Let $u \in \mathbb{C}$ open & connected.

**Definition**: $f: u \to \mathbb{C}$ is complex differentiable (CD) if

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z)$$

exists and is finite.

**Examples**

1. $f, g$ complex differentiable $\Rightarrow f + g, fg$ are also complex differentiable.

2. $1, e, e^2, \ldots, e^n, \ldots$ complex differentiable

$\bar{z}$ is not

\[
\text{CD} = \text{complex differentiable}
\]

\[
\text{RD} = \text{real differentiable}
\]
**Remark**

We have seen the same definition for

\[ f: \mathbb{R} \to \mathbb{R}, \quad u \subseteq \mathbb{R} \text{ open.} \]

The two definitions have very different consequences.

If \( f \) is CD \( \Rightarrow f' \) is CD \( \Rightarrow f'' \) is CD \( \Rightarrow \ldots \)

If \( f \) is RD this fails. Indeed,

\[ f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x^2}, & x \neq 0 \\
  0, & x = 0 
\end{cases} \]

Then

\[ f'(x) = \begin{cases} 
  2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & x \neq 0 \\
  0 & 
\end{cases} \]

is not even continuous.
If $f$ is CD, we will show

$$f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \text{ in some } \Delta (a, r) \subseteq U.$$ 

If $f$ is RD, this fails. Take

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

We have $f$ is $C^\infty$, $f^{(n)}(0) = 0$ in so the Taylor series at $0$ is $0$. Thus $f$ does not equal its Taylor series in any interval $(-r, r)$, $r > 0$.

If $f$ is CD for $u = a$ & $f$ bounded $\Rightarrow f$ constant.

If $f$ is RD, $f(x) = \sin x$ is bounded.

If $f$ is CD and $f = 0$ for $v \subseteq U$ open $\Rightarrow$

$$\Rightarrow f \equiv 0.$$ 

This fails if $f$ is RD.
A more appropriate comparison is with functions of two real variables.

Identify \( \mathbb{C} \cong \mathbb{R}^2 \), \( z = x + iy \) \( \mapsto \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \).

\[ |z| = \sqrt{x^2 + y^2} \]

**Definition** \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is real differentiable (RD) if

\[ \forall 2 \in \mathbb{R}^2 \exists A : \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{\textit{\textbf{R}} - \text{\textbf{linear}}} \]

\[ \lim_{\| h \| \to 0} \frac{f(z + h) - f(z) - Ah}{\| h \|} = 0. \]

We write \( A = Df(z) \).

**Remark** \( f \) is CD \( \Rightarrow \) \( f \) is RD.

Indeed, \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) is multiplication by \( f'(z) \).
Remark: If \( f = u + iv \) is RD then

\[
\begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix}
\] exist and \( A = \begin{bmatrix} u_x & u_y \\
  v_x & v_y \end{bmatrix} \) is the Jacobian.

Indeed, by definition

\[
\lim_{h \to 0} \frac{|f(x+h,y) - f(x,y) - h A \begin{bmatrix} u_x \\
  v_x \end{bmatrix}|}{|h|} = 0
\]

\[
\Rightarrow A \begin{bmatrix} 1 \\
  0 \end{bmatrix} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = u_x + iv_x \Rightarrow \begin{bmatrix} u_x \\
  v_x \end{bmatrix}
\]

Similarly, \( A \begin{bmatrix} 0 \\
  1 \end{bmatrix} = \begin{bmatrix} u_y \\
  v_y \end{bmatrix} \), as needed.

Conversely, if \( u_x, u_y, v_x, v_y \) exist & are continuous \( \Rightarrow f \) is RD.

See Math 140 c or Rudin 9.21.
Lemma \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). \( A \) is \( \mathbb{C} \)-linear. TFAE

\[ \frac{[a]}{[b]} \quad A \text{ is } \mathbb{C} \text{-linear} \]

\[ \frac{[b]}{[c]} \quad A(z) = \alpha^2 \text{ for } \alpha \in \mathbb{C} \]

\[ \frac{[c]}{[a]} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ for } \alpha = a + bi. \]

Proof \( [a] \Rightarrow [b] \)

Take \( \alpha = A(i) \Rightarrow A(z) = i \cdot A(i) = \alpha^2. \)

\[ \frac{[b]}{[c]} \quad A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A(i) = \alpha = \begin{bmatrix} a \\ b \end{bmatrix} \]

\[ A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A(i) = \alpha i = a_i - b \rightarrow \begin{bmatrix} -b \\ a \end{bmatrix} \]

\[ \frac{[c]}{[a]} \quad \text{Let } \alpha = a + bi. \text{ Then } A(z) = \alpha^2 \text{ by the argument above. Thus } A \text{ is } \mathbb{C} \text{-linear.} \]
Remark. By the lemma, TFAE

1) $f$ is CD

2) $f$ is RD & $Df(z)$ is $C$-linear $\forall z \in U$

Remark. (Cauchy–Riemann equations).

If $f$ is CD, $Df(z) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ is $C$-linear

so by the lemma

$u_x = v_y$  \hspace{1cm} (CR equations)

$u_y = -v_x$

Conversely if CR equations hold & $u, v$ are of class $C^1$

then $f = u + iv$ is CD.
Indeed, \( f \) is RD in this case and

\[
Df(z) = \begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix}
\]

is \( C \) linear by the Lemma part \( \square \) & CR equations. Thus \( Df(z) \) is multiplication by \( \alpha = f'(z) \) \& \( f \) is CD.
Harmonic functions

If $u, v$ satisfy CR and of class $C^2$ then

$$u_x = v_y \Rightarrow u_{xx} = v_{yx} \Rightarrow u_{xx} + u_{yy} = 0,$$

$$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$$

Similarly $v_{xx} + v_{yy} = 0$

A function $h$ of class $C^2$ with

$$h_{xx} + h_{yy} = 0$$

is said to be harmonic.

Conclusion

Thus if $f$ is CD and of class $C^2$ then

$$u = \text{Re} f, \quad v = \text{Im} f$$

are harmonic.

Pairs $(u, v)$ arising this way are called harmonic conjugates.
**Notation**

\[
\frac{\partial}{\partial x} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)
\]

\[
\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

**Remark**

\[z = x + iy\]
\[\bar{z} = x - iy\]

\[x = \frac{1}{2} (z + \bar{z})\]
\[y = \frac{1}{2i} (z - \bar{z})\]

Think of \(z, \bar{z}\) as independent variables. Then

\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z}
\]

\[= \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \frac{1}{2i}
\]

\[= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)\]
Lemma

"f depends on \( z \) but not on \( \bar{z} \)"

\[
f \text{ is } \mathbb{C} \Rightarrow \frac{\partial f}{\partial \bar{z}} = 0
\]

Proof

\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( f_x + i f_y \right) \quad \text{by definition}
\]

\[
= \frac{1}{2} \left( u_x + i v_x + i (u_y + i v_y) \right)
\]

\[
= \frac{1}{2} \left( u_x - v_y \right) + i \left( \frac{1}{2} (v_x + u_y) \right) = 0
\]

\[
\iff u_x = v_y \quad \text{These are the CR equations.}
\]

\[
u_y = -v_x
\]
Last time

1. \( f: \mathbb{C} \to \mathbb{C} \) is holomorphic provided \( v \in \mathbb{C} \)

   \[
   f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
   \]

2. \( f = u + iv \) holomorphic \( \Rightarrow \begin{align*}
   \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
   \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{align*} \)

   (Cauchy-Riemann)

   \( \iff \frac{\partial f}{\partial z} = 0 \)

3. \( u, v \) are harmonic conjugates

\[
D_f(z) = \begin{bmatrix}
  u_x & u_y \\
  v_x & v_y
\end{bmatrix} = \begin{bmatrix}
  a & -b \\
  b & a
\end{bmatrix}
\]

Today:

1. geometric consequences

2. analytic functions & power series

3. logarithm
**Def** \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) \( \text{R-linear, invertible} \)

**1.** \( T \) is **orientation preserving** if \( \det T > 0 \).

**2.** \( T \) is **angle preserving**, if for any vectors \( \vec{u}, \vec{v} \in \mathbb{R}^2 \)

\[
\hat{x}(\vec{u}, \vec{v}) = \hat{x}(T \vec{u}, T \vec{v}).
\]

\[
\vec{u} \quad \rightarrow T \quad \rightarrow T \vec{u}
\]

**Remark** \( T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) is both **orientation and angle preserving** (unless \( a = b = 0 \)).

\[
\det T = a^2 + b^2 > 0 \quad \text{if} \quad (a, b) \neq (0, 0)
\]

\[
{T_T T} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = 1 \times 1^2 \cdot 1^1 , \quad \alpha = a + b.
\]
\[ T_u \cdot T_v = T \cdot T_u \cdot v = |a|^2 \ u \cdot v \]

If \( u = v \Rightarrow T_v \cdot T_v = |a|^2 \ u \cdot v \Rightarrow \| T_v \| = |a| \| v \| \]

\[
\cos \theta (u, v) = \frac{u \cdot v}{\|u\| \|v\|}
\]

\[
\cos \theta (T_u, T_v) = \frac{T_u \cdot T_v}{\|T_u\| \|T_v\|} = \frac{|a|^2 \ u \cdot v}{|a| \|u\| \|v\|}
\]

\[ \Rightarrow \theta (u, v) = \theta (T_u, T_v) \]

**Remark**

If holomorphic \( \Rightarrow \) either \( f'(z) = 0 \) or else \( \partial f(z) \) is orientation & angle preserving.

\( \Rightarrow \) "\( f \) preserves angles" \( \Leftrightarrow \) "conformal"
\[ z = iy \quad z = 1 + iy \]

\[ z = x + i \cdot 1 \quad \rightarrow \quad 2 \rightarrow 2^z \]

\[ z = x + i \cdot 0 \]

\[ \chi(z) = 2^z \]

\[ 2 = x + i \cdot 1 \quad \Rightarrow \quad 2^z = \frac{x^2 - 1 + 2x}{u} \quad \frac{y}{v} \]

\[ \Rightarrow \quad u = \frac{v^2}{4} - 1 \quad \text{parabola} \]

\[ z = iy \quad \Rightarrow \quad \text{half line} \]

\[ 2^z = 1 + iy \quad \Rightarrow \quad 2^z = \frac{1 - y^2 + 2y}{u} \quad \frac{y}{v} \]

\[ \Rightarrow \quad u = 1 - \frac{y^2}{4} \quad \text{parabola} \]

\[ \text{Check: Angles are preserved.} \]
Power series & Analytic functions \( c \in \mathbb{C}, a \in \mathbb{C} \)

\[ f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n \quad (*) \]

**Definition** (Abel): \( \exists R \quad 0 \leq R \leq \infty \) such that

- If \( |z-c| < R \Rightarrow (\ast) \) converges.
- If \( 0 \leq r < R \Rightarrow (\ast) \) converges absolutely & uniformly.
- If \( |z-c| > R \Rightarrow (\ast) \) diverges.

