \overline{F} \).

\[
\frac{|f(z) - f(w)|}{|z - w|} = \left| \frac{1}{2\pi i} \int_{\overline{\Delta}(a, 2r)} \frac{f(s)}{s - z} \, ds - \frac{1}{2\pi i} \int_{\overline{\Delta}(a, 2r)} \frac{f(s)}{s - w} \, ds \right| \\
\leq \left| \frac{1}{2\pi i} \int_{\overline{\Delta}(a, 2r)} \frac{f(s)}{s - z} \left( \frac{1}{s - z} - \frac{1}{s - w} \right) \, ds \right| \\
\leq \frac{1}{2\pi} \int_{\overline{\Delta}(a, 2r)} \left| f(s) \right| \frac{2 - \frac{2}{|z - w|}}{(3 - 2)\overline{\Delta}(a, r)} \, ds.
\]

\[ \overline{\Delta}(a, 2r) \]

\[ \Delta(a, r) \]
\[
\leq \frac{1}{2\varepsilon} \cdot M \cdot \frac{1}{r^2} \cdot \pi \cdot (2\pi) \\
= \frac{2M}{r^2} \cdot \frac{1}{2w} = K \frac{1}{2w} \text{ for } K = \frac{2M}{r}.
\]

The claim follows by Example 10 above, or directly,

\[
\text{let } s = \frac{\varepsilon}{K}. \text{ If } \frac{1}{2w} < s \Rightarrow |f(x) - f(w)| \leq K \frac{1}{2w} < \varepsilon.
\]

\[\text{Q.E.D.}\]

**Conclusion**

**Proposition + Arzola-Ascoli \implies Montel**

We only prove Arzola-Ascoli (next time).
Last time

Montel

Cauchy

Terminology

Arzela–Ascoli

Today we give the proof.

All functions today are continuous.

$F$ family of continuous functions in $U$

$F$ normal $\iff$ $F$ locally equicontinuous and locally bounded.
Notation & Preliminaries

\[ f: U \to \mathbb{R} \text{ continuous, } K \subseteq U \text{ compact} \]

\[ \|f\|_K = \sup_{x \in K} |f(x)| \]

Note

\[ \|f + g\|_K \leq \|f\|_K + \|g\|_K \]

\[ f_n \xrightarrow{K} f \iff \|f_n - f\|_K \to 0 \quad \text{as } n \to \infty. \]

Def \( f_n \) is uniformly Cauchy in \( K \) if

\[ \forall \varepsilon > 0 \in \mathbb{R} \forall n, m \geq N, \quad \|f_n - f_m\|_K < \varepsilon. \]
Lemma \( f_n \) converges uniformly in \( K \)

\[ \iff f_n \text{ uniformly Cauchy in } K. \]

Proof We will only use \( \Rightarrow \) so we only give its proof.

Fix \( \varepsilon > 0 \) \( \Rightarrow \exists N \) with \( |f_n(z) - f_m(z)| < \varepsilon \) \( \forall n, m \geq N. \)

Thus \( \{ f_n(z) \} \) is Cauchy for fixed \( z \). Then \( \{ f_n(z) \} \) converges pointwise to \( f(z) \). Make \( m \to \infty \) in (\( \ast \)) to conclude that

\[ \forall \varepsilon > 0 \exists N \text{ with } |f_n(z) - f(z)| < \varepsilon \text{ } \forall n \geq N, z \in K. \]

Thus \( f_n \to f \) in \( K \).
Proof of Arzelà–Ascoli

Let $F$ be normal.

1. $F$ locally bounded

Let $K \subseteq U$ compact. We show $F|_K$ bounded, i.e.

$$\exists M > 0 \forall f \in F \implies \|f\|_K < M.$$ 

Assume not for a contradiction. Then

$$\forall M > 0 \exists f_m \in F \text{ with } \|f_m\|_K \geq M$$

Letting $M = n$, we obtain a sequence $f_n$ with $\|f_n\|_K \geq n$.

Since $F$ normal, we can find a subsequence $f_{n_k} \xrightarrow{k} f$

Thus $\|f_{n_k} - f\|_K < 1$ if $k$ sufficiently large.

Not $f_{n_k}$ continuous $\implies$ $f$ continuous, so $\|f\|_K < M$. Then

$$M > \|f\|_K \geq \|f_{n_k}\|_K - \|f_{n_k} - f\|_K \geq n_k - 1 \to \infty \text{ as } k \to \infty$$

This gives a contradiction.
(2) \( f \) locally equicontinuous

Let \( K \subset \mathbb{R} \) compact. We show \( f|_K \) equicontinuous.

that is \( \forall \varepsilon \exists \delta : \forall x,y \in K, |x-y| < \delta \Rightarrow \forall f \in F \text{ then } |f(x) - f(y)| < \varepsilon. \)

Assume not, then

\( \exists \varepsilon \forall \delta \exists x, y \in K \text{ with } |x-y| < \delta \exists f \in F \text{ but } |f(x) - f(y)| \geq \varepsilon. \)

Take \( \delta = \frac{1}{n}. \) Then

\( \exists x_n, y_n \in K, |x_n - y_n| < \frac{1}{n} \exists f_n \in F \text{ with } |f_n(x_n) - f_n(y_n)| \geq \varepsilon. \)

After passing to a subsequence & relabelling, we arrange

\[ f_n \xrightarrow{K} f \text{ because } \bar{F} \text{ normal} \]

\[ |x_n - y_n| < \frac{1}{n} \]

\[ |f_n(x_n) - f_n(y_n)| \geq \varepsilon. \]
Using $f_n$ continuous, $f_n \to f$ we get $f$ continuous.

Since $K$ compact $\Rightarrow f|_K$ uniformly continuous.

Then $\exists \varepsilon > 0$ with

$$|x - y| < \frac{\varepsilon}{3}, \quad x, y \in K \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}. \quad (1)$$

Let $N$ be so that $n \geq N$, we have $\frac{1}{n} < \varepsilon$ and

$$\|f_n - f\|_K < \frac{\varepsilon}{3}. \quad (2)$$

Then $|x_n - y_n| < \frac{1}{n} < \varepsilon \Rightarrow |f(x_n) - f(y_n)| < \frac{\varepsilon}{3}$ by (1).

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_n(y_n) - f(y_n)| < \frac{\varepsilon}{3} \quad \text{by} \; (2).$$

By triangle inequality (see picture)

$$|f_n(x_n) - f_n(y_n)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

contradicing
The Converse

Assume $F$ is locally equicontinuous & locally bounded.

$\Rightarrow F$ normal

Let $f_n \in F$. We wish to find a subsequence converging locally uniformly?

How do we find such a subsequence?

Better Plan I. arrange pointwise convergence of $f_n$ only at a countable dense set

II. show local uniform convergence
Let \( \{ \mathbf{a}_k \} \) be the set of points in \( \mathbb{R}^n \) with rational coordinates enumerated in any order. Dense!

**Claim 1** After passing to a subsequence of \( f_n \) and relabelling, we may assume

\[ (*) \quad \forall \mathbf{k}, \text{ the sequence } f_n (a_k) \text{ converges as } n \to \infty. \]

**Claim 2** If \( \{ f_n \} \) equicontinuous & \((*) \Rightarrow f_n \text{ converges locally uniformly.} \]

We win!
Proof of Claim II: Cantor diagonalization

We only use pointwise boundedness of \( \{f_n\} \).

Consider \( f_1(a_i) \), \( f_2(a_i) \), ... \( f_n(a_i) \) ... bounded.

Find a subsequence \( (s_i) \)

\[ f_{s_1}, f_{s_2}, ... f_{s_n}, ... \]

\text{converges at } a_i.

Look at the values of \( (s_i) \) at \( a_2 \) and repeat. We find

\[ (s_2) \]

\[ f_{s_1}, f_{s_2}, ... f_{s_n}, ... \]

\text{converges at } a_2 \text{ and } a_i.

Look at the values of \( (s_2) \) at \( a_3 \) and repeat.
We obtain an array:

\[
\begin{bmatrix}
  f_{11} & f_{12} & \cdots & f_{1n} & \cdots \\
  f_{21} & f_{22} & f_{23} & \cdots & f_{2n} & \cdots \\
  f_{31} & f_{32} & f_{33} & \cdots & f_{3n} & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\end{bmatrix}
\]

Each row is a subsequence of the previous one.

Consider the diagonal subsequence

\[
f_{11}, f_{22}, f_{33}, \cdots f_{nn}, \cdots
\]

It is a subsequence of the original sequence. It converges at each \( a_k \). Indeed

\[
f_{11}, f_{22}, \cdots f_{k-1,k-1}, / f_{kk}, f_{k+1,k+1}, \cdots
\]

initial terms

part of \((s_k)\) so we have convergence at \( a_k \).

Know \( \{a_k\} \) dense in \( U \) and

\( \forall k, \) the sequence \( \{f_n(a_k)\} \) converges

\( \text{fn locally equicontinuous} \)

Wish \( \forall \alpha \in U, \exists \Delta = \text{bounded open ball in } U, \alpha \in U \)

\( f_n/\Delta \) converges uniformly.

1) \( \forall \alpha \exists \alpha \in \overline{\Delta}, \ f/\overline{\Delta} \text{ equicontinuous.} \)

Thus \( \forall \varepsilon \exists S: \forall \left| x-y \right| < S, x, y \in \overline{\Delta}, \forall f \in F \)

\( |f(x) - f(y)| < \varepsilon/3 \)

2) \( \overline{\Delta} \) can be covered by \( \Delta_i = \Delta(a_i, S) \) for \( a_i \in \overline{\Delta} \)

This because \( \{a_i\} \cap \overline{\Delta} \) is dense in \( \overline{\Delta} \).

By compactness, we may assume

\( \overline{\Delta} \subset \bigcup_i \Delta(a_i, S) \).
(3) Since \{f_n(a_i)\} is convergent, it is Cauchy. Hence
\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n,m \geq N, \forall 1 \leq i \leq l
\]
\[
|f_n(a_i) - f_m(a_i)| < \varepsilon/3
\]

(4) Let \( z \in \overline{D} \) by (2), \( \exists i \) with \( |z - a_i| < \delta \). Let \( n, m \geq N \), as in (3). Then
\[
\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n,m \geq N
\]
\[
|f_n(z) - f_m(z)| \leq |f_n(z) - f_n(a_i)| + |f_n(a_i) - f_m(a_i)| + |f_m(a_i) - f_m(z)|
\]
\[
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3
\]

(5) Conclusion
\[
\|f_n - f_m\|_{\overline{D}} < \varepsilon/3, \forall n,m \geq N.
\]
\[
\Rightarrow \{f_n\} \text{ uniformly Cauchy in } \overline{D}
\]

Lemma
\[
\Rightarrow \{f_n\} \text{ converges uniformly in } \overline{D}.
\]
This completes the proof.
Remark  The converse only used pointwise boundedness.  

\[ F \text{ normal } \iff F \text{ pointwise bounded } \iff F \text{ locally bounded } \iff F \text{ locally equicont} \]

The second version bears connections with Montel & it is more uniform.
0. Logistics

(1) Poll regarding Math 220C

[ ] MW 3 - 4:20

[ ] Live/recorded/half live - half recorded?

(2) Midterm - Friday 12 - take home

will cover everything up to and including Monday

Conflicts?

Topics we covered:

- Infinite Products, $\Gamma$ function, sine
- Weierstrass factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz Lemma & applications
The goal is to frame the discussion & formulate guiding questions.

Given \( u, v \subseteq \mathbb{C} \) we wish to study holomorphic

\[ f: u \to v. \]

This may be too general. We can ask

- \( f \) injective
- \( f \) finite to one
- \( f \) bijective
- \( f \) proper \( \ldots \) etc.
We will focus on bijective holomorphic maps.

**Remark**

In Final Exam, Math 220A, we showed

Let $U \subset \mathbb{C}$ be an open set containing 0. Let $f : U \rightarrow \mathbb{C}$ be an injective holomorphic function.

Show that $f'(0) \neq 0$.

The same argument works for any $u$ & any point of $u$:

$f : u \rightarrow v$ injective holomorphic $\Rightarrow f'$ has no zeroes.

In Math 220A, Lecture 11, we showed

**Example** $f : u \rightarrow v$ bijective, holomorphic & $f'(a) \neq 0$ for $a \in u$. Then $f^{-1}$ holomorphic.

**Conclusion** $f : u \rightarrow v$ holomorphic & bijective

$\Rightarrow f^{-1}$ holomorphic

Bi-holomorphism = holomorphic + bijective
We focus on biholomorphisms.

Question: Given $U, V \subseteq \mathbb{C}$ are $U, V$ biholomorphic?

Remark: This has implications in topology and differential geometry. In particular $U, V$ are homeomorphic, diffeomorphic.

Example:

1) $U = \mathbb{C}, \ V = \Delta (0,1), \ U \not\sim V$. This follows by Liouville’s theorem.

2) $U = \mathbb{C}^+, \ V = \Delta, \ c : \mathbb{C}^+ \to \Delta$. Math 220A

Cayley transform: $c(z) = \frac{z - i}{z + i}$, $c^{-1}(w) = i \cdot \frac{1 - w}{1 + w}$.

$\mathbb{C}^+ \cong \Delta$
This is Homework 2, Math 220A.

**Very Important Theorem (Riemann Mapping Theorem)**

Given \( u, v \neq \mathbb{C} \), \( u, v \) simply connected \( \iff u, v \) are biholomorphic.

In particular, if \( v = \Delta(0, 1) \), then any \( u \neq \mathbb{C} \) simply connected then \( u \) is biholomorphic to \( \Delta(0, 1) \).
Riemann’s dissertation (1851) sketched a proof

Referred by Gauss

"The whole is a solid work of high quality, not merely fulfilling the requirements usually set for doctoral thesis, but far surpassing them."

It took the effort of many great minds—Weierstrass, Carathéodory, Hilbert, Schwarz, Koobe, Feyer, Riesz & others—to finalize the proof.
**Question B**

Given $U, V \subseteq \mathbb{C}$ biholomorphic can we construct

- one biholomorphism $U \to V$ explicitly?
- all biholomorphism $U \to V$ explicitly?