Furthermore, \( R^{-1} = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \). \( R = \) radius of convergence.

**Definition** \( f: \mathbb{U} \to \mathbb{C} \) is analytic if \( \forall z \in \mathbb{U} \ \exists R > 0 \) such that

\[ f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \Delta(z_0, R) \subseteq \mathbb{U}. \]
Proof

\[ w \log r c = 0, \text{ else work } 2^{-cw} = 2^{-c}. \]

\[ \sum_{n=0}^{\infty} a_n 2^n. \text{ Let } R = \limsup \sqrt[2n]{|a_n|}. \text{ Let } |z| < r. \]

\[ \text{Let } r < \rho < R. \] \[ \Rightarrow \limsup \sqrt[2n]{|a_n|} = \frac{1}{R} < \frac{1}{\rho} \Rightarrow \]

\[ \Rightarrow \sqrt[2n]{|a_n|} < \frac{1}{\rho} \text{ if } n \geq N. \]

\[ \Rightarrow |a_n| < \frac{1}{\rho^n} \text{ if } n \geq N. \]

\[ \Rightarrow |a_n 2^n| < \left( \frac{r}{\rho} \right)^n \text{ if } n \geq N. \]

By Weierstrass M-test

If \( f_n \leq M_n, \sum M_n < \infty \Rightarrow \sum f_n \text{ converges absolutely and uniformly.} \]

\[ \Rightarrow \sum a_n 2^n \text{ converges absolutely and uniformly in } A(0, r). \]

\[ \text{If } |z| > \rho > R. \Rightarrow \limsup \sqrt[2n]{|a_n|} = \frac{1}{R} > \frac{1}{\rho} \]

\[ \Rightarrow \sqrt[2n]{|a_n|} > \frac{1}{\rho} \text{ for infinitely many } n. \]

\[ \Rightarrow |a_n| > \frac{1}{\rho^n} \text{ for infinitely many } n. \]

\[ \Rightarrow |a_n 2^n| > \left( \frac{r}{\rho} \right)^n \text{ for infinitely many } n. \]

\[ \Rightarrow a_n 2^n \to 0. \]

\[ \Rightarrow \sum a_n 2^n \text{ diverges.} \]
Differentiation

Recall that if \( f_n \to f \) it doesn't follow \( f_n' \to f' \) in general. However, for power series we have

**Theorem (Rudin 8.1).**

If \( \sum_{n=0}^{\infty} a_n (z - c)^n \) has radius of convergence \( R \), then

\[
\sum_{n=1}^{\infty} n a_n (z - c)^{n-1} \] has radius of convergence \( R \) as well.

Furthermore, if

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - c)^n \quad \Delta (c, R)
\]

\[
\implies f'(z) = \sum_{n=1}^{\infty} n a_n (z - c)^{n-1} \quad \Delta (c, R).
\]

**Corollary**

\[
f^{(k)}(z) = \sum_{n=k}^{\infty} a_n \frac{n(n-1) \ldots (n-k+1) (z - c)^{n-k}}{n!}
\]

\[
z = c, \quad f^{(k)}(c) = a_k k! \implies a_k = \frac{f^{(k)}(c)}{k!}
\]

\[
\implies f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z - c)^n \quad \text{if } f \text{ is analytic in } \Delta (0, R).
\]

**Remark**

If \( f \) is analytic \( \implies f \) is holomorphic.
Examples: \( \exp, \cos, \sin \)

\[ \exp x := 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots, \quad R = \infty. \]

\[ \limsup_{n \to \infty} \frac{n}{\sqrt{n!}} \geq \limsup_{n \to \infty} \sqrt{n} \left( \frac{n}{2} \right)^{n/2} = \limsup_{n \to \infty} \sqrt{n} = \infty. \]

\[ f'(x) = 1 + 1 + 1 + \ldots + \frac{2^{n-1}}{(n-1)!} + \ldots = f(2) \]

\[ \Rightarrow (e^x)' = e^x. \]

\[ \Rightarrow e^x + c = e^x. \quad (\text{Both sides satisfy } y' = y, y(0) = e^c \quad \text{so they are equal}) \]

\[ \cos \alpha := \frac{e^{i\alpha} + e^{-i\alpha}}{2} = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \ldots \]

\[ \sin \alpha := \frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \frac{1}{i} - \frac{2^1}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \ldots \]

\[ (\sin \alpha)' = 1 - \frac{2^1}{2!} + \frac{2^3}{4!} - \frac{2^5}{6!} + \ldots = \cos \alpha. \]

\[ \sin^2 \alpha + \cos^2 \alpha = 1. \]

Beware! \( \sin \alpha, \cos \alpha \) are not bounded functions as \( \alpha \in \mathbb{R} \).

\[ \cos \pi n = \frac{e^{-in\pi} + e^{in\pi}}{2} \rightarrow 0 \quad \text{as } n \rightarrow \pm \infty. \]

\[ 2^n \text{ can be defined for all } n \in \mathbb{Z} \text{ if } 2 > 0. \]
Remark

$e^{2\pi i} = 1 \implies \text{exponential is not invertible.}$

$log 1 = \infty, \pm 2\pi i, \pm 4\pi i, \ldots, \pm 2n\pi i$

Question

Define $\log 2$.?

Remark

Issues also arise with $\sqrt{2}$ and $2^\alpha$.

These are related to the logarithm.

$\sqrt{2} \leftrightarrow 2^\alpha$ for $\alpha = \frac{1}{n}$

$2^\alpha := \exp (\alpha \log 2)$
Def: A logarithm function \( l : U \to \mathbb{C} \) is a continuous function such that
\[
\forall z \in U, l(z) = \arg z.
\]

Naturally, for this to make sense, we need \( U \subseteq \mathbb{C} \setminus \{0\} \).

Any two logarithms \( l', l \) on \( U \) differ by \( 2\pi \) in \( \mathbb{C} \).

Example A \( U = \Delta (1,1) \),
\[
l(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n.
\]
HWK: \( l \) is a logarithm in \( U \).

Example B \( U = \mathbb{C} \setminus \{0\} \),
\[
z \in U \quad z = r e^{i\theta}, \quad \theta \in (-\pi, \pi), \quad r \neq 0.
\]
\[
\log z = \log r + i\theta.
\]
\[
\Rightarrow e^{\log z} = e^{\log r + i\theta} = r e^{i\theta} = z \quad \Rightarrow \log \text{ is a logarithm in } \mathbb{C} \setminus \{0\}.
\]
Remark The two examples above give the same answer in \( \Delta (1,1) \).

Indeed, the two logarithms \( \ell(z) \) and \( \log \theta \) differ by

\[
2 \pi \sin \th \Rightarrow \log 2 - \ell(2) = 2 \pi \sin \theta, \quad \text{Set } \theta = 1
\]

\[
\Rightarrow \log 1 - \ell(1) = 2 \pi \sin \theta \Rightarrow n = 0
\]

\[
\Rightarrow \log 2 = \ell(2) \quad \text{in } \Delta (1,1).
\]
**Logistics**

- 13 votes for MWF 3-3:50
- 5 votes for WF 3-4:15
- 4 votes indifferent

**New time**: MWF 3-3:50

**No lecture**: Monday, Oct 26.

**Today**: Loose ends

- power series
- logarithm
- Mobius transformations

Conway
I loose ends from last time

**Theorem** Assume that \( \sum_{k=0}^{\infty} a_k z^k \) has radius of convergence \( R \). Then \( \sum_{k=1}^{\infty} k a_k z^{k-1} \) has radius of convergence \( R \) as well.

Furthermore, if \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) then
\[
f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}.
\]

**Proof** Radius of convergence for 2nd power series
\[
R' = \lim_{k \to \infty} \sup_{z \in \Delta(0, R)} \sqrt[k]{|a_k|} = \limsup_{k \to \infty} \sqrt[k]{|a_k|} \quad \text{since} \quad \frac{k}{R} \to 1.
\]
Fix \( \alpha \in \Delta(0, R) \). We show \( f'(\alpha) = g(\alpha) \),

where \( g = \sum_{k=1}^{\infty} k a_k z^{k-1} \).

Let \( s_N = \sum_{k=0}^{N} a_k z^k \), \( R_N = \sum_{k=N+1}^{\infty} a_k z^k \).

Know \( s_N \to f \), \( s_N' \to g \).
Fix $\varepsilon > 0$. We wish to find $\delta > 0$

$$\left| \frac{f(\varepsilon) - f(x)}{\varepsilon - x} - g(x) \right| < \varepsilon \quad \text{if } \varepsilon \in \Delta(x, \delta)$$

Let $|\alpha| < \rho < R$. For $\varepsilon \in \Delta(0, \rho)$ we have

$$(*): \left| \frac{f(\varepsilon) - f(x)}{\varepsilon - x} - g(x) \right| \leq \left| \frac{S_{\varepsilon}(x) - S_{\varepsilon}(\alpha)}{\varepsilon - \alpha} - S_{\varepsilon}'(\alpha) \right|$$

$$+ \left| S_{\varepsilon}'(\alpha) - g(\alpha) \right|$$

$$+ \left| R_{\varepsilon}(x) - R_{\varepsilon}(\alpha) \right| < \varepsilon.$$ 

We estimate each of these terms. Term $\text{III}$:

$$\left| \frac{R_{\varepsilon}(x) - R_{\varepsilon}(\alpha)}{\varepsilon - \alpha} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \left| \frac{\varepsilon^k}{\varepsilon - \alpha} \right|^k$$

$$\leq \sum_{k=N+1}^{\infty} |a_k| \left( \frac{\varepsilon}{\rho} \right)^k \leq \frac{\varepsilon}{3}$$

if $N \geq N_{\varepsilon}$. 
Term II: \[ \left| \frac{S_{n+1}(\alpha) - g(\alpha)}{x - \alpha} - S_n'(\alpha) \right| < \varepsilon/3 \] for \( N \geq N_2 \).

Fix \( N \geq N_1 \) & \( N_2 \). For this \( N \), find \( S \) such that

Term I: \[ \left| \frac{S_n(x) - S_n(\alpha)}{x - \alpha} - S_n'(\alpha) \right| < \varepsilon/3 \] if \( x \in \Delta(\alpha, S) \).

Then

\[ (\varepsilon) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \] for

\( x \in \Delta(\alpha, S) \cap \Delta(0, \rho) \).

QED.
11. **Logarithm**  
$U \subseteq \mathbb{C} \setminus \{0\}$ open & connected  
\[ l : U \to \mathbb{C} \text{ continuous & } z \mapsto l(z) = z. \]

**Example A**  
$U = \Delta(1,1)$

\[ l(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z-1)^k. \]

**Example B**  
$U = \mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ slit plane  
$z = r e^{i\theta}$

**Logarithm**  
\[ \log z = \log r + i\theta \]

$\theta \in (-\pi, \pi) \Rightarrow z = \log z \neq z \log w.$

(Principal branch of logarithm).

**Beware**  
$\log (zw) \neq \log z + \log w$

This holds if $Re \ z > 0, Re \ w > 0.$
Example

\[ \log (1 - i) = \log \sqrt{2} + i \left( -\frac{\pi}{4} \right). \]

principal branch

Example C

Other branches

\[ U = \mathbb{C} \setminus \mathbb{R}_{>0} e^{i\pi}. \]

\[ z = re^{i\theta}, \quad \theta \in (\alpha, \alpha + 2\pi). \]

\[ \log_a z = \log r + i\theta. \]

Remark

[1a] \[ U = \mathbb{C} \setminus \{ 0 \} \Rightarrow \text{impossible to define logarithm} \]

[1b] \[ U \subseteq \mathbb{C} \setminus \{ 0 \} \text{ simply connected} \Rightarrow \text{we can define logarithm (later).} \]

Examples A - C are simply connected.
Remark \[ Z^\alpha = \exp (\alpha \cdot \ln(z)) \text{ is multi-valued} \]
- differ by \[ \exp (\alpha \cdot 2\pi i \cdot n) \]

Example
Principal value of \[ z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0} \]

\[
(1-i)^\prime = \exp \left( i \cdot \log (1-i) \right)
\]

\[
= \exp \left( i \cdot \left( \log \sqrt{2} - i \frac{\pi}{4} \right) \right)
\]

\[
= \exp \left( i \log \sqrt{2} + \frac{\pi}{4} \right)
\]

\[
= e^{\pi i / 4} \left( \cos \log \sqrt{2} + i \sin \log \sqrt{2} \right)
\]
II. Geometry of holomorphic maps

We have seen holomorphic maps with \( f'(z) \neq 0 \) preserve angles.

**Remark** Given \( U, V \subseteq \mathbb{C} \), a **bibiholomorphic** map

\( f: U \rightarrow V \) is

\[
\begin{align*}
\text{i} & f \text{ bijective, holomorphic} \\
\text{ii} & g = f^{-1}: V \rightarrow U \text{ holomorphic.}
\end{align*}
\]

If \( f'(p) = 0 \) \( \Rightarrow \) \( f \circ g(z) = z \)

\[
\Rightarrow g'(z) = \frac{1}{f'(p)}, \quad f'(p) \neq 0.
\]
Important Question

Given $U, V \subseteq \mathbb{C}$, are they biholomorphic?

Today we study a class of transformations which are important for geometric arguments.

- Möbius transformations (MT)
- Fractional linear transformations (FLT)
- Linear fractional transformations (LFT)
August Ferdinand Möbius (1790–1868)

Möbius strip, Möbius inversion, Möbius transform

Möbius published important work in astronomy.
Definition \( \mathbb{C}_\infty = \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \)

Riemann sphere

Definition Möbius transformations \( MT \).

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \quad \hat{\mathbb{C}} \ni \frac{1}{z} = \hat{\mathbb{C}}.
\]

\( A \in \text{GL}_2 \).

\[
\begin{align*}
2 & \mapsto \frac{a \cdot 2 + b}{c \cdot 2 + d} \\
\infty & \mapsto \lim_{z \to \infty} \frac{a \cdot z + b}{c \cdot z + d} = \frac{a}{c}.
\end{align*}
\]

biholomorphism \( \mathfrak{h}_A : \mathbb{C} \setminus \{ -\frac{d}{c} \} \to \mathbb{C} \setminus \{ \frac{a}{c} \} \).

Remark

\([\text{ii}]\) \( A = I \Rightarrow \mathfrak{h}_A = \text{id} \).

\([\text{iii}]\) \( A = \alpha B \iff \mathfrak{h}_A = \mathfrak{h}_B \). \( \forall \alpha \neq 0 \).

\([\text{iv}]\) \( \mathfrak{h}_{AB} = \mathfrak{h}_A \circ \mathfrak{h}_B \). \( \forall B = A^{-1} \Rightarrow \mathfrak{h}_A^{-1} = \mathfrak{h}_A^{-1} \).
**Most famous example**  Cayley transform

\[ C(z) = \frac{z - i}{2 + i}, \quad C^{-1}(w) = i \cdot \frac{1 - w}{1 + w}. \]

**Notation:**  \[ \Delta = \Delta (0,1) \]

\[ \mathbb{H}^+ = \{ z : \text{Im} z > 0 \}. \]

**Claim**  \( C \) is a biholomorphism

\[ C : \mathbb{H}^+ \to \Delta. \]

Suffices to show

\[ z \in \mathbb{H}^+ \iff C(z) \in \Delta. \]

Write \( z = x + iy \)

\[ 1.2 - i/ < 1.2 + i/ \]

\[ x^2 + (y-1)^2 < x^2 + (y+1)^2 \]
Arthur Cayley (1821 - 1895)

- worked in algebraic geometry, Group theory
- Cayley – Hamilton theorem
- modern definition of a group
**Remark**

\[
\frac{az + b}{cz + d} = \frac{bc - ad}{c^2} \cdot \frac{1}{z + \frac{d}{c}} + \frac{a}{c}
\]

\[
c = 0: \quad \frac{az + b}{d} = \frac{a}{d} \cdot z + \frac{b}{d}
\]

---

**Types of Möbius transforms**

1. **Translation** \( T_z = z + \lambda \)
2. **Rotations** \( R_z = e^{i\theta}z \)
3. **Dilations** \( D_z = mz, m \in \mathbb{R} \)
4. **Inversion** \( S_z = \frac{1}{z^2} \)

---

**Lemma** All Möbius transforms are compositions of \( [1] - [4] \).
Generalized circles in \( \mathbb{C} \)

1. Circles in \( \mathbb{C} \)

2. Line \( \mathbb{C} \cup \{ \infty \} \) = circle in \( \mathbb{C} \) through \( \infty \).

Main theorems about Möbius transforms

**Theorem A** Any Möbius transformation maps generalized circles to generalized circles.

**Theorem B** Given two triples \( (Z_1, Z_2, Z_3) \) and \( (Z_1', Z_2', Z_3') \) of distinct points in \( \mathbb{C} \), \( \mathcal{C} \) Möbius transformation \( h \) with

\[
h(Z_i) = Z_i'.
\]
Last time

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \phi_A : \mathbb{C} \to \mathbb{C}, \quad z \to \frac{az + b}{cz + d} \]

**Generalized circles in \( \mathbb{C} \)**

- \( \mathbb{C} \) circles in \( \mathbb{C} \)

- Line \( L \cup \{ \infty \} \)

**Theorem A** Any Möbius transform maps generalized circles to generalized circles.

**Theorem B** \( PGL_2 \) acts triply transitively on \( \mathbb{C} \).

Given \( (z_1, z_2, z_3) \), \( (z_1', z_2', z_3') \) triples of distinct \( \in \mathbb{C} \), \( \exists ! \ h \) with \( h(z_1) = z_1' \).
Proof of Thm A: Suffices to consider the cases:

1. **Translation** \[ z \rightarrow z + i \] is clear.

2. **Rotation** \[ z \rightarrow e^{i \theta} z \] is clear.

3. **Dilation** \[ z \rightarrow mz \] is clear.

4. **Inversion** \[ z \rightarrow \frac{1}{z} \] is clear.

**Claim** A generalized circle is given by

\[
(*) \quad A \bar{z} z + B \bar{z} + C \bar{z} + D = 0, \quad \text{where } A, D \in \mathbb{R},
\]

and \( B, C \) are conjugates.

**Proof** A circle in \( \mathbb{C} \) is given by

\[
|z - z_0| = r \quad \iff \quad (z - z_0) \cdot (\bar{z} - \bar{z}_0) = r^2
\]

\[
\iff \quad z \bar{z} - \bar{z}_0 z - z_0 \bar{z} + (z_0 \bar{z}_0 - r^2) = 0
\]

\[
\iff \quad (*) \quad \text{for } A = 1, \ B = 2 \bar{z}_0 - r^2, \ C = -\bar{z}_0, \ D = -z_0
\]

Conversely, if \( A \neq 0 \), (*) can be brought into this form.

When \( A = 0 \): \[ B \bar{z} + C \bar{z} + D = 0 \quad \iff \quad \text{linear.} \]
Proof \[ \text{ preserves generalized circles.} \]

\[ A \bar{z}^2 + Bz^2 + C\bar{z} + D = 0. \]

Let \( w = \frac{1}{z} \Rightarrow A \cdot \frac{1}{w\bar{w}} + B + \frac{C}{w} + D = 0 \)

\[ \Rightarrow A + B\bar{w} + Cw + D\bar{w}w = 0. \]

\[ \Rightarrow \text{generalized circle. } \Rightarrow \text{Thm A.} \]

In the case of lines \( L \cup \infty, \) 0 and \( \infty \) correspond under \( \text{IV}. \)
Proof of Uniqueness

Assume $\exists h, h'$

\[ h \rightarrow h' \]

\[ h \rightarrow h' \]

\[ (\Leftrightarrow) \quad \frac{a^2 + b}{c + d} = \frac{a}{d} \quad \text{has 3 roots } 2_1, 2_2, 2_3 \]

\[ (\Rightarrow) \quad a, b, c, d \quad \text{has 3 roots} \]

\[ (\Rightarrow) \quad a = d, b = c \quad \Rightarrow \quad T = \text{Id} \quad \Rightarrow \quad h = h' \]

Existence

Suffices: $\exists h$ with

\[ h(2_1) = 0 \]

\[ h(2_2) = 1 \]

\[ h(2_3) = 10 \]

If $(2_1', 2_2', 2_3')$ is another triple, find $h'$ with

\[ h'(2_1') = 0, \quad h'(2_2') = 1, \quad h'(2_3') = \text{as needed} \]

Define $T = h' \circ h \quad \Rightarrow \quad T(2_i) = 2_i'$ as needed.
To deal with \((z_1, z_2, z_3)\) and \((0, 1, \infty)\).