**Special cases of II**

We saw some specific examples above e.g.
the Cayley transform for $\mathbb{G}^+$ and $\mathbb{D}(0,1)$. 
When \( u = V \), Question B \( \text{IV} \) becomes.

**Question C**

What are all biholomorphisms \( f : u \to u \)?

**Remarks**

\[
\text{Def.} \quad \text{Aut} (u) = \{ f : u \to u : f \text{ holomorphic & bijective} \}
\]

is a group. Indeed \( f \in \text{Aut} (u) \Rightarrow f^{-1} \in \text{Aut} (u) \) using that \( f^{-1} \) is automatically holomorphic by the above remarks.

**Examples:** We can consider this question for \( u = \Delta, \mathbb{R}^+, \mathbb{C}, \Delta^*, \mathbb{C}^* \) etc...
If \( f, g : u \to v \), \( f \) is known biholomorphism, then any other biholomorphism \( g : u \to v \) differs from \( f \) by automorphisms:

\[ g = \Phi \cdot f, \quad \Phi \in \text{Aut}(v) \]

Indeed, \( \Phi = g \circ f \). 

In the same fashion,

\[ g = f \circ \psi \text{ where } \psi = g \circ f^{-1} \]

and \( \psi \in \text{Aut}(u) \).

Thus, knowledge of Question c helps with aspects of Question B.
**Question D**

Is the action of $\text{Aut}(u)$ on $u$ transitive i.e.

$$u, a, b \in u \quad \exists f \in \text{Aut}(u) \text{ with } f(a) = b$$

**Example**  $U = \{ u \}$. FLT are automorphisms of $u$. 

A action is transitive. (Math 220A)

**Question E**

Given $a \in u$, describe $f : u \to u$ biholomorphism, with $f(a) = a$.

Many other questions can be asked.
We begin the discussion with the case
\[ u = \Delta (0, 1) = \Delta. \]

The crucial statement is the **Schwarz Lemma**

**Theorem** Given \( f : \Delta \rightarrow \Delta, \Delta = \Delta (0, 1) \) holomorphic, \( f(0) = 0 \).

then \( |f'(0)| \leq 1 \) and \( |f(z)| \leq |z| \).

If \( |f'(0)| = 1 \) or if \( |f(z)| = |z| \) for some \( z \in \Delta \setminus \{0\} \) then \( f \) is a rotation, \( f(z) = e^{i\pi} z, z \in \Delta. \)

**Proof** - next time.
Midterm Exam

(1) 4 - 5 Questions

- Infinite Products, $\Gamma$ function, sine
- Weierstrass factorization
- Mittag Leffler
- Normal families & Montel
- Schwarz lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the questions for as long as you wish in this interval.
**Theorem**

Given \( f : \Delta \rightarrow \Delta \), \( f(0) = 0 \) then

1. \( |f'(0)| \leq 1 \), and

2. \( |f(z)| \leq |z| \)

If either \( |f'(0)| = 1 \) or \( f(z_0) \neq 0 \) with \( |f(z_0)| = |z_0| \) then

\( f \) is a rotation i.e. \( f(z) = e^{i\theta}z \)

**Proof**

Let \( g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \). By the removable singularity theorem (Lecture 13, Math 220A), \( g \) is holomorphic.

This uses \( f(0) = 0 \).

Let \( 0 < r < 1 \). Then for \(|w| = r\),

\[ |g(w)| = \frac{|f(w)|}{|w|} \leq \frac{1}{r} \text{ since } \lvert \text{Im} f \rvert \leq \Delta. \]
By maximum modulus principle,

\[ \sup_{|w| \leq r} |g(w)| = \sup_{|w| = r} |g(w)| \leq \frac{1}{r}. \]

In particular, for all \( 1 \leq r < 2 \), we have

\[ |g(2)| \leq \frac{1}{r}. \]

Make \( r \to 1 \) keeping \( g \) fixed. Then \( |g(2)| \leq 1 \). In particular,

\[ |g(2)| = |f'(o)| \leq 1 \quad \text{and} \quad |f(o)| \leq |z|. \]

If \( |f'(o)| = 1 \) or \( |f(z)| = |z| \) for \( z \neq 0 \) then either

\[ |g(0)| = |g(z)| = 1. \quad \text{Since} \quad |g(z)| \leq 1 + z^2, \quad \text{then} \ g \text{ must be constant by MMP again. Thus} \ g(z) = e^{1X} \Rightarrow f(z) = e^{1X}z. \]
Corollary \[ f : \Delta \to \Delta \text{ biholomorphism, } f(0) = 0 \text{ then } f \text{ is a rotation.} \]

**Proof** \[ \text{Note } f(0) = 0 \implies f^{-1}(0) = 0. \text{ We apply Schwarz to both } f, f^{-1}. \text{ We obtain} \]

\[ |f(z)| \leq 2 \quad \text{and} \quad |f^{-1}(w)| \leq |w|. \text{ Let } w = f(z) \text{ to get} \]

\[ \frac{1}{2} \leq |f(z)|. \text{ Therefore } |f(z)| = \frac{1}{2} \implies f \text{ rotation.} \]
II. Automorphisms of the unit disc $\Delta = \Delta (0, 1)$.

**Question**
What can we do if we are given

$$f: \Delta \to \Delta \text{ with } f(0) = a \neq 0, \ |a| < 1.$$ 

**Key Idea**

$$\exists \gamma_a: \Delta \to \Delta \text{ with } \gamma_a(a) = 0.$$ 

We can then recenter $f$ by considering $\tilde{f} = \gamma_a \circ f$.

Specifically

$$\gamma_a(z) = \frac{z - a}{1 - \overline{a} z}.$$
Important Properties

1. \( \gamma_a : \Delta \to \Delta \), \( \gamma_a : \partial \Delta \to \partial \Delta \)

2. \( \gamma_a (0) = -a \), \( \gamma_a (a) = 0 \)

3. \( \gamma_a, \gamma_{-a} \) are inverses

\[ \gamma_a' (0) = 1 - 1/|a|^2, \quad \gamma_a' (a) = \frac{1}{1 - 1/|a|^2}. \]

\( \gamma_a \) shrinks \( \leq 1 \), \( \gamma_a \) expands \( > 1 \).

Proof \( \text{II} \rightarrow \text{IV} \) follow by direct calculation.

4. Note that \( \gamma_a (z) = \frac{z - a}{1 - \overline{a}z} \) has pole at \( \frac{1}{\overline{a}} \) but this is met in \( \overline{\Delta} \) since \( |z| < 1 \). Thus \( \gamma_a \) is holomorphic in \( \Delta \), continuous in \( \overline{\Delta} \). If we show

\( (*) \) \( |\gamma_a (z)| = 1 \) if \( |z| = 1 \), by the maximum modulus, it follows \( |\gamma_a (z)| < 1 \) if \( |z| < 1 \) so \( \gamma_a : \Delta \to \Delta \).

To see \( (*) \) we show \( |z - a| = |1 - \overline{a}z| \) if \( |z| = 1 \).
Let \( \frac{1 - a \bar{z}}{1 - \bar{a} z} \) be the conjugation

\[
= \frac{1}{1 - \frac{a}{z}} \quad \text{since} \quad 2\bar{z} = |z|^2 = 1
\]

\[
= \frac{12 - a}{12z} = 12 - a \quad \text{as needed.}
\]

---

**Theorem**

If \( f : \Delta \to \Delta \) is biholomorphic then

\[
f(z) = c \cdot \frac{z - a}{1 - \bar{a} z} \quad \text{for} \quad |a| < 1.
\]

---

**Proof**

Let \( a \) be such that \( f(a) = 0 \). Let

\[
f \circ \varphi_a = f \circ \varphi = 0.
\]

Next, \( \tilde{f} \) is a biholomorphism. Then \( \tilde{f} \) is a rotation.

\[
\Rightarrow \tilde{f}(w) = e^{i\theta} w \Rightarrow f \circ \varphi_a(w) = e^{i\theta} w = f(a) = e^{i\theta} \varphi_a(a).
\]

Setting \( w = \varphi_a(z) \).
Remark. We have seen $\varphi_a$'s in HWK 1.

**Blaschke's products**

$f: \Delta \rightarrow \Delta, \ a \Delta \rightarrow a \Delta$ then

$$f(z) = c \prod_{k=1}^{N} \frac{1}{1 - \varphi_{a_k} z}, \quad |c| = 1.$$ **Exercise**

Assume $f: \Delta \rightarrow \Delta, \ a \Delta \rightarrow a \Delta$.

where only zeroes are at $\frac{1}{2}$ and $\frac{1}{4}$ with multiplicities 2 & 3

Find $|f(0)|$.

**Solution**

$f(z) = c \prod_{k=1}^{N} \frac{1}{1 - \varphi_{a_k} z}$. Then

$$f(0) = c \prod_{k=1}^{N} \frac{1}{1 - \varphi_{a_k}} \frac{1}{2} \frac{1}{4} \frac{1}{4}, \quad |c| = 1 \Rightarrow |f(0)| = \frac{1}{2^8}.$$
Understanding the action of $\text{Aut} (\Delta)$ on $\Delta$

**Important Remark**

The action of $\text{Aut} (\Delta)$ on $\Delta$ is transitive.

\[ \forall a, b \in \Delta \exists f \in \text{Aut} \Delta, \ f(a) = b. \]

![Diagram of action with points a, 0, and b, with arrows indicating the action of $f$.]

Note: $f = \varphi_b \circ \varphi_a$ is an automorphism and

\[ f(a) = \varphi_b \circ \varphi_a (a) = \varphi_b (0) = b. \]

**Application**

Show if $f: \Delta \to \Delta$ holomorphic, $f \neq 1 \Rightarrow f$ has at most 1 fixed point.
Proof Assume \( f(a) = a \) & \( f(b) = b \). & \( a \neq b \).

If \( a = 0 \) then \( f(0) = 0 \) & \( f(b) = b \) \( \Rightarrow \) \( f \) rotation via 

Schwarz \( f(2) = e^{i\theta} \). Using \( f(b) = b \) \( \Rightarrow \) \( e^{i\phi} = 1 \) \( \Rightarrow \)

\( f = I \) which is disallowed.

For \( a \neq 0 \), we reduce to this case. Let

\[
\tilde{f} = \gamma_a f \gamma_a^{-1} \quad \& \quad \lambda = \gamma_a (b) \neq 0 = \gamma_a (a)
\]

\[
\begin{array}{c}
\gamma_a \\
\gamma_a \\
\gamma_a \\
\gamma_a
\end{array}
\]

Then \( \tilde{f}(0) = 0 \) and \( \tilde{f}(\lambda) = \lambda \) \( \Rightarrow \) \( \tilde{f} = I \)

\( \Rightarrow \) \( \gamma_a f \gamma_a^{-1} = I \) \( \Rightarrow \) \( f = I \), again a contradiction.

Thus \( f \) has at most one fixed point.
Recap

• if \( f(0) = 0 \) then
  
  - we proved Schwarz Lemma
  
  - we determined \( f \in \text{Aut } \Delta, \ f(0) = 0 \)

• if \( f(0) \neq 0 \)
  
  - we determined \( f \in \text{Aut } \Delta \)

**Question** Is there a version of Schwarz if \( f(0) \neq 0 \)?

Yes - Schwarz-Pick Lemma.

- we illustrate it for derivatives
**Proposition**  \( f : \Delta \rightarrow \Delta \) holomorphic, \( \forall \in \Delta \)

\[
\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}
\]

**Remark**  If \( a = 0 \) this gives \( |f'(0)| \leq 1 - |f(0)|^2 \).

If \( f(0) = 0 \) this gives \( |f'(0)| \leq 1 \). Thus the Proposition generalizes Schwarz lemma.

The proof will be given next time.

**Remark**  This is naturally formulated in hyperbolic geometry.
Math 2208 - Lecture 16

February 10, 2021
0. **Midterm Exam**

(1) 5 Questions

- Infinite Products, \( R \) function, \( \sin \)
- Weierstrass factorization
- Mittag Leffler
- Normal families & Monohl
- Schwarz Lemma & applications

(2) Available on Friday at noon, due Tuesday at noon.

You can think about the Questions for as long as you wish in this interval.

(3) Closed book / closed notes / no internet / no collaboration

(4) Email if questions arise
(5) you may use theorems proved in lecture but no homework problems can be used without proof.

(c) Office hour 4 - 5:30 today.
Last time

- if $f(0) = 0$ then
  - we proved **Schwarz Lemma**
  - we determined $f \in \text{Aut } \Delta$, $f(0) = 0$

- if $f(0) \neq 0$
  - we determined $f(0) \neq 0$

**Idea** Use $g_0$ to recenter $f$ so that $o$ maps to $0$.

**Question** Is there a version of **Schwarz** if $f(0) \neq 0$?

Yes — **Schwarz - Pick Lemma**.

- we illustrate it for derivatives
Schwarz–Pick: $f: \Delta \to \Delta$ holomorphic, $\forall a \in \Delta = \Delta(0,1)$.

\[
\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.
\]

Remark: $f(0) = 0$, $a = 0$ recovers Schwarz: $|f'(0)| \leq 1$.

Example: Conway VI.2.3 $f: \Delta \to \Delta$ holomorphic.

If $f\left(\frac{1}{2}\right) = \frac{1}{4}$, find the maximum value of $|f\left(\frac{i}{2}\right)|$. 

We know this when \( a = 0 \) & \( \alpha = f(a) = 0 \).

We use \( \text{Aut} \ (\Delta) \) to reduce to this case.

Let \( f(a) = \alpha \). Let \( f = \gamma_\alpha \cdot f \cdot \gamma_{-\alpha} \Rightarrow f'(0) = 0 \) as the diagram shows.