**Cross ratio**  
If \(z_1, z_2, z_3 \neq \infty\),

\[
\lambda(z) = \frac{z - z_1}{z - z_3} \div \frac{z_2 - z_1}{z_2 - z_3}
\]

This is sometimes denoted \([z_1 : z_2 : z_2 : z_3]\).

Check \(\lambda(z) = 0\)

\[
\lambda(z_1) = 0, \quad \lambda(z_2) = 1, \quad \lambda(z_3) = \infty.
\]

There are 3 remaining case \(z_1 = \infty, z_2 = \infty\) or \(z_3 = \infty\).

For example, when \(z_1 = \infty\). The above expression is

\[
\lambda(z) = \frac{z_2 - z_3}{z_2 - z_3}, \quad \lambda(z_1) = 0, \quad \lambda(z_2) = 1, \quad \lambda(z_3) = \infty.
\]
II. Cauchy theory & Integration

The theory of integration is crucial to complex analysis. Many important results have as starting point Cauchy's integral formula.

§1. Complex integration

1. \( U \subseteq \mathbb{C} \) open & connected

2. \( \gamma : [a, b] \to U \) \( C' \)-path

3. \( \text{length} (\gamma) = \int_a^b |\gamma'(t)| \, dt \).

4. \( C' \)-reparametrization \( \hat{\gamma} : [\hat{a}, \hat{b}] \to u \)

\( \hat{\gamma} = \gamma \circ \Phi \), \( \Phi : [\hat{a}, \hat{b}] \to [a, b] \)

Orientation preserving: \( \Phi' > 0 \).
A piecewise $C^1$-path

\[ Y = Y_1 + \ldots + Y_n, \quad Y_i \text{ of class } C^1 \]

if \( a = a_0 < a_1 < \ldots < a_n = b \)

\[ Y_i \big|_{[a_{i-1}, a_i]} = Y_i \]

\[ f : u \rightarrow \text{a continuous}, \quad \text{Define} \]

\[ \int_a^b f(z) \, dz := \int_{\gamma(a)}^{\gamma(b)} f(\gamma(t)) \cdot \gamma'(t) \, dt \]

This is independent of orientation preserving reparametrization.

\[ \int_a^b f(\gamma(t)) \gamma'(t) \, dt = \int_{\alpha}^{\beta} f(\hat{\gamma}(s)) \cdot \hat{\gamma}'(s) \, ds \]

This is change of variables: \( f(\gamma(t)) = f(\hat{\gamma}(s)) \)

\[ \gamma'(t) \, dt = \hat{\gamma}'(s) \, ds. \]
Remark \[ \int_{-\gamma} f \, dz = -\int_{\gamma} f \, dz \] after changing orientation.

Remark The definition extends to piecewise \( C^1 \) paths.

\[ \int_{\gamma} f \, dz = \int_{\gamma_1} f \, dz + \ldots + \int_{\gamma_n} f \, dz. \]

In particular, we can define \( \int f \, dz \) \( R \) rectangle.

Remark Conway works with rectifiable paths.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

Ahlfors - Complex Analysis, 3rd edition

Page 105
Assume $|f| \leq M$ along $\gamma$

$$\Rightarrow \left| \int_{\gamma} f \, d\mathbf{z} \right| \leq \text{length}(\gamma) \cdot M.$$  

**Proof**

$$\left| \int_{\gamma} f \, d\mathbf{z} \right| = \left| \int_{a}^{b} f(\gamma(t)) \cdot \gamma'(t) \, dt \right|$$

$$\leq M \int_{a}^{b} |\gamma'(t)| \, dt$$

$$= M \cdot \text{length}(\gamma)$$
Augustin-Louis Cauchy (1789 - 1857) was a French mathematician who made contributions to several branches of mathematics including complex analysis.

Cauchy was a prolific writer: 800 research articles and 5 textbooks.
**Example A**

\[ \gamma = \text{circle of radius } r, \quad \gamma(t) = re^{it} \]

\[
\int_{\gamma} \frac{z^n}{z} \, dz = 
\int_{0}^{2\pi} \frac{re^{it}}{re^{it}} \cdot re^{it} \, dt = 
\int_{0}^{2\pi} e^{i(n+1)t} \, dt = 
\frac{e^{i(n+1)t}}{i(n+1)} \bigg|_{t=0}^{t=2\pi} = 0, \quad n \neq -1.
\]

When \( n = -1 \)

\[
\int_{\gamma} \frac{dz}{z} = 
\int_{0}^{2\pi} \frac{re^{it}}{re^{it}} \, dt = 
\int_{0}^{2\pi} i \, dt = 2\pi i.
\]

**Example B**

\( f \) admits primitive \( F \), \( f = F' \).

\[
\int_{\gamma} f \, dz = \int_{a}^{b} F'(\gamma(t)) \cdot \gamma'(t) \, dt = 
\int_{a}^{b} (F(\gamma(t)))' \, dt = F(\gamma(b)) - F(\gamma(a)).
\]

*Path independence!*
III. Existence of primitives

U ⊆ C open connected, f continuous. We show three results.

**Proposition A**

TFAE

1. f admits a primitive

2. \[ \int f \, dz = 0 \quad \forall \gamma \text{ piecewise C' loop}. \]

**Remark**

11. \[ \Rightarrow \] is clear by Example B.

**Remark**

1/2 doesn't admit a primitive in \( U = \mathbb{C}^x \).

since \[ \int \frac{dz}{2} = 2\pi i \quad \text{by Example A}. \]

\[ \Rightarrow \] no logarithm in \( U = \mathbb{C}^x \).

**Proposition B**

If \( U = \Delta = \text{disc} \). TFAE

1. f admits primitive

2. \[ \int f \, dz = 0 \quad \text{for all rectangles } \overline{R} \subseteq U. \]
**Compare:**

<table>
<thead>
<tr>
<th>Prop. A</th>
<th>Prop. B</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u \subseteq \mathbb{C} )</td>
<td>( u = \Delta )</td>
</tr>
<tr>
<td>( \gamma ) piecewise ( C^1 )</td>
<td>( \gamma = \partial R )</td>
</tr>
</tbody>
</table>

**Proposition C**

If \( f: U \rightarrow \mathbb{C} \) holomorphic \( \Rightarrow \int_{\partial R} f \, dx = 0 \)

for all rectangles \( R \subseteq U \).

**Corollary**

If \( f: \Delta \rightarrow \mathbb{C} \) holomorphic \( \Rightarrow f \) admits a

primitive.
Last time — existence of primitives

$U \subseteq \mathbb{C}$ open and connected, $f: U \to \mathbb{C}$ continuous

**Proposition A** \hspace{1cm} TFAE

- $f$ admits a primitive
- $\int f \, dz = 0 \Leftrightarrow \gamma$ piecewise $C^1$ loop.

**Corollary** \hspace{1cm} $f$ no logarithm in $U = \mathbb{C}^x$

**Proposition B** \hspace{1cm} If $U = \Delta = \text{disc.}$ \hspace{1cm} TFAE

- $f$ admits primitive
- $\int f \, dz = 0$ for all rectangles $R \subseteq U$. or

**Compare**:

<table>
<thead>
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<tr>
<td>$U \subseteq \mathbb{C}$</td>
<td>$U = \Delta$</td>
</tr>
<tr>
<td>$\gamma$ piecewise $C^1$</td>
<td>$\gamma = \partial R$</td>
</tr>
</tbody>
</table>
**Proposition c**  
If \( f: U \rightarrow \mathbb{C} \) holomorphic \( \Rightarrow \int f \, dz = 0 \)  
for all rectangles \( R \in U \). (Goursat's lemma).

**Remark**  
\( f' \) is not assumed to be continuous.  
If \( f' \) is continuous an easier proof is possible.

**Corollary**  
\( f : \Omega \rightarrow \mathbb{C} \) holomorphic  
\( B \subset \Omega \)  
\( \Rightarrow \) \( f \) admits a primitive.  
\( \Rightarrow \) \( \int f \, dz = 0 \) \( \forall \gamma \) piecewise \( C^1 \) loop.  
This is a form of Cauchy's theorem.
J'ai reconnu depuis longtemps que la démonstration du théorème de Cauchy, que j'ai donnée en 1883, ne supposait pas la continuité de la dérivée.

(I have recognized for a long time that the demonstration of Cauchy's theorem which I gave in 1883 didn't really presuppose the continuity of the derivative.)

Sur la définition générale des fonctions analytiques, d'après Cauchy. Trans. AMS, 1900, 14-46.
Proposition A  \( \text{TFAE} \)

1. \( f \) admits a primitive
2. \( \int f \, dz_2 = 0 \) \( \forall \gamma \) piecewise \( C' \) loop.

\[ \int \gamma \text{ is a piecewise } C' \text{ path in } U \text{ joining } p \text{ to } q. \]

Proof \( \text{1} \Rightarrow \text{2} \) follows by path independence

\( \text{2} \Rightarrow \text{1} \). Fix \( p \in U \). Let 

\[ f (g) = \int_{\gamma} f \, dz \text{ where } \gamma \text{ is a piecewise } C' \text{ path in } U \text{ joining } p \text{ to } q. \]

This is well-defined \( \iff \int_{\gamma_1} f \, dz = \int_{\gamma_2} f \, dz \).

\[ \iff \int f \, dz_2 = 0 \] where \( \gamma = \gamma_1 + (-\gamma_2). \)

which is true by assumption \( \text{1} \).

Claim \( f' = f \).

Proof \( \forall \gamma \in U \). Let \( \varepsilon > 0 \). Let \( s > 0 \) with

\( (*) \quad \lvert f (x) - f (y) \rvert < \varepsilon \) if \( x \in \Delta (g, s) \).
We compute

\[
\frac{\int_0^{2+h} f(2) - f(2)}{2} = \frac{1}{\epsilon} \int_2^{2+h} (f(z) - f(2)) dz.
\]

\[
= \frac{1}{\epsilon} \int_2^{2+h} (f(z) - f(2)) dz.
\]

\[
< \epsilon \quad \text{by (a)} \quad \text{if } 1/\epsilon < \delta.
\]

\[
\leq \frac{1}{\epsilon} \cdot \text{length } [2, 2+h]. \quad \exists.
\]

\[
= \frac{1}{\epsilon} \cdot 1 \cdot \pi = \frac{\pi}{\epsilon} = \epsilon \Rightarrow F' = f.
\]

**Question** Why can we always find a piecewise \( C^1 \) path?

Let

\[
\mathcal{X} = \{ z \in U : \exists \text{ piecewise } C^1 \text{ path from } p \text{ to } z \}.
\]

\( \mathcal{X} \neq \emptyset \) since \( p \in \mathcal{X} \).

\( \mathcal{X} \) open. Let \( z \in \mathcal{X} \Rightarrow \exists \epsilon > 0 \text{ with } \Delta(z, \epsilon) \subseteq U. \)

For \( z' \in \Delta(z, \epsilon) \), join \( p \) to \( z' \) (since \( z \in \mathcal{X} \)) \( \Delta(z, \epsilon) \).

Join \( z \) to \( z' \) (via line segment).

\( \Rightarrow z' \in \mathcal{X} \Rightarrow \Delta(z, \epsilon) \subseteq \mathcal{X} \Rightarrow \mathcal{X} \) open.
Let $x \in \mathbb{R}$. We show $x \in X$.

Let $x' \in X$, $x' \in \Delta(x, \varepsilon) \subseteq U$.

Join $p$ to $x'$ by a piecewise $C^1$ path. $x'$ to $x$ by line segment thus joining $p$ to $x$

$\Rightarrow x \in X$.

Since $U$ connected $\Rightarrow X = U$. 

$X$ closed.
Proposition B

If \( U = \Delta = \text{disc} \), \( TFAE \)

1. \( f \) admits primitive

2. \( \int_{R \in U} f \, d^2 = 0 \) for all rectangles

Proof: We only need (1) \( \Rightarrow \) (2). Let \( p \in U \).

Define \( F(g) = \int_{\gamma} f \, d^2 \) where \( \gamma \) is a path from \( p \) to \( g \) consisting of 2 segments parallel to the axes. Such a path exists since \( U = \Delta = \text{disc} \).

The proof \( F' = f \) is similar. We have

\[
F(g + h) - F(g) = \int_{\gamma} f \, d^2 - \int_{\gamma} f \, d^2 = \int_{\gamma} f \, d^2 - \int_{\gamma} f \, d^2
\]

because \( \int_{R \in U} f \, d^2 = 0 \).

For the red path from \( g \) to \( g + h \), the same argument applies, the length of the path is \( 2 \| h \| \).
Proposition c. If $f : U \to \mathbb{C}$ is holomorphic, then \(\int f \, dz = 0\) for all rectangles $\overline{R} \subseteq U$. (Goursat's lemma).

Proof. Let $A = \left| \int f \, dz \right|$. Let $\varepsilon > 0$ arbitrary. With $A = 0$, we will show $A < K \varepsilon$ for some $K > 0$.

Subdivide rectangle $R$ into 4 equal rectangles $R^1, R^2, R^3, R^4$.

$$A = \left| \int f \, dz \right| = \left| \sum_{j=1}^{4} \int f \, dz \right| \leq \sum_{j=1}^{4} \left| \int f \, dz \right|$$

For each rectangle (out of $R^1, R^2, R^3, R^4$), call it $R^{(j)}$, with

$$\frac{A}{4} \leq \left| \int f \, dz \right|_{\partial R^{(j)}}$$
Continue inductively. We obtain a sequence of rectangles

\[ R \supseteq R^{(n)} \supseteq R^{(2)} \supseteq \ldots \], \; \text{diam } R^{(n)} \to 0.

such that

\[ \frac{A}{4^n} \leq \oint_{\partial R^{(n)}} f \, d\sigma. \]

By compactness, \( \bigcap_{n=0}^{\infty} R^{(n)} = \{ c \} \). Since \( f \) is holomorphic

\[ \left| \frac{f(z) - f(c)}{z - c} - f'(c) \right| < \varepsilon \quad \text{if} \quad z \in \Delta(c, \varepsilon), \quad \text{for some } \varepsilon > 0. \]

\[ \Rightarrow \quad \int_{\partial R^{(n)}} f \, d\sigma < 2 \varepsilon \quad \text{and} \quad f(z) = f(c) + (z - c) f'(c) + (z - c) \chi(z). \]

\[ \Rightarrow \quad \frac{A}{4^n} \leq \oint_{\partial R^{(n)}} f \, d\sigma = \left| \int_{\partial R^{(n)}} f(c) + (z - c) f'(c) + (z - c) \chi(z) \, d\sigma \right|. \]

\[ = \left| \int_{\partial R^{(n)}} (z - c) \chi(z) \, d\sigma \right| < \varepsilon \chi(z) \quad \text{if } n \gg 0. \]

\[ \leq \text{diam } (R^{(n)}) \cdot \varepsilon \leq \text{length } (\partial R^{(n)}). \]

\[ = \varepsilon \cdot \frac{\text{diam } (R)}{2^n} \cdot \frac{\text{length } (\partial R)}{2^n} = \frac{\varepsilon}{4^n} \cdot \text{K}. \]

\[ \Rightarrow \quad A < k \varepsilon \quad \forall \varepsilon > 0 \quad \Rightarrow \quad A = 0. \]
October 16, 2020
Last time \( \Delta = \text{disc} \)

Proposition If \( f: U \to \mathbb{C} \) holomorphic \( \Rightarrow \int_{\partial R} f \, dz = 0 \) for all rectangles \( R \subseteq U \). (Goursat's lemma).

Corollary \( f: \Delta \to \mathbb{C} \) holomorphic

\[ \Rightarrow f \text{ admits a primitive.} \]

\[ \Rightarrow \int_{\gamma} f \, dz = 0 \quad \forall \gamma \text{ piecewise } C' \text{ loop} \]

We seek improvements

New assumption \( \quad \star \)

\[ (\star) \quad f: U \to \mathbb{C} \) continuous, holomorphic in \( U \). \]
Proposition C+ \[ f \text{ satisfies (**) then } \int f \, d\mathcal{H} = 0 \]
for all \( \bar{R} \subseteq U \).

Proof

If \( a \) is outside \( \bar{R} \), let \( U^{\text{new}} = U \setminus \{a\} \) and apply Proposition C to \((f, U^{\text{new}})\)

\[ \implies \int f \, d\mathcal{H} = 0 \]

If \( a \in \bar{R} \), after subdividing \( \bar{R} \) we may assume \( a \) is a vertex.

If \( a \) is a vertex, let \( R_0 \) be a square of side \( \varepsilon \) with vertex \( a \).

\[ \int f \, d\mathcal{H} = 0 \implies \int f \, d\mathcal{H} = \int f \, d\mathcal{H}. \]

Suffices \( \int f \, d\mathcal{H} \to 0 \) as \( \varepsilon \to 0 \).

Since \( f \) cont. at \( a \) \[ |f(a)| / < |f(a)|/ \] if \( 2 \in \partial R_2 \)

for small \( \varepsilon \) \[ \implies \int f \, d\mathcal{H} / \leq (|f(a)|/ + 1) \lim_{\varepsilon \to 0} |\partial R_2| \to 0 \]
**Corollary** \[ f : \Delta \rightarrow \mathbb{C} \text{ continuous, holomorphic in } \Delta \setminus \{a\}. \]

**Prop B** \[ \Rightarrow f \text{ admits a primitive.} \]

**Prop A** \[ \Rightarrow \int f dz = 0 \text{ for piecewise } C^1 \text{ loop } \gamma. \]

**Local Cauchy Integral Formula**

\[ f : \Delta \subset \mathbb{C} \text{ holomorphic, } \Delta \subset U \text{ and } a \in \Delta, \]

\[ f(a) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(z)}{z-a} \, dz. \]

**Remark** \[ f/\partial \Delta \text{ determines } f \text{ in } \Delta. \]
\textbf{Proof} \hspace{1em} \text{Let}

\[ F(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases} \]

\[ \Rightarrow F \text{ continuous on } U \text{ and holomorphic in } U \setminus \{a\}. \]

\text{Let } \Delta \text{ s.t. } \Delta \subseteq \tilde{\Delta} \subseteq \Omega \subseteq U. \text{ Apply Corollary to } \Delta \text{ with } \sigma = \Omega \Delta.

\[ \Rightarrow \int_{\partial \Delta} F \, dz = 0 \Rightarrow \int_{\partial \Delta} \frac{f(z) - f(a)}{z - a} \, dz = 0. \]

\[ \Rightarrow \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(z)}{z - a} \, dz = f(a) \cdot \frac{1}{2\pi i} \int_{\partial \Delta} \frac{dz}{z - a}. \]

\[ \Rightarrow \text{Local Cauchy follows.} \]

\[ 1. \text{ (next lemma)} \]
**Lemma**  If $a \in \Delta$ \Rightarrow \int \frac{dz}{z-a} = 2\pi i \\
\forall \Delta \\

**Proof**

Let $c$ be the center of $\Delta$. \\

\[ \Rightarrow \int \frac{d^2 w}{w} = 2\pi i \]

\[ \Leftarrow \int \frac{d^2 w}{w} = 2\pi i \]

which we have seen before.

It suffices to show \[ \int \left( \frac{d^2}{z-a} - \frac{d^2}{z-c} \right) = 0 \iff \int \Re d^2 = 0 \]

Let \( h(z) = \frac{1}{z-a} - \frac{1}{z-c} \). We show that $h$ admits a principal branch

primitive in $\mathbb{C} \setminus \{ac\}$. Let \( \log \frac{z-a}{z-c} = g(z) \)

\[ \Rightarrow g' = h. \]

**Issue** We need to show \( \frac{z-a}{z-c} \in \mathbb{C}^-=\mathbb{C} \setminus \mathbb{R}_{\leq 0}. \)

\[ \frac{z-a}{z-c} = -u, \; u \in \mathbb{R}_{\geq 0} \iff z = a - \frac{1}{u+1} + c \cdot \frac{u}{u+1} \in \text{segment from } a \text{ to } c. \] (false)
Index (winding number) \( a \notin \{ y \} \). Define

\[
n(y, a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z - a}
\]

**Example A**

\( \gamma \) circle

\( n(y, a) = 1 \) if \( a \in \text{Int} \gamma \).

by the Lemma.