By Schwarz, \( |f'(0)| \leq 1 \). We compute using the chain rule:

\[
\tilde{f}'(0) = \gamma_{\alpha}'(f(\gamma_{-\alpha}(0))) \cdot f'(\gamma_{-\alpha}(0)) \cdot \gamma_{-\alpha}'(0)
\]

\[
= \gamma_{\alpha}'(\alpha) \cdot f'(a) \cdot \gamma_{-\alpha}'(0)
\]

\[
= \frac{1}{1 - |a|^2} \cdot f'(a) \cdot (1 - |a|^2) & \quad & \text{as needed.}
\]

\[
|f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}
\]
**Remark**

<table>
<thead>
<tr>
<th>Schwarz</th>
<th>$f(0)=0$</th>
<th>Schwarz - Pick</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>f'(0)</td>
<td>\leq 1$</td>
</tr>
<tr>
<td>$</td>
<td>f(z)</td>
<td>\leq</td>
</tr>
</tbody>
</table>

Define $d(z, w) = \sqrt{\frac{z - w}{1 - \bar{z} \cdot w}} = \text{pseudo hyperbolic distance}$

**Schwarz - Pick**

Holomorphic maps decrease pseudo hyperbolic distance.

This will be made precise in HWK 5.
2. Further applications of Schwarz

We can use Schwarz to study other domains e.g.

1) \( u = \Delta^* = \Delta \backslash \{0\} \)

2) \( u = \mathbb{H}^+ = \text{upper half plane} \)

Example: All automorphisms of \( \Delta^* \) are rotations.

Proof: Let \( f: \Delta^* \to \Delta^* \). Since \( \text{Im} f \) is bounded, \( f \) can be extended across 0 by the removable singularity theorem. The extension \( \tilde{f}: \Delta \to \overline{\Delta} \) is holomorphic.

Its image \( \text{Im} \tilde{f} \subseteq \Delta \) by the open mapping theorem (draw picture).
We claim \( \tilde{f}(0) = 0 \). Then \( f: \Delta^x \to \Delta^x \) shows \( \tilde{f} \) bijective. Then \( f \) from \( \Delta \to \Delta \) hence a biholomorphism preserving \( 0 \). Then \( \tilde{f} \) is a rotation.

To show \( \tilde{f}(0) = 0 \) assume otherwise \( \tilde{f}(0) = \alpha \neq 0 \).

Since \( \alpha \in \Delta^x \) we can find \( a \in \Delta^x \). \( f(a) = \alpha \).

By the open mapping theorem, we can find small discs \( \Delta_0, \Delta_a, \Delta_\alpha \) near \( 0, a, \alpha \) with \( \Delta_0 \cap \Delta_a = \emptyset \) and:

\[
\Delta_\alpha \subseteq f(\Delta_0), \Delta_\alpha \subseteq f(\Delta_a). \quad (\text{why?})
\]

Let \( b \in \Delta_\alpha \); \( \exists \alpha \) \( \Rightarrow b \in f(\Delta_0) \Rightarrow b = f(u), u \neq 0, u \in \Delta_0 \)

\( \Rightarrow b \in f(\Delta_a) \Rightarrow b = f(v), v \in \Delta_a \)

\( \Rightarrow f(u) = f(v) = b \)

\( \Rightarrow f \) not injective (contradiction).

\( u \neq v \) since \( \Delta_0 \cap \Delta_a = \emptyset \)
Key idea

Use \( \mathbb{H}^+ \rightarrow \Delta \),

\[
c(z) = \frac{z - i}{2 + i}.
\]

\[
c^{-1}(z) = i \cdot \frac{1 + 2i}{1 - 3}.
\]

Questions we can answer:

[1] \( \text{Aut} (\mathbb{H}^+) \) at next time

Schwarz lemma for \( f: \mathbb{H}^+ \rightarrow \mathbb{H}^+ \)

Schwarz-Pick for \( f: \mathbb{H}^+ \rightarrow \mathbb{H}^+ \)

[3] Biholomorphisms \( \Delta \rightarrow \mathbb{H}^+ \)

Schwarz lemma for \( f: \Delta \rightarrow \mathbb{H}^+ \)

Schwarz-Pick for \( f: \mathbb{H}^+ \rightarrow \Delta \)

for derivatives or for distance ...

It is impossible to record them all.
Example \( f: \Delta \rightarrow \mathbb{S}^n, f(v) = v \) Show \( |f'(v)| \leq 2 \).

Let \( f = c \cdot f \). Then \( f'(0) = 0 \) since
\[
|c(\xi)| = \frac{2 - \xi}{2 + \xi} \rightarrow 0
\]
\( \Rightarrow |f'(0)| \leq 1 \) by Schwarz.

We compute
\[
|f'(0)| = |c'(f(0)) \cdot f'(0)| = |c'(\xi)| \cdot |f'(0)| \leq 1.
\]

Since \( c'(\xi) = \frac{1}{2^\xi - 1} \Rightarrow |f'(0)| \leq 2 \).

3. Further discussion of Aut. - Loose ends

1. \( \text{Aut} \, \mathbb{C} \)
2. \( \text{Aut} \, \mathbb{C} \)
3. \( \text{Aut} \, \Delta \)
4. \( \text{Aut} \, \mathbb{S}^n \)
5. \( \text{Aut} \, \Delta^x \)
1. Further discussion of Aut. — Loose ends

\[ \text{Aut } \mathcal{C} = \{ ax + b : a \neq 0, b \in \mathbb{C} \} \cong \text{Aff.} \]

\[ \text{Aut } \hat{\mathcal{C}} = P\text{SL}_2 \]

\[ \text{Aut } \Delta \cong SU(1,1)/\pm 1 = P\text{SU}(1,1) \]

\[ \text{Aut } J^+ \cong SL(2,\mathbb{R})/\pm 1 = P\text{SL}(2,\mathbb{R}) \]

\[ \text{Aut } \Delta^x \cong \text{Rotations} \]

Case \( \boxdot \) \quad U = \emptyset

7. Assume that \( f : \mathbb{C} \to \mathbb{C} \) is entire and injective. Show that \( f(z) = az + b \). You can solve this problem using the notions introduced in Problem 6 above.
\[ u = \hat{c} = c \cdot u / \infty \]


\[
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad \tau_m(z) = \frac{az + b}{cz + d}, \quad \tau_m : \hat{c} \rightarrow \hat{c}
\]

\[ \tau_M = \tau_N \iff M = \lambda N. \]

\[ \tau_M \tau_N = \tau_{MN}. \]

\[ \tau_M \text{ bijective} \iff M \text{ invertible since } \tau_m \cdot \tau_m^{-1} = \mathbb{I} \]

Define \( PGLA = GL_2 / \{ \lambda \cdot I, \lambda \neq 0 \} \) = invertible \( 2 \times 2 \) matrices up to scaling.

Recall from Math 220 A, Lecture 3, the action of Mobius transforms is transitively on \( \hat{c} \).
Theorem \[ \text{Aut} \; \hat{\mathfrak{C}} = PGL_2. \]

Proof: \( f \in \text{Aut} \; \hat{\mathfrak{C}}, \; f(\infty) = \infty \) then \( f : \mathfrak{C} \to \mathfrak{C} \) is bijective. Thus \( f(z) = az + b = \lambda z \) for the matrix
\[
M = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}
\]

If \( f(\infty) \neq \infty \) then \( f(\infty) = \lambda \in \mathfrak{C} \). Let
\[
g(z) = \frac{1}{f(z) - \lambda} \Rightarrow g(\infty) = \infty \Rightarrow g(z) = az + b
\]

\( \Rightarrow f(z) = \lambda + \frac{1}{az + b} = \text{fractional linear transformation}, \)
as needed.
Case (iii): $\text{Aut } (\Delta)$.

**Question**

What is $\text{Aut } (\Delta)$ as an abstract group?

$f \in \text{Aut } \Delta$

$$f(z) = e^{18} \cdot \frac{z - a}{1 - \bar{a}z} = e^{18/2} \cdot \frac{z - a}{e^{-18/2} - \bar{a}z} = h_{\infty}.$$ 

$$M = \begin{bmatrix} e^{18/2} & -a e^{18/2} \\ -\bar{a} e^{-18/2} & e^{-18/2} \end{bmatrix} = \begin{bmatrix} A & B \\ \bar{B} \bar{A} \end{bmatrix} \text{ invertible.}$$

**Note** $\det M = 1 - |a|^2 > 0$. Let $\lambda = (1 - |a|^2)^{-1/2}$.

Rescale $A \rightarrow \lambda A$, $\lambda \in \mathbb{R}.$

$B \rightarrow \lambda B$, $\lambda \in \mathbb{R}.$

$A \bar{A} - B \bar{B} = |A|^2 - |B|^2 = 1.$

**Conclusion** $\text{Aut } \Delta = \left\{ \begin{bmatrix} A & B \\ \bar{B} \bar{A} \end{bmatrix} : 1|A|^2 - |B|^2 = 1 \right\}/\pm 1 = \mathbb{S}U (1, 1)/\pm 1 = \mathbb{P}SU (1, 1).$
**Case II** \( Aut \mathcal{G}^+ \)

**Key idea** Use Cayley transform:

\[ \mathcal{G}^+ \xrightarrow{C} \Delta \xrightarrow{c^{-1}} \mathcal{G}^+ \]

\[ c(2) = \frac{2 - i}{2 + i}. \]

\[ c^{-1}(2) = \frac{1 + 2}{1 - 2}. \]

\[ \mathcal{G}^+ \xrightarrow{f} \Delta \xrightarrow{g} \mathcal{G}^+ \]

\[ g = c^{-1} \circ f \circ C \text{ is an automorphism} \]

Any \( g \in Aut \mathcal{G}^+ \) is of this form for \( f = CgC^{-1} \).
\[
C^{-1} \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix} C = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}
\]

\[
\alpha = \text{Re} A + \text{Re} B, \quad \delta = \text{Re} A - \text{Re} B
\]

\[
\beta = \text{Im} A - \text{Im} B, \quad \gamma = -\text{Im} A - \text{Im} B
\]

\[
\Rightarrow \alpha, \beta, \gamma, \delta \in \mathbb{R}.
\]

\[
1|A|^2 - |B|^2 = 1 \iff (\text{Re} A)^2 + (\text{Im} A)^2 - (\text{Re} B)^2 - (\text{Im} B)^2 = 1.
\]

\[
\iff \alpha \delta - \beta \gamma = 1.
\]

\[
\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, \mathbb{R}).
\]

**Conclusion**

\[
\text{Aut}(\mathbb{F}_2^+) \cong \frac{SL(2, \mathbb{R})}{\{1, -1\}} = PSL(2, \mathbb{R}).
\]
"Two given simply connected planar surfaces can always be related to each other in such a way that every point of one corresponds to one point of another, which varies continuously with it, and their corresponding smaller parts are similar."

(Translation by R. Remmert.)
Theorem: If $\mathcal{U}$ is simply connected, then $\mathcal{U}$ is biholomorphic to the unit disc, $\Delta = \Delta(0,1)$.

Remarks: $\mathcal{U} = \mathbb{C}$ is not biholomorphic to $\Delta$.

By Liouville's theorem, there cannot exist a holomorphic nonconstant map $\mathbb{C} \to \Delta$.

Implications in topology:

$\mathcal{U}$ simply connected, $\mathcal{U} \subseteq \mathbb{C}$, so $\mathcal{U}$ is topologically $\Delta$, i.e., there exists a bijective continuous map $\mathcal{U} \to \Delta$ (homeomorphism).

This holds even for $\mathcal{U} = \mathbb{C}$ using the map:

$$z \to \Delta, \quad 2 \to \frac{z}{\sqrt{z + 13}^2}.$$
Why is the proof difficult? Imagine the domain $u$.

It is hard to construct explicit maps (even in the topological category).

**Examples**

1. $c : \mathbb{D}^+ \rightarrow \Delta$, $c(z) = \frac{2 - i \cdot z}{z + 1}$.

2. Biholomorphism between $\Delta$ and the slit plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ (both simply connected).
We use simple geometric moves:

\[ \Delta \rightarrow \tilde{g}^+ \text{ via } c^{-1}(x) = \frac{i}{1+i}. \]

\[ \tilde{g}^+ \rightarrow c \setminus R_{\geq 0} \text{ via } w \rightarrow w^2. \]

\[ c \setminus R_{\geq 0} \rightarrow c \setminus R_{\leq 0} = c^- \text{ via } s \rightarrow -s. \]

Composition:

\[ -\left( \frac{1+i}{1-i} \right)^x = \left( \frac{1+i}{1-i} \right)^x : \Delta \rightarrow c^- \]
So. **Riemann Mapping Theorem**

Theorem $U \neq \mathbb{C}$ simply connected $\Rightarrow U$ biholomorphic to the unit disc $\Delta = \Delta (0,1)$.

**Ingredients in the proof**

1. Montel & normal families
2. Hurwitz's Theorem
3. Aut $\Delta$ & Schwarz Lemma
4. Square root trick of Carathéodory–Koebe

& standard tools: Open Mapping & Weierstrass.
§1. **Strategy**

*Fix* \( a \in U \)

\[ u \quad \rightarrow \quad \Delta \]

**Goal #1**

First, \( f: U \rightarrow \Delta, \ f(a) = 0 \) & \( f \) *injective*

---

**Main Actor in the Proof**

*Consider the family*

\[ F = \{ f: U \rightarrow \Delta, \ f(a) = 0, \ f \text{ injective} \} \]

*Want*

\( F \neq \bar{F} \)
**Question** How to achieve $f$ bijective?

Imagine $U = \bigcup_{n} K_{n}$, $\alpha \in K_n \subseteq \text{Int} \ K_{n+1}$.

We hope $\bigcup_{n} f(K_{n})$ cover $\Delta$. We expect that this has a chance if $|f'(a)|$ is as large as possible.

Let $M = \sup \{ |f'(a)| : f \in F \}$.

**Goal #2** Show $\exists f \in \overline{F}$ with $|f'(a)| = M$.

**Goal #3** Show that for this choice, $f : U \to \Delta$ is bijective.
Why might this actually work?

Example \( u = \Delta, \ a = 0 \).

\[
F = \{ f: \Delta \to \Delta, \ f(0) = 0, \ f \text{ injective} \}.
\]

By Schwarz Lemma, \( |f'(0)| \leq 1 \). If the maximum value \( |f'(0)| = 1 \) then \( f \) is a rotation so \( f \) is bijective.

Remark

We can also consider points \( a \in \Lambda, \ a \neq 0 \). Let

\[
F = \{ f: \Delta \to \Delta, \ f(a) = 0, \ f \text{ injective} \}.
\]

Schwarz - Pick

\[
|f'(a)| \leq \frac{1}{1 - |a|^2} \quad \text{with equality iff}
\]

\[ f = \text{Rot} \circ \varphi_a \Rightarrow f \text{ bijective.} \]
2. Assume \( f : U \to \mathbb{C} \) is a holomorphic function on a simply connected open set \( U \) such that \( f(z) \neq 0 \) for all \( z \in U \). Let \( n \geq 2 \) be an integer. Show that there is a holomorphic function \( g : U \to \mathbb{C} \) such that

\[ g(z)^n = f(z). \]

*Hint:* This has something to do with problem 1(ii).

We only need \( n = 2 \).

\[ U \text{ simply connected } \Rightarrow \text{ any } f : U \to \mathbb{C} \text{ holomorphic, } \]

\( \text{no where zero, admits a holomorphic square root } g : U \to \mathbb{C} \)

\[ f = g^2. \quad (\ast) \]

"Root domain"

\( U \subseteq \mathbb{C} \) is a root domain if \((\ast)\) is satisfied.

**Remark**

simply connected \( \implies \) root domain
Remark: This turns out to be equivalent to a simply connected domain.

We will prove the seemingly stronger form:

Riemann Mapping Theorem

\[ U \neq \Delta \implies \text{U is biholomorphic to } \Delta. \]
§ 2. **Study of the family** $\mathcal{F}$. **Fix** $a \in u$.

$$\mathcal{F} = \{ f : u \to \Delta : f \text{ holomorphic, injective, } f(a) = 0 \}.$$

---

**Claim 1** $g \text{ injective.}$

*Indeed, if* $g(z_1) = g(z_2)$ *then* $g(z_1)^2 = g(z_2)^2$ *so*

$$\Rightarrow z_1 - b = z_2 - b \Rightarrow z_1 = z_2.$$

---

**Claim 2** $g(u) \cap (-g)(u) = \emptyset$.

*Indeed, if* $\exists z_1, z_2 \in u$ *with* $g(z_1) = -g(z_2)$
\[ g(x_1)^2 = g(x_2)^2 \Rightarrow x_1 - b = x_2 - b \Rightarrow x_1 = x_2. \]

But then \( g(x_1) = -g(x_2) \Rightarrow g(x_1) = -g(x_2) \Rightarrow g(x_1) = 0 \)

\[ g(x_1)^2 = 0 = x_1 - b \Rightarrow x_1 = b. \text{ But } x_1 \in U, b \notin U. \]

**Claim 3**

For \( c, r \) with \( |g(x) - c| > r \) \( \forall x \in U \).

Indeed, by the open mapping theorem, \((-g)(\mathbf{u})\) is open so it contains a disc \( \Delta(c,r) \). By Claim 2,

\[ g(\mathbf{u}) \subseteq c \setminus \Delta(c,r) \iff |g(x) - c| > r \forall x \in U. \]

**Construction**

Let \( f(x) = \frac{r}{g(x) - c} \). \( \Rightarrow f \) injective since \( g \) is by Claim 1 & \( f: U \to \Delta(0,1) \). by Claim 3.

\( \text{To achieve } f(a) = 0, \text{ define } \tilde{f}(x) = \frac{f(x) - f(a)}{2} \).

\( \Rightarrow \tilde{f} \) injective since \( f \) is, \( \tilde{f}(a) = 0 \).
Note that since \( f \) takes values in \( \Delta \), the same is true for \( \tilde{f} \):

\[
|\tilde{f}(z)| \leq \frac{1}{2} \left( |f(2)| + |f(a)| \right) < \frac{1}{2} (1+1) = 1.
\]

Thus \( \tilde{f} \in F \Rightarrow F \neq \emptyset \).

**Step 1** Let \( M = \sup \{ |f'(a)|, f \in F \} \)

**Show:** The supremum is achieved by some \( f \in F \).

**Proof:** Indeed, take \( f_n \in F \) with \( |f_n'(a)| \to M \) as \( n \to \infty \).

The family \( F \) is bounded by \( 1 \) since the functions in \( F \) take values in \( \Delta \). \( \Rightarrow \) \( F \) normal. \( \Rightarrow \) passing to a subsequence, we may assume \( f_n \to f \) locally uniformly.

**Claim:** \( f \) holomorphic, \( f(a) = 0, |f'(a)| = M \).
Indeed, by Weierstrass convergence, $f$ is holomorphic.

and $f_n' \to f'$ locally uniformly. In particular,

$$f_n'(a) \to f'(a) \quad \text{so} \quad |f'(a)| = M.$$  

Since $f_n(a) = 0$ and $f_n \to f$ at $a$, we have

$$f(a) = 0.$$  

**Claim 5.** \( f : U \to \Delta \) is injective.

Indeed, $f_n$ injective and $f_n \to f'$ shows $f$ is either injective or $f$ constant by Hurwitz's theorem (Math 220 A, Lecture 24).

If $f = \text{constant} \implies f'(a) = 0 \implies M = 0 \implies$

$$g'(a) = 0 \quad \forall \, g \in \mathcal{F} \quad \text{since} \quad M \text{ is the supremum}.$$  

But if $g \in \mathcal{F}$, $g$ injective and $g'(a) \neq 0$ by Math 220 A, Final Exam, Problem 7.
Thus $f$ injective.

Note that since $f_n \rightarrow f$ and $f_n : U \rightarrow \Delta$
shows $f : U \rightarrow \Delta$. By the open mapping theorem,

$f : U \rightarrow \Delta$ ($f \neq \text{not constant}$).

By Claims 4 & 5, $f \in F$ and $|f'(a)| = M \Rightarrow \text{Step 2 \: V}$.

---

**Step 3**

For a function $f \in F$ which achieves the supremum

$f$ is bijective.

Proof: next time.
Let a ∈ U. Wish \( U \sim \Delta \) biholomorphically.

\[ F = \{ f: U \rightarrow \Delta, \; f(a) = 0, \; f \text{ injective} \} \]

**Step 1** \( U \not\sim \text{a root domain} \rightarrow F \neq \emptyset \).

**Step 2** Let \( M = \sup \{ |f'(a)|, \; f \in F \} \).

Then \( M \) is achieved by some function \( f \in F \).
Today

**Step 3** For the extremal function $f$ in Step 2, we show $f$ surjective, then $f$ biholomorphism.

If $f: U \to \Delta$ not surjective, then we show $\exists \tilde{f} \in F$ with $\left| f'(a) \right| > \left| f'(a) \right|$ contradicting maximality of $|f'(a)|$.

**Strategy**

We will in fact show that if $f: U \to \Delta$ not surjective, then $\exists \tilde{f}: U \to \Delta$, $F: \Delta \to \Delta$, $f = F \circ \tilde{f}$, $\tilde{f} \in F$, $F(0) = 0$, $F \notin \text{Aut} \Delta$. 
Assume this can be done. The proof is then completed.

Indeed, by Schwarz lemma \( |F'(0)| < 1 \). (The inequality is strict since \( F \) is not a rotation as \( F \notin \text{Aut} \Delta \).

Then we indeed contradict maximality since

\[
|f'(a)| = |F'(0)| \cdot |\tilde{f}'(a)| < |\tilde{f}'(a)|.
\]

How do we execute the above strategy?

Assume \( f: \mathbb{D} \to \Delta \) is not surjective.

Let \( a \in \Delta \setminus f(\mathbb{D}) \).

Construction of the function \( \tilde{f} \) "square root trick"
We carry out the following moves:

1. **recenter**

   The function \( \gamma \circ f : \Delta \rightarrow \Delta \) omits the value \( \gamma(\alpha) = 0 \) since \( f \) omits \( \alpha \) and \( \gamma \in \text{Aut} \Delta \).

2. **square root**. Since \( \Delta \) is a root domain and \( \gamma \circ f \) is nowhere zero, we can find \( g : \Delta \rightarrow \Delta \) holomorphic with \( g^2(x) = \gamma \circ f \).

   **Claim.** \( g \) injective.

   Indeed \( g(x) = g(w) \Rightarrow g(x)^2 = g(w)^2 \Rightarrow \gamma \circ f(x) = \gamma \circ f(w) \Rightarrow f(x) = f(w) \Rightarrow x = w \) since \( f \in F \) injective.

3. **recenter**. Let \( \beta = g(\alpha) \). We define

   \( \tilde{f} = \gamma \circ g \Rightarrow \tilde{f}(\alpha) = \gamma \circ g(\alpha) = \gamma(\beta) = 0 \).

   \( \tilde{f} : \Delta \rightarrow \Delta \) injective, then \( \tilde{f} \in F \).
\[ g^2 = \gamma_\alpha \circ f, \quad \tilde{f} = \gamma\beta \circ g, \quad \tilde{f} \in \mathcal{F}. \]

**Outcome**

**Comparison**

\[ g^2 = \gamma_\alpha \circ f \implies f = \gamma_{-\alpha} \circ g^2. \]

Let \( s : \Delta \to \Delta, \quad s(w) = w^2 \implies f = \gamma_{-\alpha} \circ s \circ g. \)

\[ \tilde{f} = \gamma\beta \circ g \implies g = \gamma_{-\beta} \circ \tilde{f} \implies f = \gamma_{-\alpha} \circ s \circ \gamma_{-\beta} \circ \tilde{f}. \]

Let \( F : \Delta \to \Delta, \quad F = \gamma_{-\alpha} \circ s \circ \gamma_{-\beta}. \implies f = F \circ \tilde{f} \]

**Claim**

\( F \notin \text{Aut} \Delta, \quad F(0) = 0. \)

Indeed, if \( F \in \text{Aut} \Delta, \quad F = \gamma_{-\alpha} \circ s \circ \gamma_{-\beta} \in \text{Aut} \Delta \)

\implies s \in \text{Aut} \Delta. \) But \( s \) is not even injective as \( s(3) = s(-3). \)

To see \( F(0) = 0 \) we compute

\[ F(0) = \gamma_{-\alpha} \cdot s \cdot \gamma_{-\beta}(0) = \gamma_{-\alpha} \cdot s(\beta) = \gamma_{-\alpha}(\beta^2) = \gamma_{-\alpha}(-\alpha) = 0 \]

when we used

\[ \beta^2 = g(0)^2 = \gamma_\alpha \circ f(0) = \gamma_\alpha(0) = -\alpha. \]
This is exactly what we needed to complete the proof of Steps 3 & the proof of Riemann Mapping.

Remarks

Uniqueness of the biholomorphism. Take two biholomorphisms $f, g: u \rightarrow \Delta$, $f(a) = g(a) = 0$ then

$$\text{consider } \Delta \xrightarrow{f^{-1}} u \xrightarrow{g} \Delta, \quad g f^{-1}(a) = 0, \quad g f^{-1} \in \text{Aut } \Delta.$$  

Then

$$g f^{-1} = \text{Rot} \Rightarrow g = \text{Rot } f.$$ 

Thus the biholomorphisms we constructed are unique up to rotations.
The extremal function $f$ we constructed maximizes the derivatives at $a$ of all functions $g: U \to \Delta$, $g(a) = 0$ not only the injective ones.

Indeed if $f: U \to \Delta$ is the function we constructed, then for $g: U \to \Delta$, $g(a) = 0$,

$$
\begin{align*}
\Delta & \longrightarrow U \quad \longrightarrow \Delta \quad , \quad F = g \circ f^{-1} : \Delta \longrightarrow \Delta. \\
F'(a) &= 0.
\end{align*}
$$

Then $g = F \circ f \implies |g'(a)| = |F'(0)|$. If $f'(a) \leq |f'(0)|$

when we used $|F'(0)| \leq 1$ by Schwarz.

\[ U, V \text{ simply connected, } U, V \neq \emptyset \implies U, V \text{ are biholomorphic. } (U \cong \Delta \cong V \text{ transitive}) \]
A "logarithm domain" is a domain where \( f: U \rightarrow \mathbb{C} \) is holomorphic, \( f \) nowhere zero, we can define \( \log f: U \rightarrow \mathbb{C} \) holomorphic.

**Proof**

1. \( \Rightarrow \) 2. Math 220A, PSet 4

2. \( \Rightarrow \) 1. Define \( \sqrt{f} = \text{Exp} \left( \frac{1}{2} \log f \right) \) for all \( f: U \rightarrow \mathbb{C} \) nowhere zero.

3. \( \Rightarrow \) 2. If \( U = \mathbb{C} \Rightarrow U \) simply connected
Let \( u \neq 0 \) \( \Rightarrow \) let \( f: u \rightarrow \Delta, \ g: \Delta \rightarrow u \) inverse biholomorphisms. If \( \gamma \) is a loop in \( u \), then

\[
\begin{align*}
\text{for loop in } \Delta = \text{simply connected} & \Rightarrow f \circ \gamma \sim 0 \\
\Rightarrow g \circ f \circ \gamma & \sim g(0) \\
& \Rightarrow \gamma \sim g(0) = \gamma \text{ null homotopic.}
\end{align*}
\]

**Question** How do we construct biholomorphism \( f: u \rightarrow \Delta \) explicitly?

**Answer** depends on \( u \).

Some examples worth knowing

\( \Delta \rightarrow \Delta, \ \zeta \rightarrow \frac{(1 + x)}{(1 - x)} \)

We will give more examples next time.
Next: More on boundary behaviour & Schwarz Reflection (Conway IX.1)

After: Runge's Theorem. (Conway VIII.1)
Extra hints added to 1 ii. & 1 iv.