**Example B**

\( y_k(t) = e^{2\pi it + \theta}, \quad 0 \leq t \leq 1 \).

\[
\Rightarrow n(y_k, o) = k.
\]

\[
n(y_k, o) = \frac{1}{2\pi i} \oint_{y_k} \frac{dz}{z} =
\]

\[
= \frac{1}{2\pi i} \int_0^1 \frac{e^{2\pi i t + \theta}}{e^{2\pi iT + \theta}} \, dt
\]

\[
= k.
\]
Cauchy (revisited) $f: \Delta \to \mathbb{C}$ holomorphic,

$\gamma$ closed $C^1$ loop in $\Delta$, $a \in U \setminus \{\gamma\}$.

$$f(a) \cdot n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz$$

The proof is identical to the previous proof.
Lemma \( n(\gamma, a) \in \mathbb{Z} \), \( a \notin \gamma \).

Proof \( n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{\gamma'(s)}{\gamma(s) - a} \, ds \) where

\( \gamma: [\alpha, \beta] \to \mathbb{C} \) is a piecewise \( C^0 \) loop \( \gamma(\alpha) = \gamma(\beta) \).

Consider

\[ t(t) = \int_{\alpha}^{t} \frac{\gamma'(s)}{\gamma(s) - a} \, ds \], \( h(\alpha) = 0 \).

Want \( t(\beta) \in 2\pi i \mathbb{Z} \).

Compute

\[ t'(t) = \frac{\gamma'(t)}{\gamma(t) - a} \].

\[ \Rightarrow \left( e^{-t(t)} \right)'(\gamma(t) - a) = e^{-t(t)} \left( -t'(t) (\gamma(t) - a) + \gamma'(t) \right) = 0. \]

\[ \Rightarrow e^{-t(t)} (\gamma(t) - a) \text{ constant. Let } t = \alpha, t = \beta: \]

\[ e^{-t(\alpha)} (\gamma(\alpha) - a) = e^{-t(\beta)} (\gamma(\beta) - a) \]

\[ \Rightarrow e^{-t(\beta)} = 1 \Rightarrow h(\beta) \in 2\pi i \mathbb{Z}, \quad a \in \mathbb{C}. \]
Math 220 A - Lecture 7

October 19, 2020
Last time: Winding number (index)

γ piecewise C¹ loop, a ≠ \{γ\}.

\[ n(γ, a) = \frac{1}{2\pi i} \oint_{γ} \frac{dz}{z - a} \in \mathbb{Z} \]

\[ n(γ, a) = 1. \]

\[ n(γ, a) = 2 \]

Properties

11. \[ n(-γ, a) = -n(γ, a) \quad \text{(change of orientation)} \]

Proof:

\[ \int_{-γ} \frac{dz}{z - a} = -\int_{γ} \frac{dz}{z - a} \]
$n(\gamma, -): \mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ is locally constant.

$n(\gamma, a) = 0$ for $a$ in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$.

**Proof.**

Let $R$ be a component of $\mathbb{C} \setminus \{\gamma\}$. If $a, b \in R 
\Rightarrow a, b$ can be joined by a polygonal path in $R$.

This is the same argument used in the past to show we can join by piecewise $C^1$ path. Sufficient to show if $\overline{ab} \subseteq R \Rightarrow n(\gamma, a) = n(\gamma, b)$

\[
\Leftrightarrow \quad \int_{\gamma} \frac{dz}{z-b} - \int_{\gamma} \frac{dz}{z-a} = 0.
\]

This is true since $\log \frac{2-a}{2-b}$ is a primitive of the
in the integrand. We showed last time \[ \log \frac{2-a}{2-b} \] is well defined in \( C \setminus \overline{ab} \).

If \( U \) is the unbounded component, let \( R > R_0 \) such that \( \{ \gamma \} \subseteq \Delta(0,R) \). Let \( m \) be the value of \( n(\gamma, \_\_ \_ \_) \) on \( U \). Pick \( 1/a \geq 2R \).

\( a \in U \Rightarrow \) \( |2-a| \geq |a|/2 \geq 2R - R = R \) if \( z \in \{ \gamma \} \) then

\[
|n(\gamma, a)| = \frac{1}{2\pi} \int_{\gamma} \left| \frac{d\zeta}{2-a} \right| \leq \frac{1}{2\pi} \cdot \frac{1}{R} \cdot \text{length}(\gamma).
\]

Make \( R \to \infty \Rightarrow n(\gamma, a) = m = 0 \).
\[ \gamma = \gamma_1 + \gamma_2 \]

\[
\Rightarrow n(\gamma, a) = n(\gamma_1, a) + n(\gamma_2, a)
\]

**Proof:**

\[
\int_{\gamma} \frac{d^2}{z - a} = \int_{\gamma_1} \frac{d^2}{z - a} + \int_{\gamma_2} \frac{d^2}{z - a}.
\]
Rudiments of algebraic topology

\[ \pi_1 (x) = \text{(based) loops in } X \sim \text{homotopy} \]

\[ \pi_1 (X \setminus \{a\}) \cong \mathbb{Z} \text{ isomorphism} \]

\[ \gamma \longrightarrow n(\gamma, a). \]

Two questions arise:

[a] Can we define integrals over \( \gamma \) continuous?

[b] \( \gamma_1 \sim \gamma_2 \implies n(\gamma_1, a) = n(\gamma_2, a). \)

Answer to [a] YES. If \( f \) holomorphic, \( \gamma \) continuous
we define \( \int f \, dx \). For instance by analytic continuation

We will not pursue this here.

Answer to [b] YES. Cauchy’s Theorem (Homotopy)

Conway IV. 6.
We reparametrize so that the domain is $I = [0, 1]$. 

\[ \text{Homotopy } \gamma_0, \gamma_1 : I \rightarrow U \text{ continuous loops} \]

\[ \gamma_0 \sim \gamma_1 \text{ if } \exists \tau : I \times I \rightarrow U \text{ continuous} \]

\[ \tau(t, 0) = \gamma_0(t), \quad \tau(t, 1) = \gamma_1(t). \]

\[ \tau(0, s) = \tau(1, s). \]

\[ \Rightarrow \gamma_s(t) = \tau(t, s). \text{ continuous loop}. \]
**Def** $\gamma_0, \gamma_1 : I \to \mathcal{U}$ continuous paths from $P$ to $Q$.

$\gamma_0 \sim \gamma_1$ if there exists $h : I \times I \to \mathcal{U}$ continuous such that

$$h(t, 0) = \gamma_0(t), \quad h(t, 1) = \gamma_1(t).$$

$h(0, s) = P$, $h(1, s) = Q$. 

\[
\begin{array}{c}
P \quad \gamma_0 \quad \gamma_1 \quad \gamma_s \quad \gamma \quad Q \\
\hline
s \\
\hline
Q \\
\hline
P \\
\hline
t
\end{array}
\]
Remark \( \Box \) \( \sim \) is an equivalence relation.

\[ \gamma_0 \sim \gamma_1, \gamma_1 \sim \gamma_2 \Rightarrow \gamma_0 \sim \gamma_2 \]

\[ u \notin \text{constant loop} \]

(b) Check \( \gamma + (-\gamma) \sim 0, \ \forall \gamma \text{ path in } U \)

(c) If \( \gamma_0 \not\in \text{EP} \), let \( \gamma = \gamma_0 + (-\gamma_1) \) loop

\( \Rightarrow \gamma \sim 0 \). as loops. Indeed let

\[ \Gamma_s = \gamma_s + (-\gamma_1). \]

\[ \Gamma_0 = \gamma. \text{ By } \Box, \Gamma_1 \sim 0. \]

\[ \text{By } \Box \Rightarrow \gamma \sim 0. \]

**Def** \( U \) is simply connected if \( \forall \gamma \text{ loop in } U, \)

\[ \gamma \sim 0 \iff \pi_1(U) = 0. \]
Example: $U$ is star convex $\Rightarrow U$ simply connected

Def $U$ star convex if $\exists p_0 \in U$

such that $u \in U \Rightarrow \overline{p_0 u} \subseteq U$.

Let $\gamma$ be a loop in $U$.

$\tilde{\gamma}(t,s) = s p_0 + (1-s) \gamma(t) \subseteq U$

$\tilde{\gamma}(t,0) = \gamma(t)$

$\tilde{\gamma}(t,1) = p_0 \Rightarrow \gamma \sim 0$. 
Cauchy’s Theorem (Homotopy version)

Let $f: U \to \mathbb{C}$ be holomorphic, and let $\gamma_0 \sim \gamma_1$ be piecewise $C^1$ loops in $U$.

$$\int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz$$

Remarks

If $\gamma \sim 0$, then $\int_{\gamma} f \, dz = 0$.

If $U$ is simply connected, then $\int_{\gamma} f \, dz = 0$ for any piecewise $C^1$ loop $\gamma$ in $U$.

Let $\gamma_1, \gamma_2$ be piecewise $C^1$ paths, $\gamma_1 \sim \gamma_2$.

$$\int_{\gamma_1} f \, dz = \int_{\gamma_2} f \, dz$$

Indeed, let $\gamma = \gamma_1 + (-\gamma_2)$.

By the above, $\int_{\gamma} f \, dz = 0$.

Let $\gamma_0 \sim \gamma_1$, and let $\{a\}$ be piecewise $C^1$ loops in $U$.

$$\int_{\gamma_0} \frac{d\gamma}{2\pi i} = \int_{\gamma_1} \frac{d\gamma}{2\pi i}$$

$$\Rightarrow \gamma(\gamma_0, a) = \gamma(\gamma_1, a)$$

This proves a previous assertion.
Remark  The homotopy in Cauchy's theorem is not assumed to be $C^1$.

Existence of primitives in simply connected sets

If $U$ simply connected, $f: U \rightarrow \mathbb{C}$ holomorphic

\[ \Rightarrow \int f \, dz = 0. \text{ by Remark } 1 \]

\[ \Rightarrow \text{ Prop } A, \ f \text{ has a primitive} \]

Corollary  Any holomorphic function in a simply connected set admits a primitive.

Take $f(z) = \frac{1}{z}$. A primitive is a branch of logarithm.

Corollary  Let $U \subseteq \mathbb{C} \setminus \{0\}$ simply connected. We can define a branch of logarithm in $U$. 
Cauchy’s Theorem (Homotopy version).

\[ f : U \rightarrow \mathbb{C} \text{ holomorphic, } \gamma_0, \gamma_1 \text{ piecewise } C^1 \]

loops in \( U \), \( \gamma_0 \sim \gamma_1 \). Then

\[ \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz \]

Remark We prove a seemingly stronger result

Cauchy’s Theorem (Homotopy version).