Office Hour: 4 - 5:30 today

1. More examples of biholomorphisms

Example 4

Squaring in \( \mathbb{R}^+ \)

"Half \( \mathbb{R}^+ \rightarrow \mathbb{R}^+ \)"

\[ c(z) = \frac{z - i}{z + i} \]

\[ c: \mathbb{R}^+ \rightarrow \Delta. \]
Example

\[ \Delta^+ = \text{upper half disc (open)} \]

\[ \mathbb{D} \overset{\omega}{\sim} \mathbb{D} \]

\[ z \rightarrow 2 \]

Question

Find \[ \mathbb{D} \]

Answer

Not done by squaring since \( 0 \in \Delta, \) \( 0 \notin \Delta^+ \)

Instead

We use the Cayley transform & work in \( \mathbb{C}^+ \)

Idea

\[ \Delta^+ \overset{c^{-1}}{\rightarrow} \mathbb{C}^+ \]

\[ \text{the map we want} \]

\[ \text{half disc} \]

\[ \text{square} \]

\[ \text{"half" } \mathbb{C}^+ \]
Concretely consider \( c : \mathbb{H}^+ \to \Delta \), the Cayley transform

\[
\mathcal{C} - 1 : \Delta \to \mathbb{H}^+, \quad c^{-1}(z) = i \cdot \frac{1 + z}{1 - z}
\]

Check Under \( c^{-1} \) we map

\[
\begin{align*}
-1 & \rightarrow 0 \\
1 & \rightarrow i \\
0 & \rightarrow \text{i}
\end{align*}
\]

\[
\begin{align*}
i & \rightarrow -1 \\
-\text{i} & \rightarrow 1
\end{align*}
\]
Conclusions

1. Diameter $b$ \rightarrow imaginary axis $\hat{b}$

2. Arc $a$ \rightarrow negative real axis $\hat{a}$

3. Arc $c$ \rightarrow positive real axis $\hat{c}$

4. $\Delta^+$ \rightarrow $L = 2^{nd}$ quadrant (left)

5. $\Delta^-$ \rightarrow $R = 1^{st}$ quadrant (right)
Construction of biholomorphism $\Delta^+ \rightarrow \Delta$
as a composition of three moves:

1. $\Delta^+ \rightarrow \mathcal{L}$, $z \rightarrow \frac{1 + z}{1 - 2}$.
2. $\mathcal{L} \rightarrow \mathcal{R}^*$, $2 \rightarrow -2^2$.
3. $\mathcal{R}^* \rightarrow \Delta$, $C(2) = \frac{2 - i}{2 + i}$.
Conclusion

The biholomorphism $\Delta^+ \rightarrow \Delta$ extends to $\partial \Delta^+ \rightarrow \partial \Delta$ continuously and bijectively.

(the upper arc $a$ is sent to the upper arc $\partial \Delta^+$ and the diameter $b$ is sent to the lower arc $\partial \Delta$).
2. Extension to the boundary

Question: Given $f: U \rightarrow \Delta$ biholomorphism, does it extend $\tilde{f}: \tilde{U} \rightarrow \tilde{\Delta}$ bicontinuously?

Answer: Yes if $U$ bounded and $\partial U =$ simple closed curve.

Caratheodory's theorem

We will not give the proof in this course.
3. **Beyond the boundary**

**Question** Can we extend beyond the boundary?

The easiest instance is provided by

**Schwarz Reflection Principle**

**Conway IX. 1.**

There are several versions but two stand out:

1. Reflection across line segments (book)
2. Reflection across circular arcs (HWKC)

**Applications**

3. Biholomorphic maps between rectangles, annuli:

4. Analytic continuation...
Reflection across segments

Given \( f : u^+ \rightarrow \mathbb{C} \)

\( u^+ = u \cap \mathbb{D}^+ \)

\( u^- = u \cap \mathbb{D}^- \)

\( u^0 = u \cap \mathbb{R} = (a, b) \)

4. Reflect across segments

Open, \( U \subseteq \mathbb{C} \) symmetric \( z \rightarrow \overline{z} \), \( \forall z \in U \rightarrow \overline{z} \in U \).
Theorem: The function \( F: U \rightarrow \mathbb{C} \) is a holomorphic extension of \( f \) beyond the boundary.

Remarks

Visualization
The condition

\[ f(\omega_0) \in \mathbb{R} \]

ensures we reflect across real axis on both sides.

More generally, we can reflect across arbitrary lines.

This can be deduced via rotations.
Using the Cayley transform

\[ C : \Delta \to \mathbb{D}^+ \]

we can also reflect across arcs in the unit disc. (HWK c).
Last time

Given \( f : \mathbb{C} \rightarrow \mathbb{C} \) holomorphic in \( \mathbb{U}^+ \), \( \mathbb{U}^- \) extends continuously to \( \mathbb{U}^0 \) such that the values \( f(\mathbb{U}^0) \subseteq \mathbb{R} \).

Define

\[
F(z) = \begin{cases} 
  f(z) & \text{if } z \in \mathbb{U}^+ \\
  f(z) & \text{if } z \in \mathbb{U}^0 \\
  f(\bar{z}) & \text{if } z \in \mathbb{U}^- 
\end{cases}
\]
**Theorem**  The function $F: U \to \mathbb{C}$ is a holomorphic extension of $f$ beyond the boundary.

**Visualization**
Proof of Schwarz

I. \( F \) continuous

II. \( F \) holomorphic in \( U^+ \)

III. \( F \) holomorphic in \( U^- \)

IV. \( F \) holomorphic at points of \( U^0 \).

Proof of IV

Let \( z_0 \in U_0 \Rightarrow z_0 = \overline{z_0} \).

We show \( \lim_{z \to z_0} F(z) = \lim_{z \to \overline{z}_0} F(z) \).

\[
\begin{align*}
&\quad \lim_{z \to z_0} F(z) = \lim_{z \to \overline{z}_0} F(z) \\
\iff &\quad \lim_{z \to z_0} f(z) = \lim_{z \to \overline{z}_0} f(z) \\
\iff &\quad f(z_0) = f(\overline{z}_0)
\end{align*}
\]

which holds since \( z_0 = \overline{z}_0 \) and \( f(z_0) = f(\overline{z}_0) \).
Proof of \textit{iii.} We show $F$ holomorphic in $\mathbb{U}^-$. \\
Let $c^- \in \mathbb{U}^-$. Let $c^+ = \overline{c^-} \in \mathbb{U}^+$. Since $f$ is holomorphic at $c^+ \Rightarrow \exists \Delta(c^+, r) \subseteq \mathbb{U}^+$. Taylor expand in $\Delta(c^+, r)$:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - c^+)^k, \text{ radius of convergence } 2r.$$ \\
Let $z \in \Delta(c^-, r) = \Delta(c^+, r)$. Then

$$F(z) = \overline{f(\overline{z})} = \sum_{k=0}^{\infty} a_k (\overline{z} - c^+)^k$$

$$= \sum_{k=0}^{\infty} a_k (\overline{z} - c^+)^k$$

$$= \sum_{k=0}^{\infty} a_k (z - c^-)^k, \text{ radius of convergence } 2r.$$ \\
\Rightarrow $F$ holomorphic in $\mathbb{U}^-$. 
We show $F$ is holomorphic.

By Math 220A, Lecture 5.

Goal: $F = G'$ for some holomorphic $G$ in $\Delta$.

\[ \int_{\partial \Delta} F = 0 \]

or

\[ \int_{\partial \Delta} G = 0 \]

or

$F \text{ holomorphic in } \Delta$.

This will complete the proof.

Proof of $F$ is holomorphic.

We show $F$ is holomorphic.

By Math 220A, Lecture 5.

Goal: $F = G'$ for some holomorphic $G$ in $\Delta$.

\[ \int_{\partial \Delta} F = 0 \]

This will complete the proof.

Proof of $F$ is holomorphic.

We show $F$ is holomorphic.

By Math 220A, Lecture 5.

Goal: $F = G'$ for some holomorphic $G$ in $\Delta$.

\[ \int_{\partial \Delta} F = 0 \]

This will complete the proof.

Proof of $F$ is holomorphic.
If $\bar{R} \subseteq U^+$ or $\bar{R} \subseteq U^-$ this is clear (Goursat / Cauchy).

Assume $\bar{R}$ intersects the real axis. We assume that $\bar{R}$ is not a side of $R$. Otherwise the argument is simpler.

We may assume $\delta < \varepsilon$.

We show $\exists K > 0$ such that for all $\varepsilon > 0$,

$$\left| \int_{\partial R} F \, dz \right| < K \varepsilon \Rightarrow \int_{\partial R} F \, dz = 0.$$

1. If continuous in $\overline{\Delta} = \{ F(z) \mid z \in \Delta \}$ for all $\zeta \in \overline{\Delta}$.

11. If uniformly continuous in $\overline{\Delta} = \text{compact}$.

$$\Rightarrow \exists \delta \exists \varepsilon, \ |x - y| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon.$$
Construct $R^+, R^-, R^0$ where $R^+ \subseteq U^+$, $R^- \subseteq U^-$.

$$R^0 = [\alpha, \beta] \times \left[-\frac{\delta}{2}, \frac{\delta}{2}\right].$$

$$\int_{\partial R^+} F \, d\mathbf{z} = 0, \quad \int_{\partial R^-} F \, d\mathbf{z} = 0 \text{ by Goursat.}$$

$$\Rightarrow \int_{\partial R} F \, d\mathbf{z} = \int_{\partial R^0} F \, d\mathbf{z}.$$ 

Estimates:

Sides of $R^0$: $s_1, s_2, s_3, s_4.$
\[
\begin{align*}
(1) \quad & \left| \int_{S_4} F \, dx + \int_{S_4} F \, dy \right| \leq \left| \int_{S_4} F \, dx \right| + \left| \int_{S_4} F \, dy \right| \\
& \leq M \cdot \text{length } S_4 + M \cdot \text{length } S_4 \\
& = 2MS < 2M \varepsilon. \\
\end{align*}
\]

\[
\begin{align*}
(2) \quad & \left| \int_{S_4} F \, dx + \int_{S_4} F \, dy \right| \leq \int_{\Omega} \left| F \left( t - \frac{i \delta}{2} \right) - F \left( t + \frac{i \delta}{2} \right) \right| \, dt \\
& < \varepsilon \quad \text{(uniform continuity)}.
\end{align*}
\]

\[
\begin{align*}
(1) + (2) \quad & \Rightarrow \left| \int_{\mathbb{R}^2} F \, dx \right| \leq \left| \int_{S_4} F \, dx + \int_{S_4} F \, dy \right| + \left| \int_{S_4} F \, dx + \int_{S_4} F \, dy \right| \\
& \leq 2M \varepsilon + \varepsilon \cdot \text{diam } (\Delta) = K \varepsilon.
\end{align*}
\]

This completes the proof.
2. **Application**

**Conformal maps of rectangles**

**Example**

A biholomorphic map $f: \mathbb{R} \to \mathbb{R}$ such that

- $f$ extends continuously and bijectively to the boundary.
- Sending corners to corners and edges to edges.

*If and only if*

$$\frac{a'}{a} = \pm \frac{b'}{b} \quad \text{or} \quad aa' = \pm bb'.$$
Remark  Condition (5) is automatic by Caratheodory, while condition (6) is really need.

We first assume

\[ f: S_1 \rightarrow S'_1, \quad 0 \rightarrow 0, \quad a \rightarrow a'. \]

\[ f(a) = 0, \quad f(a) = a' \]

- \( S_4 \) is sent to a side containing \( f(0) = 0 \), hence \( S'_4 \)
- \( S_2 \) is sent to a side containing \( f(a) = a' \), hence \( S'_2 \)
- \( S_3 \) is sent to the remaining side \( S_3 \)
We use Schwarz Reflection along \( s_1 \) & \( s_1' \).

Note

\[
S_4 \rightarrow S_4', \quad S_2 \rightarrow S_2', \quad S_3 \rightarrow S_3'.
\]

from the explicit formula for the extension

The extension is still bijective, (as the picture shows).
Reflect the new rectangle one more time, across $s_3$ & $s_3'$. 

and continue until we get two strips mapping to each other so that their boundaries are mapped respectively.

Now reflect the strips across their sides.
In the end, we obtain \( f: \mathbb{C} \to \mathbb{C} \) bijective & holomorphic.

We saw in Math 220A, PSets that \( f(z) = az + b \).

Since \( f(a) = 0 \Rightarrow b = 0 \Rightarrow f(z) = az \).

\[
\begin{align*}
f(a) &= a' \Rightarrow a = a' \\
f(b) &= b' \Rightarrow ab = b'
\end{align*}
\]

\[
\Rightarrow \frac{a'}{a} = \frac{b'}{b}.
\]

The remaining cases are part of Homework 6.
Part I: Weierstraß & Mittag-Leffler Series & Products

Part II: Riemann & Schwarz Mapping Theory

Part III: Runge Approximation Theory

< Conway VIII. 1.
§1. Context for Runge

In real analysis (Math 140B), we learn

**Weierstrass Approximation Theorem**

\[ f : [a, b] \rightarrow \mathbb{R} \text{ continuous, } \exists P_n \text{ polynomials} \]

\[ P_n \rightarrow f. \]

This was proven by Weierstrass at age 70 in 1885.

There are many applications of this theorem.

e.g. in Fourier analysis, functional analysis etc.

**Remark**

This can be generalized in \( \mathbb{R}^n \).

If \( K \subseteq \mathbb{R}^n \) compact, \( f : K \rightarrow \mathbb{R} \text{ continuous, then} \)

\[ \exists P_n \text{ polynomials, } P_n \rightarrow f \text{ in } K. \]
Runge (age 29, Ph. D. 1850, student of Weierstrass):

**Question** What about $f$ holomorphic? Can it be approximated by polynomials in $z$?

**Answer** was given in 1855 as well.

**Remark** This doesn’t follow from Weierstrass. Weierstrass produces polynomials in $x, y$ for $z = x + iy$.

E.g. polynomials in $ar{z}$ and $z$. 
Zur Theorie der eindeutigen analytischen Functionen

Von

C. Runge

in Berlin.

Seit dem Bekanntwerden der Modulfunctionen, weise man, dass der Gültigkeitsbereich einer analytischen Funktion nicht notwendig von diskreten Punkten begrenzt zu sein braucht, sondern dass auch kontinuierliche Linien als Begrenzungspunkte auftreten können und einen Teil der komplexen Ebene von dem Gültigkeitsbereich ausschliessen können.

Hier entspricht nun die Frage, ob der Gültigkeitsbereich analytischer Funktionen nicht in irgendwelchen Beschränkungen unterliegt oder nicht. Diese Frage bildet, so weit sie sich auf eindeutige analytische Funktionen bezieht, den Gegenstand der nachfolgenden Untersuchung. Es wird sich ergeben, dass der Gültigkeitsbereich einer eindeutigen analytischen Funktion d. h. die Gesamtheit aller Stellen an denen sie sich regulär oder ausnahmslos singular verhält keiner anderen Beschränkung unterliegt als derjenigen, zusammenhängend zu sein. In dem ersten Theile


Der Herausgeber.

Acta Mathematica 6 (1885)

Carl Runge (1856 - 1927)

- Runge - Kutta

- Runge's Approximation

- mathematics, astrophysics, spectroscopy.
2. Phrasing the Question more carefully

Beware: A holomorphic function is defined over open sets. (See Math 220A).

Definition: \( K \subseteq \mathbb{C} \) compact. A holomorphic function in \( K \) is a function \( f : K \to \mathbb{C} \) that extends holomorphically to a neighborhood \( U \supseteq K \).
Two versions of the question

**Runge C** (compact sets) \( K \subseteq \mathbb{C} \) compact

Given \( f \) holomorphic in \( K \), are there polynomials \( P_n \) such that \( P_n \rightarrow f \) in \( K \)?

**Runge O** (open sets) \( U \subseteq \mathbb{C} \) open

Given \( f \) holomorphic in \( U \), are there polynomials \( P_n \) such that \( P_n \rightarrow f \) in \( U \)?
**Emphasis**

Runge C: approximation on a single compact $K$

Runge O: approximation on all compacts $K$ in the domain of a holomorphic function

Runge C is more basic.

\[ \text{complex analysis} \]

\[ \text{Runge C} \quad \Rightarrow \quad \text{Runge O} \]

point-set topology.

The two versions are very similar.
Example: Runge C.

\[ K = \{ 1 \leq 12 \leq 2 \} \], \quad f(z) = \frac{1}{z}. \text{ holomorphic in } K.

Can we find \( P_n \rightarrow f \) in \( K \)?

\[ |z| = \frac{3}{2} \]

\[ |z| = 2 \]

\[ |z| = \frac{3}{2} \]

\[ |z| = 1 \]

\[ \int_{|z| = \frac{3}{2}} P_n \, dz = 0 \quad \text{and} \quad \int_{|z| = \frac{3}{2}} f \, dz = 2 \pi i \quad \text{by the residue theorem.} \]

The failure is due to the "hole" in \( K \).
What is a "hole"?

**Definition** \( K \subseteq \mathbb{R} \), compact

A **hole** is a bounded connected component of \( \mathbb{R} \setminus K \).

**Example**

\[ K \quad \mathbb{R} - K = A \cup B \cup C \]

\( \mathbb{R} \) unbounded

\( A, B \) bounded

\( A, B \) are holes for \( K \).

\[ K = \bigcup \left\{ \frac{1}{n+1} \leq |z| \leq \frac{1}{n} \right\} \cup \{0\} \]

\( K \) closed & bounded \( \Rightarrow \)

\( \Rightarrow K \) compact.

\( \infty - \text{many holes} \)

\[ H_n = \left\{ \frac{1}{n+1} \leq |z| < \frac{1}{n} \right\} \]
§ 3. Runge's Theorem - Compact Sets

We give three versions. The simplest version is:

**Runge's Little Theorem (Case c)**

If \( K \) has no holes \((\iff \partial \setminus K \text{ connected})\)

then \( \forall f \text{ holomorphic in } K, \exists \text{ polynomials } P_n \text{ with} \)

\[ P_n \sim f \text{ in } K. \]
**Question**

How about arbitrary $K$?

**Answer**

Polynomial approximation fails (Example).

Are we even asking the right question?

**Better**

Rational Approximation.

---

**Question**

Given $f$ holomorphic in $K$,

exists $R_n$ rational functions, $R_n \rightarrow f$ in $K$ & the poles of $R_n$ are outside $K$?

**Question**

Can we prescribe the location of the poles of $R_n$?
Runge C (Almost final) \( K \subseteq \mathbb{C} \) compact.

**Thm** Let \( S \) be a set of points, at least one from each hole of \( K \).

Then \( \forall f \) holomorphic in \( K \).

\[ \exists R_n \rightarrow f \text{ in } K \]

\[ R_n \text{ are rational functions whose poles are in } S. \]

**Remark** The poles of \( R_n \) are contained in \( S \), but it may happen that not all points of \( S \) are poles.

**Remark** If \( K \) has no holes then \( S = \bar{K}. \) Thus \( R_n \) has no poles \( \Rightarrow \) \( R_n \) have no denominators \( \Rightarrow \)

\[ \Rightarrow R_n \text{ are polynomials. We recover Little Runge.} \]
We replace \( \mathbb{C} \) by \( \widehat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \).

**Theorem** Let \( K \subseteq \mathbb{C} \) compact. Let \( S \subseteq \widehat{\mathbb{C}} \) be a set of points, at least one chosen from each component of \( \widehat{\mathbb{C}} \setminus K \).

Let \( f \) be holomorphic in \( K \). Then

\[ \exists R_n \overset{\text{in } K}{\longrightarrow} f \] in \( \widehat{\mathbb{C}} \)

\[ R_n \text{ are rational with possible poles in } S. \]
Remark: An interesting case allowed by the Final Version is to pick \( z \in S \) from the unbounded component. Thus, when \( S \) consists in

- \( \infty \) from the unbounded component of \( \mathbb{C} \setminus K \)
- a point from each bounded component of \( \mathbb{C} \setminus K \) (holes)

we recover the Almost Final Version.

The two versions are even equivalent in this case since the condition that a rational function \( R \) have at worst a pole at \( \infty \) is vacuous. Indeed,

\[
R(z) = \frac{\prod_{i=1}^{n} (z-a_i)}{\prod_{i}^{m} (z-b_i)} \Rightarrow R(\frac{1}{2}) = 2^{m-n} \frac{\prod_{i=1}^{n} (1-a_i, 2)}{\prod_{i=1}^{m} (1-b_i, 2)}
\]

has at worst a pole at 0.
Summary

Runge C (Final) \(\Rightarrow\) Runge C (Almost Final)

Conway VIII.1.7

- rational approximation
- version for \(\mathbb{C}\)

Little Runge C

- polynomial approximation
- \(K\) has no holes
Example / Review

\[ f(z) = \frac{z^3}{(z-2)(z-7)} \]

\[ K = \{ 3 \leq |z| \leq 4 \} \]

\( f \) is holomorphic in \( K \) because it extends holomorphically to

\[ U = \left\{ \frac{5}{2} < |z| < \frac{9}{2} \right\} \supseteq K \]

Can we approximate \( f \) uniformly on \( K \) by:

1. Rational functions with poles in \( \mathbb{C} \) at 1?
   - YES Almost Final Version. Poles in \( \mathbb{C} \) are 1, \( \infty \).

2. Rational functions with poles at 0, \( \infty \)
   - YES Final Version

3. Rational functions with poles at \( \infty \)?
   - NO. Such rational functions would have to be
polynomials (if they had denominators, there would be poles). But if \( P_n \rightarrow f \) then

\[
\int P_n \, dz - \int f \, dz = 2\pi i \cdot \text{Res}(f, z) 
\]

\[
|z| = \frac{7}{2} \quad |z| = \frac{3}{2}
\]

\[
\int 0 = 2\pi i \cdot \frac{2^3}{2 - 3/2} \neq 0
\]

using the Residue theorem. Contradiction!
Math 220 B — Lecture 13

March 3, 2021
Math 220C Survey

- first half: MWF 3-3:50, Live
- second half: TBD

Remaining Topics in Math 220B

- Proof of Runge C (today, Friday)
- Runge 0 (Monday)
- Summary & loose ends (Wednesday)
- Review (Friday)
Runge C (Final) => Runge C (Almost Final)

Conway VIII.1.7

- rational approximation
- version for $C$

Little Runge C

- polynomial approximation
- $K$ has no holes
Thm Let $K \subseteq \mathbb{C}$ be a compact set, let $S \subseteq \mathbb{C}$ be a set of points, at least one chosen from each component of $\mathbb{C} \setminus K$.

Let $f$ be holomorphic in $K$. Then

\[ \exists \quad R_n \to f \quad \text{in} \quad K \]

\[ \text{in} \quad \hat{\mathbb{C}} \]

\[ R_n \text{ are rational with possible poles in } S. \]

Strategy

Step 1 Touchy Integral Formula for compact sets.

Step 2 Approximation without prescribed poles.

Step 3 Push the poles to prescribed location.
Step 1

Recall (Math 220A).

If $f$ is holomorphic in $U$, $\overline{R} \subseteq U$, then

$$\frac{1}{2\pi i} \int_{\partial R} \frac{f(z)}{z-a} \, dz = \begin{cases} f(a), & a \in \mathbb{R} \\ 0 & a \notin \mathbb{R} \end{cases}$$

We wish to do the same for any compact $K \subseteq U$. 
Lemma 8.1.1.

Let $K \subseteq \mathcal{U}$ be compact. There exist segments $\Gamma_j$ such that

$$\Gamma = \Gamma_1 + \ldots + \Gamma_n \subseteq \mathcal{U} \setminus K$$

and such that for all functions $f$ holomorphic in $\mathcal{U}$

$$f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a} \, dz, \quad \forall a \in K.$$
We will construct $\mathcal{C}$ as a union of closed polygons.

Remark: If $K$ has a simple structure this is not so bad. We'd need

$$n(\mathcal{C}, a) = 1 \quad \forall \quad a \in K.$$

and argue using Cauchy’s formula from Math 220A.

The issue is if $K$ has complicated (fractal) structure.

Idea: Lay a grid!
Proof

(1) Construction

\[ WLOG \ u \neq \emptyset = \emptyset \cup u \neq \emptyset \] is closed. Note

\[ k \cap (\emptyset \cup u) = \emptyset. \] Let \( d = d \ (k, \emptyset \cup u) > 0. \)

\[ \text{Lay a grid of squares of side } < \frac{d}{\sqrt{2}}. \]
Consider the closed squares $Q_1$, $Q_2$, ..., $Q_m$ that intersect $K$.

Then there are only finitely many squares since $K$ is compact.
Claim 1 \[ K \subseteq \bigcup_{j=1}^{m} Q_j \subseteq U. \]

Proof. If \( k \in K \) then \( k \) is contained in a square of the grid. This square intersects \( K \) at \( k \) so it must be one of the \( Q_j \) and \( k \in Q_j \). This gives the first inclusion.

For the second inclusion, let \( g \in Q_j \) where \( Q_j \cap K \neq \emptyset \). Let \( h \in Q_j \cap K \). If \( g \notin U \Rightarrow \)
\( \Rightarrow g \in \mathcal{S} \setminus U \) and \( h \in K \) so
\[ d(g, h) = d(\mathcal{S} \setminus U, K) = d. \]

But \( g, h \in Q_j \Rightarrow d(g, h) < \text{diam}(Q_j) = d \) contradiction!

Thus \( g \in U \), as needed.

Construction of \( \mathcal{T} \)

- \( \mathcal{T}_1, \ldots, \mathcal{T}_n \) sides of \( Q_1, \ldots, Q_m \) which are not shared by two squares \( Q_i, Q_j \), \( 1 \leq i \neq j \leq m \).
Claim 2: \( \bigcup_{j=1}^{n} v_j \subseteq u \setminus K \).

**Proof**

Note \( v_j \subseteq u \) by Claim 1. Assume \( v_j \cap K \neq \emptyset \).

Let \( k \in v_j \cap K \). Then \( v_j \) is a side of two squares. These squares must intersect \( K \) necessarily since \( v_j \) does.

These squares must be some of the \( Q_x, Q_y \)'s, contradicting the definition of \( v_j \).
Claim 3 \( \forall a \in U \setminus \bigcup_{j=1}^{m} Q_j \), then

\[
\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{Q_j} \frac{f(z)}{z-a} \, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-a} \, dz.
\]

This follows because the common sides of the \( Q_j \)'s cancel out, leaving only the integral over \( \gamma_j \)'s.

Assume \( a \in \text{Int} \, Q_e \). By Cauchy for rectangles

\[
\frac{1}{2\pi i} \int_{Q_j} \frac{f(z)}{z-a} \, dz = f(a) \quad \text{if} \ j = e
\]

and 0 otherwise.

\( \Rightarrow \ \forall a \in \text{Int} \, Q_e \),

\[
f(a) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(z)}{z-a} \, dz \quad (\star)
\]

This is almost the lemma. We have one more step.
\[ f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(z)}{z-a} \, dz \quad (**). \]

**Proof** The only issue is the case when \( a \notin \text{Int } Q_e \),

\[ \Rightarrow \ a \text{ must be on a side of some } Q_j \text{ b/c } K \subseteq \bigcup_{j=1}^{m} Q_j \],

by Claim 1. By Claim 2, \( a \notin \gamma_j \).

Find an \( \rightarrow a \) with \( a_n \) in the interior of the squares \( Q_s \). Both sides of (**4) agree at \( a_n \) by (*).

\[ f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(z)}{z-a} \, dz \]

Both sides are continuous in \( a \). This is clear for LHS & RHS is explained below. Make \( n \to \infty \) to conclude

\[ f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} \frac{f(z)}{z-a} \, dz \],

proving the Lemma completely.
Continuity of RHS is a consequence of:

**Key Fact** (Math 220A, Homework 3, Problem 3).

\[
\Phi : \mathbb{T} \times \mathbb{U} \setminus \mathbb{T}_j \rightarrow \mathbb{C} \text{ continuous}
\]

then \( a \rightarrow \int_{\mathbb{T}} \Phi (\mathbb{t}, a) \, d\mathbb{t} \) is continuous.

Apply to \( \Phi : \mathbb{T}_j \times \mathbb{U} \setminus \mathbb{T}_j \rightarrow \mathbb{C} \)

\[
\Phi (\mathbb{t}, a) = \frac{f(a)}{\mathbb{t} - a}, \quad \mathbb{t} \in \mathbb{T}_j, \ a \in \mathbb{U} \setminus \mathbb{T}_j
\]

to conclude.