(+) \( f : U \rightarrow \mathbb{C} \text{ continuous, holomorphic in } U \setminus \{a\} \)

\[ \Rightarrow \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz \text{ if } \gamma_0 \sim \gamma_1 \text{ on piecewise } C^1 \text{ loops.} \]

We need this stronger form to prove:
**Cauchy's Integral Formula (cif)**

\[ f: U \rightarrow \mathbb{C} \text{ holomorphic}, \quad \gamma \sim 0, \quad a \in U \setminus \{x\} \]

\[ n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz \]

**Remark** This generalizes **Local Cauchy's Integral Formula**.

We proved before. In that case, \( \gamma = \partial \Delta \) where \( \Delta \subseteq U \).
Proof of CIF

Let \( F(z) = \begin{cases} \frac{f(z) - f(a)}{z - a}, & z \neq a \\ f'(a), & z = a \end{cases} \)

\( \Rightarrow \) \( F \) continuous in \( U \), holomorphic in \( U \setminus \{a\} \).

\( \Rightarrow \) \( \int_{\gamma} F \, dz = 0 \) by Cauchy

\( \Rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz = 0 \)

\( \Rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} = f(a) \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} = f(a) \cdot n(z, a) \).

QED.

Remark

Homotopy Cauchy \( \Rightarrow \) CIF

\[ \xymatrix{ \text{Homotopy Cauchy} \ar@{<->}[r] & \text{CIF} } \]

In fact CIF \( \Rightarrow \) Homotopy Cauchy by using CIF

for \( \gamma = \gamma_0 + (-\gamma_1) \) & the function \((z - a) f(z)\)
Proof of Cauchy

Recall the assumption

\((+) f \text{ cont} \& f \text{ hol. in } U \setminus \{a\}\)

For the proof we only use

1. \(f\) continuous

2. \(\forall \Delta \subseteq U, \{ \gamma \} \subseteq \Delta \) piecewise \(C^1\) loop

\[
\Rightarrow \int_{\gamma} f \, \mathrm{d}z = 0 \quad (*).
\]

Under assumption \((+), \) item 2 follows from a previous

Corollary \((+) \quad f : \Delta \rightarrow \mathbb{C} \text{ continuous, holomorphic in } \Delta \setminus \{a\}\)

\[
\Rightarrow \int_{\gamma} f \, \mathrm{d}z = 0 \quad \forall \gamma \text{ piecewise } C^1 \text{ loop}
\]

(see Lecture 6).
We want \( \int_{\gamma_0} f \, dz = \int_{\gamma_1} f \, dz \).

Let \( h : I \times I \rightarrow U \) be the homotopy from \( \gamma_0 \) to \( \gamma_1 \).

Let \( h \) be continuous, \( I \times I \) compact \( \Rightarrow \) \( h \) uniformly cont.

\[ \Rightarrow \exists \delta > 0 \text{ such that} \]
\[ |t - t'| < \delta, \ |s - s'| < \delta \Rightarrow |h(t,s) - h(t',s')| < \delta. \]

Let \( n \in \mathbb{N}_+ \) with \( \frac{1}{n} < \delta \). Subdivide \( I \) into equal intervals \( \left[ \frac{i}{n}, \frac{i+1}{n} \right] \) of length \( < \delta \).
Let $P_{ij}$ have coordinates $\left( \frac{i}{n}, \frac{j}{n} \right)$. Let $Q_{ij} = T \left( P_{ij} \right)$.

Let $R_{ij} = \left[ \frac{i}{n}, \frac{i+1}{n} \right] \times \left[ \frac{j}{n}, \frac{j+1}{n} \right]$. Let $\Delta_{ij} = \Delta \left( Q_{ij}, d \right)$.

Note $\Delta_{ij} \subseteq \mathcal{U}$ by the choice of $d$.

Since sides of $R_{ij}$ have length $< s \Rightarrow T \left( R_{ij} \right) \subseteq \Delta_{ij}$ by uniform continuity.

Let $\Pi_{ij}$ be the polygon through $Q_{oij}, Q_{ij}, \ldots, Q_{nij} = Q_{oij}$.
Claim [10] \[ \int f \, d\gamma = \int f \, d\gamma_0 \quad \text{and} \quad \int f \, d\gamma = \int f \, d\gamma_1. \]

Let \( l_0, l_1, \ldots, l_n \) be the edges of the polygon \( \gamma_0 \)

\( m_0, m_1, \ldots, m_n \) be the arcs of the curve \( \gamma_0 \).

\[ m_j = \gamma_0 \left\lfloor \frac{j}{n} \right\rfloor \]

By construction both \( l_j, m_j \) are contained in \( \Delta_{j0} \subseteq \Omega_i \).

By (10) we have \( \int f \, d\gamma_0 = 0 \rightleftharpoons \int f \, d\gamma = \int f \, d\gamma_1 \)

\[ l_j + (-m_j) \]

Adding for all \( j \), we find \( \int f \, d\gamma = \int f \, d\gamma_0 \).
Claim \[ \underbrace{\int_{\Gamma_j} f \, d\mathbf{z}} = \int_{\Gamma_{j+1}} f \, d\mathbf{z} \]

Let \( s_0, \ldots, s_{n-1} \) be the edges of \( \Gamma_j \).
\( \tilde{s}_0, \ldots, \tilde{s}_{n-1} \), the edges of \( \Gamma_{j+1} \).
\( t_0, \ldots, t_{n-1} \) the segments joining \( Q_{ij} \) to \( Q_{ij+1} \).

Since \( \tilde{r}_{ij} \leq \Delta_{ij} \Rightarrow \tilde{s}_i + t_i + t_{i+1} = (-s_i) + (-t_i) \) is a loop in \( \Delta_{ij} \). By (\#)
\[ \Rightarrow \int f \, d_2 = 0 \]

\[ \tilde{s}_i + \tilde{t}_{i+1} + (-\tilde{s}_i) + (-\tilde{t}_{i+1}) \]

\[ \Rightarrow \int f \, d_2 - \int f \, d_2 = \int f \, d_2 - \int f \, d_2. \]

Add these for all \( i \); we find

\[ \int f \, d_2 - \int f \, d_2 = 0 \Rightarrow \text{Claim 15.} \]

From Claims (2) & (6),

\[ \int f \, d_2 = \int f \, d_2 = \ldots = \int f \, d_2 = \int f \, d_2. \]

Q.E.D.

Last time

Cauchy's Integral Formula (cif)

\[ f: U \rightarrow \mathbb{C} \text{ holomorphic}, \gamma \not= 0, \ a \in U \setminus \{ \gamma \} \]

\[ n(\gamma, a) f(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \, dz \]

Example 1: \( |a| < |b| \). We compute

\[ \int_{|z|=r} \frac{z^{2}}{(z-a)(z-b)} \, dz \]

1. \( r < |a| \), the integrand is holomorphic so answer = 0.

2. \( |a| < r < |b| \), write \( z = r e^{i \theta} \)

\[ \int_{|z|=r} \frac{r^{2} e^{2i \theta}}{(r e^{i \theta} - a) (r e^{i \theta} - b)} \, dz = 2\pi i \cdot \frac{z^{2}}{z - a} \bigg|_{z=a} = 2\pi i \cdot \frac{a^{2}}{a - b} \]
Let \( r_r = \{ z \mid |z| = r \} \).

Let \( f(z) = \frac{e^z}{(z-a)(z-b)} \).

Let \( \gamma, \gamma_b \) be two circles centered at \( a, b \) and \( s \) a segment joining them.

Let \( \gamma = \gamma_a + \delta + \gamma_b + (-\delta) \).

Note \( \gamma \sim \gamma_r \) in \( \mathbb{C} \setminus \{ a, b \} \).

By homotopy Cauchy

\[
\int_{\gamma_r} f(z) \, dz = \int_{\gamma} f(z) \, dz = \int_{\gamma_a} f(z) \, dz + \int_{\gamma_b} f(z) \, dz + \int_{\delta} f(z) \, dz + \int_{-\delta} f(z) \, dz
\]

\[
= \int_{\gamma_a} \frac{e^z}{2-a} \, dz + \int_{\gamma_b} \frac{e^z}{2-b} \, dz
\]

\[
= 2\pi i \cdot \frac{e^2}{2-b} \bigg|_{z=a} + 2\pi i \cdot \frac{e^2}{2-a} \bigg|_{z=b}
\]

\[
= 2\pi i \cdot \frac{e^a - e^b}{a-b}
\]
Taylor Expansion

**Theorem** \( f : U \rightarrow \mathbb{C} \) holomorphic, \( a \in U \), \( \Delta(a, R) \subseteq U \).

Then in \( \Delta(a, R) \):

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \quad (z).
\]

\( \Rightarrow f \) analytic \( \Rightarrow f \) is \( \infty \) many times differentiable.

**Proof** Let \( \Delta(a, R) \subseteq U \). We pick \( 0 < r < R \). Let \( z \in \Delta(a, r) \). By CIF

\[
f(z) = \frac{1}{2\pi i} \int_{1\{t-a\}=r} \frac{f(t)}{t - z} \, dt
\]

\( K_{z,y} : \frac{1}{t - z} = \frac{1}{t - a - (z - a)} = \frac{1}{t - a} \cdot \frac{1}{1 - \frac{2-a}{t-a}} \)
\[ \frac{1}{t-a} \sum_{k=0}^{\infty} \frac{(t-a)^k}{(t-a)^k} \text{ converges since } \left| \frac{t-a}{t-a} \right| = \frac{|2-a|}{r} < 1. \]

\[ \Rightarrow \frac{f(t)}{t-2} = \sum_{k=0}^{\infty} f(t) \cdot \frac{(t-a)^k}{(t-a)^{k+1}}. \tag{+} \]

**Claim**: This converges uniformly in \( t \) over \( |t-a|=r \).

Indeed, let \( f_k(t) = f(t) \cdot \frac{(t-a)^k}{(t-a)^{k+1}}. \)

\[ \Rightarrow \left| f_k(t) \right| \leq M \cdot \frac{|2-a|^k}{r^{k+1}} = M_k, \quad \left| f(t) \right| \leq M \text{ for } |t-a|=r. \]

Note \( \sum M_k < \infty \) since \( |2-a| < r \). Thus the claim follows by Weierstrass M-test.