Step 1 is now established. Steps 2 & 3 next time.
Where are we?

K compact, K ⊂ U, \( f : U \to \mathbb{C} \) holomorphic

With \( \forall \varepsilon > 0 \) rational functions with prescribed poles

\[ \left| f - R \right| < \varepsilon \quad \text{in } K \]

in a suitable set \( S \).

\[ \text{Step 1} \]
We found segments \( \Gamma_1, \ldots, \Gamma_n \subset U \setminus K \)

\[ f(z) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(w)}{w - z} \, dw \quad \forall z \in K. \]

\[ \text{Step 2} \]
Find rational functions \( R \) with

\[ \left| f - R \right| < \varepsilon \quad \text{in } K, \text{ poles of } R \text{ are on the segments } \Gamma_j. \]

\[ \text{Step 3} \]
Push the poles to prescribed locations.

Conway VIII. 1.1

Conway VIII. 1.5

Conway 1.6 - 1.13.
Visualization of the strategy

\[ S = \{ \ast, \ast \} \]

prescribed poles

Steps will push poles to \( S \)

Step 2 will produce poles on \( \overline{T_j} \).

For step 2 we argue one segment \( \overline{T_j} \) at a time showing

\[ F_j(z) = \frac{1}{2\pi i} \int_{\overline{T_j}} \frac{f(w)}{w-z} \, dw \]

can be approximated by rational functions, with poles in \( \overline{T_j} \).
Proof of Step 2

- $K$ compact, $\pi$ segment (compact), $\pi \cap K = \varnothing$
- $f$ continuous in $K$

Main Claim (Conway VIII. 1.5)

\[ F(\omega) = \int_{\pi} \frac{f(\omega)}{\omega - 2} \, d\omega \text{ can be approximated uniformly on } K \text{ by rational functions with poles in } \pi. \]
Proof \ Let \ \varphi(w,z) = \frac{f(w)}{w-z} \ : \ \pi \times K \to \mathcal{A}, \ w \in \pi, \ z \in K.

Since \ \pi \cap K = \emptyset \ \Rightarrow \ \varphi \ \text{is continuous hence uniformly cont.}

\Rightarrow \ \exists \ \delta > 0 \ such \ that

\|w - w'\| < \delta \ \Rightarrow \ \|\varphi(w,z) - \varphi(w',z)\| < \varepsilon.

- Subdivide \ \pi \ \text{into subsegments} \ \pi_1, \ldots, \pi_c \ \text{of length} \ < \delta.

- Pick \ \{p_k \in \pi_k\}

- Let \ \mathcal{C}_k = \int_{\pi_k} f(p_k) \ \frac{1}{w-p_k} \ \text{rational function with pole at} \ p_k \ \in \pi.
Claim

\[ \left| F(2) - R(2) \right| = \left| \int_{\pi} \frac{f(w)}{w - \epsilon} \, dw - \sum_{k=0}^{t} \frac{f(p_k)}{p_k - \epsilon} \int_{\pi_k} dw \right| \]

\[ = \left| \sum_{k=0}^{t} \int_{\pi_k} \left( \frac{f(w)}{w - \epsilon} - \frac{f(p_k)}{p_k - \epsilon} \right) \, dw \right| \]

\[ \leq \sum_{k=0}^{t} \left| \int_{\pi_k} \varphi(w, 2) - \varphi(p_k, 2) \, dw \right| \]

\[ \leq \sum_{k=0}^{t} \varepsilon \cdot \text{length}(\pi_k) = \varepsilon \cdot \text{length}(\pi). \]

Here we used \( \left| \varphi(w, 2) - \varphi(p_k, 2) \right| < \varepsilon \) since \( \left| w - p_k \right| < \delta \) which is true as \( p_k \in \pi_k \), \( \text{length}(\pi_k) < \delta \).

The proof of Step 2 is completed.
Where are we?

1. $K \in U$, $f$ holomorphic
2. $\exists R$ with poles in $U_j$, $|f-R| < \varepsilon$ in $K$.

**Final Step** Fix $S$ a set of poles, one from each component of $\hat{a} \setminus K$.

Push the poles from $U_j$ to the points of $S$. 
Step 3 Pole pushing to prescribed location.

Let \( T \setminus K = \bigcup H_i \) = connected components.

Let \( H \) be a fixed component.

Pole produced in Step 2

prescribed location

Lemma \( \forall a, b \in H. \) Then

\[
\frac{1}{t^2 - a} \text{ can be approximated uniformly in } K \text{ by polynomials in } \frac{1}{t^2 - b}
\]

If \( H \) is unbounded \( a, b = \infty \) then

\[
\frac{1}{t^2 - a} \text{ can be approximated uniformly in } K \text{ by polynomials}
\]

Polynomials in \( t \) = Rational Functions with poles possibly only at \( \infty \).
Proof of the Lemma

Let $b$ be fixed and vary $a$. Consider the set

$$W = \{ c \in H : \frac{1}{2 - c} \text{ can be approximated uniformly in } K \}$$

We wish to prove $W = H$.

$W \neq \emptyset$, because $b \in W$.

Key Claim

($\ast$) For $c \in W$, let $\delta = d(c, K)$. Then $d(c, \delta) \leq W$.

Exercise This implies $W$ is closed and open, hence $W = H$. 

$K$
Proof of Key Claim

Let $s \in \Delta(c; \delta)$. We wish to show that $s \in W \Rightarrow \Delta \in W$ as needed.

Idea: $\frac{1}{2-s}$ is poly in $\frac{1}{2-c}$ is poly in $\frac{1}{2-\delta} \Rightarrow s \in W$.

Consider the Laurent expansion of $\frac{1}{2-s}$ at $c$ in $\Delta(c; \delta, \omega)$

$$
\frac{1}{2-s} = \frac{1}{2-c} \cdot \frac{1}{1 - \frac{s-c}{2-c}} = \frac{1}{2-c} \sum_{k=0}^{\infty} (\frac{s-c}{2-c})^k = \sum_{k=0}^{\infty} (\frac{s-c}{2-c})^k
$$

Convergence: $|2-c| > \delta > |s-c|$.

Note $z \in K$, $\delta = d(c, K) \Rightarrow K \subseteq \Delta(c; \delta, \omega)$. The Laurent expansion in $\Delta(c; \delta, \omega)$ converges locally uniformly. (Math 220 A, Lecture 12).
Pick a Laurent polynomial in \( \frac{1}{2-c} \) from the Laurent expansion above so that

\[
\left| \frac{1}{2-s} - T \right| < \frac{\varepsilon}{2} \quad \text{over } K.
\]

Since \( c \in W \Rightarrow \frac{1}{2-c} \) can be approximated by polynomials in \( \frac{1}{2-b} \). The same is then true about \( T = \text{polynomial in } \frac{1}{2-c} \). Then there exists a polynomial in \( \frac{1}{2-b} \) so that

\[
\left| T - P \right| < \frac{\varepsilon}{2} \quad \text{in } K
\]

Then

\[
\left| \frac{1}{2-s} - P \right| \leq \left| \frac{1}{2-s} - T \right| + \left| T - P \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{in } K.
\]

This shows \( s \in W \).
If \( H \) is unbounded  

Let \( K \subseteq \Delta(0, \rho) \)

- first move the poles to \( |c| > \rho \)
- Taylor expand \( \frac{1}{z-c} \) near \( z = 0 \) in

\[
\Delta(0, |c|) \subseteq \Delta(0, \rho) \subseteq K
\]

The Taylor series converges locally uniformly. Hence we can approximate \( \frac{1}{z-c} \) by polynomials uniformly on \( K \).

---

**Proof of the Exercise**

- \( W \) open. Indeed \( \forall x \in W \exists \Delta(c, \delta) \subseteq W \) by (w) showing \( W \) open

- We show \( W \) closed in \( H \).
Assume $w_n \to w$, $w_n \in W$, $w \in H$.

Let $d(w, k) = \delta$.

Fix $n$ with $d(w, w_n) < \frac{\delta}{2}$.

$\Rightarrow d(w_n, k) \geq d(w, k) - d(w, w_n) > \frac{\delta}{2}$

$\Rightarrow \Delta(w_n, \frac{\delta}{2}) \subseteq W$ since $w_n \in W$ and $*$

$\Rightarrow w \in W$. Since $w \in \Delta(w_n, \frac{\delta}{2})$. This proves the

Exercise.

Remark: This completes the proof of Runge.

Summary: Start with $f$ is Cauchy for compact sets

Skip: rational approximation with poles in $S_k$

Skip: further approximation with prescribed poles
So last time we established Runge's C

**Theorem**

If \( K \subseteq \mathbb{C} \) compact, \( S \subseteq \hat{C} \setminus K \) contains a point from each component of \( \hat{C} \setminus K \).

**Hull**

If \( f \) holomorphic in \( K \)

\[ \Rightarrow \forall \varepsilon \exists f \text{ rational}, \]

\[ |f - \mathcal{R}| < \varepsilon \text{ in } K \text{ and } \text{poles } (\mathcal{R}) \subseteq S. \]

**Remark**

For \( \varepsilon = \frac{1}{n} \Rightarrow \exists R_n \text{ with } \|f - R_n\| < \frac{1}{n} \text{ in } K \)

\[ \Rightarrow R_n \rightarrow f \text{ in } K, \text{ and } \text{poles } (R_n) \subseteq S. \]

The set \( K \) can be disconnected and quite strange.
Applications

- density in spaces of functions
- new proof of Mittag-Leffler Conway viii. 3.
- polynomial convexity Conway viii. 1.
- generalizations: Mergelyan...

Important Special Case - Little Runge C

$K$ has no holes $\Rightarrow \widetilde{\mathbb{C}} \setminus K$ has only one unbounded component $\&$ we can take $\varepsilon = 1/n$.

All $f$ holomorphic in $K$ can be approximated uniformly in $K$ by polynomials.

The set $K$ can be disconnected.
§1. How about the converse?

If \( K \) has no holes \( \Rightarrow \) polynomial approximation holds.

If \( K \) has holes \( \Rightarrow \) polynomial approximation fails in general.

How to see this? Two methods

\( K = \{ z \mid 1 \leq |z| \leq 2 \} \), \( f(z) = \frac{1}{z} \).

If \( p_n \to f \) in \( K \), \( p_n \) polynomials

then \( \int p_n \, dz \to \int f \, dz = \int \frac{dz}{z} \)

\( 12l = \frac{3}{2} \)

\( 12l = \frac{3}{2} \)

\( 12l = \frac{3}{2} \)

Both integrals follow by the residue theorem, for instance.

This contradiction shows \( f \) cannot be approximated uniformly in \( K \) by polynomials \( p_n \).
(New method).

\[
k = \{ 1 \leq |z| \leq 2 \}, \quad f(z) = \frac{1}{z}
\]

Assume \( P_n \Rightarrow f \) in \( K \), \( P_n \) polynomials.

\[
\exists N : |P_N - f| < \frac{1}{4} \quad \text{on } K
\]

\[
\iff \quad |P_N - \frac{1}{z}| < \frac{1}{4} \quad \text{on } K.
\]

\[
\iff \quad |2P_N - 1| < \frac{1}{4} \quad \text{on } K.
\]

\[
\iff \quad 12P_N - 1 < \frac{1}{4} \quad \text{when } |z| = 1.
\]

Define \( g(z) = 1 - 2P_N \Rightarrow g \) entire. Note \( |g(0)| = 1 \) and

\[
|g(z)| < \frac{1}{4} \quad \text{for } |z| = 1.
\]

This contradicts Maximum modulus for \( g \) in \( \overline{\Delta}(0,1) \).
The second method generalizes:

Let \( H \) be a hole of \( K \). Let \( a \in H \), \( f(z) = \frac{1}{2 - a} \)

\[
M = \max_{z \in K} |2 - a| > 0.
\]

If \( P_n \rightarrow f \) in \( K \), find \( N \) such that

\[
\left| P_n(z) - \frac{1}{2 - a} \right| < \frac{1}{2M} \quad \text{in } K
\]

\[
\Rightarrow \left| (2 - a) P_n - 1 \right| < \frac{|2 - a|}{2M} < \frac{1}{2} \quad \text{in } K.
\]

\[
g(z) = 1 - (2 - a) P_n \quad \text{satisfies}
\]

\[
g(0) = 1 \quad \text{& } \left| g(z) \right| < \frac{1}{2} \quad \text{on } \partial H \in K.
\]

This contradicts maximum modulus for \( g \) & the set \( \overline{H} \).

Thus \( f \) cannot be approximated by polynomials.

**Conclusion**: \( K \) has no holes \( \iff \) polynomial approximation holds in \( K \).
§2. Runge for Open Sets \& Conway VIII. 1.15.

- We approximate locally uniformly on open sets.
- The statement is similar to Runge for compact sets.

**Theorem:** \( U \subseteq \mathbb{C} \) possibly disconnected open set.

- \( S \subseteq \mathbb{C} \setminus U \) containing at least a point from each component of \( \mathbb{C} \setminus U \).
- \( f: U \to \mathbb{C} \) holomorphic.

Then \( \exists \ R_n \) rational functions, poles \((R_n) \subseteq S \) and \( \{ R_n \to f \} \) locally uniformly in \( U \).

\[
S = \{ z_1, z_2, z_3 \}
\]
**Important Special Case** (Little Runge 0)

Let $U \subseteq \mathbb{C}$, open, $\mathbb{C} \setminus U$ connected.

Any $f: U \rightarrow \mathbb{C}$ holomorphic can be approximated locally uniformly on $U$ by polynomials.

Indeed, take $S = \{\infty\}$ in Runge 0.

---

**Example** Let $U = \Delta (0, r)$, $f: U \rightarrow \mathbb{C}$ holomorphic.

We can Taylor expand $f$ in the disc. The Taylor polynomials

$$T_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

and

$$T_n \rightrightarrows f \quad (\text{Math 220A}).$$

Little Runge 0 applies to more general sets $U$. 
Proof of Runge Open

Topological Lemma
For \( \mathcal{U} \subseteq \mathbb{C} \) open, we can find \( \mathcal{K}_n \subseteq \mathcal{U} \) compact

\((*)\) \( \mathcal{U} = \bigcup_{n=1}^\infty \mathcal{K}_n \) exhausting compact sets

\[ \mathcal{K}_n \subseteq \text{Int} \mathcal{K}_{n+1} \]

\[ \forall \mathcal{K} \subseteq \mathcal{U} \text{ compact} \Rightarrow \exists n, \mathcal{K} \subseteq \mathcal{K}_n. \]

Each component of \( \mathcal{C} \setminus \mathcal{K}_n \) contains a component of \( \mathcal{C} \setminus \mathcal{U} \).

Remark: \( \mathcal{K}_n \) means holes of \( \mathcal{K}_n \) contain holes of \( \mathcal{U} \).

Good vs. bad

**Bad** (hole of \( \mathcal{K}_n \), but not of \( \mathcal{U} \)).

**Good** hole of \( \mathcal{K}_n \) containing hole of \( \mathcal{U} \).
Let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $s$ contain a point from each component of $\Omega \setminus \overline{U}$. We have

$$U = \bigcup_{n \geq 1} K_n$$

as in the lemma.

The set $s$ contains a point from each component of $\Omega \setminus K_n$.

By Runge’s theorem, applied to $f$ and $K_n$, we find

$$\| f - R_n \|_{\Omega} < \frac{1}{n}$$

in $K_n$, poles $(R_n) \subseteq S$.

We claim $R_n \to f$. Let $K$ be compact in $\Omega$. By Runge’s theorem,

$$K \subseteq K_n$$

for some $n$. For $n \geq N \Rightarrow K \subseteq K_n \subseteq K_n$ by Runge’s theorem.

Thus

$$\| f - R_n \|_{\Omega} < \frac{1}{n}$$

over $K_n$.

Thus $R_n \to f$ in $K_n$, as needed.
Proof of the Topological Lemma

Let $K_n = \{ x \in \mathbb{R}^n : 1 \leq n \leq \frac{1}{n} \}$ and $\bar{K}_n = \{ x \in \mathbb{R}^n : 1 \leq n \leq \frac{1}{n} \}$ be closed.

It is easy to see \textit{II} hold, using the above pictures.

The technical details follow (see also Conway).
\[ K_n = \{ x : 1 \leq x \leq n \text{ and } d(x, a \cup b) < \frac{1}{n} \} \].

**Claim 1**: \( K_n \subseteq U \)

**Proof**: If \( a \in K_n \Rightarrow d(a, (a \cup b)) < \frac{1}{n} \Rightarrow a \notin (a \cup b) = \emptyset \in U \). Thus \( K_n \subseteq U \).

**Claim 2**: \( U = \bigcup_{n=1}^{\infty} K_n \)

**Proof**: If \( a \in U \) then let \( n \) such that \( n \geq 1 \) \& \( d(a, (a \cup b)) < \frac{1}{n} \) which is possible since \( d(a, (a \cup b)) > 0 \). Thus \( a \in K_n \Rightarrow U = \bigcup K_n \subseteq U \).

**Claim 3**: \( K_n \) closed \& bounded \( \Rightarrow K_n \) compact.

**Proof**: \( K_n \) is closed since
\[
\complement K_n = \left\{ x : 1 \leq x \leq n \right\} \cup \left\{ x : \exists b \notin U, \ d(x, b) < \frac{1}{n} \right\}
\]
\[
= \left\{ x : 1 \leq x \leq n \right\} \cup \bigcup_{b \notin U} \Delta (b, \frac{1}{n}) = \text{open}.
\]
Claim 4  \[ K_n \subseteq \text{Int } K_{n+1} \]

Proof  Let \( z \in K_n \). Let \( r < \frac{1}{n} - \frac{1}{n+1} \). Then

\[ \Delta (z, r) \subseteq K_{n+1} \implies z \in \text{Int } K_{n+1} \text{ as needed.} \]

To see \( \Delta (z, r) \subseteq K_{n+1} \), note for \( w \in \Delta (z, r) \)

\[ |w| \leq |z| + |w-z| \leq n + r < n+1 \text{ and} \]

\[ d(w, \text{Int } K_n) \geq d(z, \text{Int } K_n) - d(z, w) \geq \frac{1}{n} - r > \frac{1}{n+1}. \]

\[ \Rightarrow w \in K_{n+1}, \text{ as needed.} \]

Claim 5  Each compact \( K \subseteq U \) is contained in some \( K_n \).

Proof  Let \( K \subseteq U = \bigcup_n K_n \subseteq \bigcup_n \text{Int } K_{n+1} \). Since \( K \) is compact we find a finite subcover by \( \text{Int } K_j \), \( j \leq n \).

\[ \Rightarrow K \subseteq \bigcup_{j=1}^n \text{Int } K_j \subseteq K_n \]

\[ \text{claim 4.} \]
Claim 6: $A = \hat{\mathbb{C}} \backslash K_n, B = \hat{\mathbb{C}} \backslash U \Rightarrow A \cong B$.

$(+)$ Each component of $A$ contains a component of $B$.

**Proof** This is a bit more technical. We will use repeatedly:

**Easy important fact** (by definition)

If $Z \subseteq A$ connected & $Z$ intersects a component $A^o$ of $A$,

$\Rightarrow Z \subseteq A^o$.

**Proof of (+)** Let $A^o$ be a component of $A$. By Claim 3 (proof):

$$A = \left\{ z \in \mathbb{C} : 1217^n \exists \cup \Delta \left( 0, \frac{1}{n} \right) \right\}$$

contains $\infty$.

**Note** $\infty \in A$. If $A^o$ is the component containing $\infty$, let $B^o$ be the component of $B$ containing $\infty \in B$.

**Note**

$$A^o \cap B^o \neq \emptyset \quad (\text{contains } \infty) \quad A \cap B^o \subseteq A \Rightarrow B^o \subseteq A^o.$$ 

This is what we wanted to show.
If \( \infty \notin A^o \), then \( A^o \) cannot be disjoint from all sets \( \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \).

Why else \( A^o \in \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \subseteq A \Rightarrow \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \subseteq A^o \Rightarrow \infty \in A^o \) connected set easy fact.

Thus \( \exists b \in B \) with \( A^o \cap \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \neq \emptyset \). Note \( \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \subseteq A \cap \in \) intersects \( A^o \) \( \Rightarrow \Delta \left( \frac{1}{n}, \frac{1}{n^2} \right) \subseteq A^o \) easy fact.

Let \( b \in B^o \) for some component \( B^o \).

Then \( B^o \cap A^o \neq \emptyset \) \( \Delta \) \( B^o \subseteq B \subseteq \cap A \Rightarrow B^o \subseteq A^o \) as needed.
Putting the pieces together

We tie up loose ends from Math 220A & B

Common theme: simply connected regions.

Topology ↔ Analysis

Review of Lecture 15, 220A

\[ U \subseteq \mathbb{C} \text{ connected} \]

13. \( U \) is simply connected iff \( \forall \gamma \) closed path in \( U \)

\[
\gamma \sim 0
\]

14. \( \gamma \) piecewise \( C^1 \) loop in \( U \), \( \gamma \sim 0 \) (null homologous) iff

\[
\forall a \notin U, \quad \operatorname{n}(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = 0
\]
Recall

\[ \gamma \sim 0 \Rightarrow \gamma \approx 0 \]

Indeed, \( a \not\in U \).

\[ n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = 0 \] by the homotopy form of Cauchy applied to the holomorphic function

\[ z \mapsto \frac{1}{z-a} \] in \( U \).

However the converse is false \( U = \mathbb{C} \setminus \{a, b\} \)

\[ \gamma \not\approx 0, \gamma \approx 0. \]
Theorem. Let $U \subseteq \mathbb{C}$ open, connected.

1. $U$ simply connected
2. $\forall \gamma$ piecewise $C^1$ loop, $\gamma \approx 0$
3. $\mathbb{C} \setminus U$ connected.
4. Polynomial approximation of $f$ holomorphic in $U$ can be approximated $p_n \to f$ in $U$
5. $\forall \gamma$ piecewise $C^1$ loop, $f$ holomorphic in $U$
   \[ \int f \, dz = 0. \]
6. Primitives: any holomorphic $f : U \to \mathbb{C}$ admits a primitive.
7. Logarithms: $\forall f : U \to \mathbb{C}$ holomorphic, nowhere zero can be written $f = e^g$, $g : U \to \mathbb{C}$ holomorphic.
8. Roots: $\forall f : U \to \mathbb{C}$ holomorphic, nowhere zero can be written $f = z^2$, $r : U \to \mathbb{C}$ holomorphic.

$U$ is homeomorphic to $\Delta(0,1)$. 

Conway VIII.2.
Recall \( u, v \subseteq \mathbb{R} \) are homeomorphic if \( \exists f : u \rightarrow v \) \( g : v \rightarrow u \) continuous and inverse to each other.

**Proof**

\[ [a] \Rightarrow [b] \text{ This is the statement } u \sim 0 \Rightarrow v \sim 0. \]
that we saw previously.

\[ [b] \Rightarrow [c] \text{ Assume } \hat{\mathbb{R}} \setminus u = A \cup B \]

\( A, B \neq \emptyset \) closed and disjoint. Assume \( x \in B. \Rightarrow \]

\( \Rightarrow A \text{ is closed in } \hat{\mathbb{R}} \setminus u \Rightarrow A \text{ closed in } \hat{\mathbb{R}} \Rightarrow A \text{ compact.} \)

\( \hat{\mathbb{R}} \text{ is } \text{closed} \)

Let \( V = U \cup \hat{\mathbb{R}} \setminus B. \Rightarrow V \text{ open subset of } \hat{\mathbb{R}}, A \subseteq V. \)
In Lecture 23, we saw Cauchy's formula for compact sets.

\[ f(a) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\Gamma_j} \frac{f(z)}{z-a} \, dz \quad \forall a \in A, \ f \text{ holom. in } V. \]

Take \( f = 1 \), then

\[ 1 = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_j} \frac{dz}{z-a} = \sum_{j=1}^{n} n(\Gamma_j, a). \]

However, by assumption \( n(\Gamma_j, a) = 0 \) \( \forall j \) since \( \Gamma_j \) is a piecewise \( C^1 \) loop in \( U \) and \( a \notin A \Rightarrow a \notin U \). This contradicts

\[ \sum_{j=1}^{n} n(\Gamma_j, a) = 1. \]

\[ \square \Rightarrow \square \quad \text{This is Little Runge 0.} \]
If \( p_n \to f \) in \( u \) then \( \int p_n \, d\lambda \to \int f \, d\lambda \).

However \( p_n \) admits a primitive \( p_n = g_n' \) so by Lecture 5, Math 220A
\[
\int p_n \, d\lambda = \int g_n' \, d\lambda = 0
\]
\[
\Rightarrow \int f \, d\lambda = 0.
\]

\( \Rightarrow \Rightarrow \) This was done in Lecture 5, Math 220A

\( \Rightarrow \Rightarrow \) Math 220A, Homework 4. Recall the argument.

Consider \( \frac{f'}{f} \) holomorphic in \( u \). Then \( \frac{f'}{f} = g' \) for some \( g \) by (14)
\[
\Rightarrow (e^{-g} f)' = 0 \Rightarrow f = c \; e^g = e^{\tilde{g}}, \; \tilde{g} = g + \log c, \; c \neq 0.
\]

\( \Rightarrow \Rightarrow \) Write \( f = e^{\tilde{g}} \) and let \( \tilde{z} = e^{\frac{3}{2}}. \)
\textbf{R1} \Rightarrow \textbf{R2} \quad \text{If } u \neq \infty, \textit{Riemann Mapping} shows \, u \text{ and } \Delta \text{ are biholomorphic hence homeomorphic.}

\text{If } u = \infty \text{ then } \Delta \to \frac{2}{\sqrt{1+|z|^2}} \text{ is a homeomorphism between } \infty \text{ and } \Delta.

\hline

\textbf{R3} \Rightarrow \textbf{R4} \quad \text{Let } f, g \text{ be the two inverse homeomorphisms } u \xrightarrow{f} \Delta \xrightarrow{g} u.

\text{Let } \gamma \text{ be a loop in } u \Rightarrow \gamma \sim 0 \Rightarrow g \circ \gamma \sim g(0). \Rightarrow \gamma \sim g(0).

\Rightarrow \text{ } u \text{ simply connected.}

\hline

\textbf{Remark} \quad \text{The implications } a \Rightarrow b, c, d, e \ldots \text{ are very useful.}

\text{For the converse, } c \Rightarrow a \text{ is important.}

\textbf{Remark}

\textbf{Topology} : \quad a, c, d, \ldots

\textbf{Analysis} : \quad d, e, f, g, \ldots
Summary of Math 220A - B

Topology

Elliptic Functions

Number Theory

Cubic curves

Algebraic Geometry

Conformal Geometry

Riemann Mapping

Milnor Zeta

Elliptic Functions

cubic curves, de function

Algebraic Properties of Hol (z)

CR Geometry

Potential Theory

Harmonic Functions

Hyperbolic Geometry

Schwarz- Pick

Möbius, Range

Poincaré, Pólya identities

Probability

Statistics

Lie Theory

Homogeneous spaces

Functional Analysis

Combinatorics
Topics for Math 220C

(1) Harmonic Functions — Conway

(2) Hadamard Factorization — Conway

(3) Picard’s Theorems — Conway

(4) Introduction to Riemann Surfaces.
Logistics

(1) **Office Hours:** *Today 4 - 5:30 PM*

(2) **Homework 7** due *Friday, 11:59 PM.*

*No Sunday afternoon extensions.*

(3) **Final Exam,** Wed *March 17, 3 - 6 PM.*

(4) **Office Hours:**

*Tuesday, March 16, 2 - 4 PM* (Dragos)

*Tuesday, March 16, 4 - 6 PM* (Shubham)

(5) **Practice Problems online**

(6) **Last Lecture – Review.**