Since the convergence is uniform, we can integrate (Rudin)

\[ \Rightarrow f(z) = \frac{1}{2\pi i} \int_{|t-a|=r} \frac{f(t)}{t-z} \, dt = \sum_{k=0}^{\infty} a_k (z-a)^k. \]
Def A holomorphic $f: \mathbb{C} \rightarrow \mathbb{C}$ is said to be entire.

**Remark** $f$ entire $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \forall z \in \mathbb{C}$

**Remark** $f: U \rightarrow \mathbb{C}, \quad \overline{\Delta(a,r)} \subseteq U.$

$$a_k = \frac{f^{(k)}(a)}{k!} = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(t)}{(t-a)^{k+1}} \, dt$$

**Lecture 2/3** from the proof of the theorem.

Thus $f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma(a,r)} \frac{f(t)}{(t-a)^{k+1}} \, dt.$

This is local CIF for derivatives.
Cauchy's Integral Formula (for derivatives)

If \( \bar{\Delta} \subseteq U \), \( a \in \Delta \), \( f : U \to \mathbb{C} \) holomorphic

\[
f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial \Delta} \frac{f(t)}{(t-a)^{k+1}} \, dt.
\]

**Proof**

If \( a \) is the center of \( \Delta \) we showed this on the previous page.

If \( a \) is not the center then

let \( \gamma_a \) be a small circle centered at \( a \). Then \( \gamma_a \sim \gamma \)

where \( \gamma = \partial \Delta \). We have

\[
f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma_a} \frac{f(t)}{(t-a)^{k+1}} \, dt = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{k+1}} \, dt.
\]

\( \gamma_a \) has center \( a \) by homotopy

Cauchy
**Remark (Homotopy version)**

\[ f: U \rightarrow \mathbb{C}, \quad \gamma \sim 0, \quad a \in \mathbb{C} \backslash \gamma \]

\[ h(\gamma, a) f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(t)}{(t-a)^{k+1}} \, dt. \]

The case \( \gamma = \partial \Delta, \quad \Delta \subseteq U \) is considered above.

A possible proof is via Conway IV. 2.2 / HWK 3

**Exercise 7.** Another proof is via the residue theorem to be stated later.

**Example**

\[ \int_{|z|=r} \frac{z^2}{(z-a)^k} \, dz, \quad r \neq 1|a|. \]

If \( |a| > r \), the answer is 0 because the integrand is holomorphic.

If \( r > |a| \), apply CIF for derivatives:

\[ \frac{1}{(k-1)!} \cdot 2\pi i \cdot \frac{\partial^{(k-1)} z^n}{\partial z^{k-1}} |_{z=a} = \frac{z^a}{(k-1)!} \cdot 2\pi i. \]
Let \( f : U \rightarrow \mathbb{C} \) be holomorphic, \( \overline{D}(a, R) \subseteq U \). Let

\[
M_R = \sup_{|z-a|=R} |f(z)|.
\]

Then

\[
|f^{(k)}(a)| \leq k! \frac{M_R}{R^k}.
\]

**Proof.** By CIF for derivatives

\[
|f^{(k)}(a)| = \frac{k!}{2\pi i} \oint_{|z-a|=R} \frac{f(z)}{(z-a)^{k+1}} \, dz
\]

\[
\leq \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot \text{Length } |z-a|=R
\]

\[
= \frac{k!}{2\pi} \cdot \frac{M_R}{R^{k+1}} \cdot 2\pi R = k! \frac{M_R}{R^k}.
\]

**Liouville's Theorem**

If \( f : \mathbb{C} \rightarrow \mathbb{C} \) entire & bounded \( \Rightarrow f \) constant.

We prove this next time.
Recall – Midterm next Friday

10 Last time (Cauchy’s Estimate)

\[ f: U \to \text{holomorphic, } \bar{D}(a, R) \subseteq U \]

\[ |f(z)| \leq \frac{M_R}{R^k}, \quad M_R = \sup_{z \in \bar{D}} |f(z)|, \quad |z - a| = R \]

Remark \quad k = 0:

\[ |f(a)| \leq \sup_{z \in \Delta} |f(z)| \]

\[ 2 \in \Delta \]

II. Liouville’s Theorem

If \( f: \mathbb{C} \to \mathbb{C} \) entire & bounded \( \Rightarrow f \) constant.
Joseph Liouville
1809 - 1882

Known for:
- Liouville's theorem
- Sturm–Liouville theory
- Liouville numbers
- Liouville function
Proof: \( f \) is bounded by \( M \), \( |f(z)| \leq M + |\epsilon| \).

Cauchy's estimate for \( k = 1 \). Take \( \bar{B}(a, R) \subseteq G \).

\[
|f'(a)| \leq \frac{M_R}{R} \leq \frac{M}{R}.
\]

Take \( R \to \infty \).

Thus \( f'(a) = 0 \). \( \forall a \Rightarrow f \) constant.

**Fundamental Theorem of Algebra**

Any nonconstant polynomial \( f \in \mathbb{C}[z] \) has at least one complex root.

Proof: WLOG \( f \) monic

\[
f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.
\]

Assume \( f \) has no roots \( \Rightarrow f(z) \to \pm \infty \).

Let \( g = f \). \( \Rightarrow g \) is entire. We show \( g \) bounded \( \Rightarrow \)

\( g \) constant. \( \Rightarrow f \) constant. This is a contradiction.
We show \( g \) bounded. If \( |z| = R \)

\[
|f(z)| = |z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0| \geq |z|^n - \sum_{k=1}^{n-1} |a_k| |z|^{n-k}
\]

\[
= R^n - \sum_{k=1}^{n-1} |a_k| R^{n-k} \to \infty \quad \text{as} \quad R \to \infty.
\]

If \( R \geq R_0 \implies |f(z)| \geq 1 \implies |\log |z|| \leq 1.

If \( R \leq R_0 \implies \) by continuity of \( g \): \( |\log |z|| \leq K. \)

\[
\implies |\log |z|| \leq M = \max(1, K), \quad \forall z.
\]

2. Zeros of holomorphic functions

Conway IV. 3.

\( f : U \to \mathbb{C} \) holomorphic, \( f \neq 0 \), \( U \) connected.

\( a \in U \) is a zero of order \( N \) if

\[
f(a) = 0, \quad f'(a) = 0, \quad \ldots, \quad f^{(N-1)}(a) = 0, \quad f^{(N)}(a) \neq 0.
\]

\( \implies \) Taylor expansion in \( \Delta (a, R) \subseteq U \)

\[
f(z) = \sum_{k \geq N} \frac{f^{(k)}(a)}{k!} (z-a)^k = (z-a)^N g(z)
\]

where \( g \) is a power series in \( \Delta (a, R) \).

\[
g(z) = \frac{f^{(N)}(a)}{N!} \neq 0.
\]

We need to rule out the case \( N = \infty \).
**Lemma**  \( f: U \to \mathbb{C}, \ U \) connected. TFAE

1. \( f \equiv 0 \)
2. \( \forall a \in U, \ f^{(k)}(a) = 0 \ \forall k \)
3. \( S = \{ z : f(z) = 0 \} \) has a limit point in \( U \).

**Proof**  \( \begin{align*} &1 \implies 2, \ 1 \implies 3 \end{align*} \)

\( 3 \implies 1 \) Let \( a \) be a limit point for \( S, \ a \in U \).

Clearly \( f(a) = 0 \). Let us assume \( a \) has finite order \( N \).

By \( (\ast) \), \( f(z) = (z-a)^N g(z) \) in \( D(a, r) \) with \( g \) power series, \( g(a) \neq 0 \). By continuity of \( g \), \( g(z) \to 0 \) in some \( D(a, r) \subseteq D(a, R) \). Then

\[
S \cap D(a, r) = \{ z : (z-a)^N g(z) = 0 \} = \{ a \}.
\]

contradiction with \( a \) being a limit point.

Thus \( N = \infty \). \( \implies 3 \).
Let \( A = \{ a : f^{(k)}(a) = 0 \text{ for all } k \} \subseteq U \).

By assumption \( A \neq \emptyset \). We show \( A \) is closed and open.

Thus \( A = U \Rightarrow f \equiv 0 \).

\[ A \text{ closed. Indeed } A = \bigcap_{k=0}^{\infty} \left( f^{(k)}(0) = \text{closed} \right) \]

Since \( f^{(k)} \) is continuous \( \Rightarrow f^{(k)}(0) \) is closed \( \Rightarrow A \) closed.

\[ A \text{ open. Let } a \in A. \text{ By Taylor if } \Delta(a, R) \subseteq U, \]

\[ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k = 0 \quad \text{since } f^{(k)}(a) = 0. \]

Since \( f = 0 \) in \( \Delta(a, R) \Rightarrow f^{(k)} = 0 \) in \( \Delta(a, R) \Rightarrow \]

\[ \Rightarrow \Delta(a, R) \subseteq A \Rightarrow A \text{ open.} \]

---

**Corollary (Identity principle)** Let \( f, g : U \to \mathbb{C} \) holomorphic.

If

\[ S = \{ z : f(z) = g(z) \} \text{ has a limit point } \Rightarrow f = g. \]

**Proof** Work with \( h = f - g \). Apply Lemma above.
Remarks

The zeros of \( f : U \rightarrow \mathbb{C} \) holomorphic cannot have a limit point in \( U \).

\[
\begin{align*}
\text{Zeros:} & \quad \frac{1 + 2}{1 - 2} = n\pi \iff z = \frac{-1 + n\pi}{1 + n\pi} \rightarrow 1. \\
\text{Thus the zeros can accumulate to } & \partial U.
\end{align*}
\]

This fails for \( C^1 \) functions:

\[
f(x) = \begin{cases} 
0, & x = 0 \\
\frac{1}{x^2} \sin \frac{1}{x}, & x \neq 0.
\end{cases}
\]

Check \( f \) is \( C^1 \). Also \( f \) has zeros at \( \frac{1}{n\pi} \rightarrow 0 \), which has a limit point.
\[ f \neq 0 \text{ has at most countably many zeros.} \]

Let \( U = \bigcup_{n=1}^{\infty} K_n \) where \( K_n \) compact. In each compact set \( K_n \), \( f \) can only have finitely many zeros. (indeed this is because \( \text{zeros}(f) \) can't accumulate in \( K_n \))

\[ \Rightarrow \text{zeros}(f) = \bigcup_{n} \text{zeros}(f) \cap K_n = \text{countable}. \]

Aufgaben und Lehrsätze,
erstere aufzulösen, letztere zu beweisen.

1. (Von Herrn N. H. Abel.)

49. Théorème. Si la somme de la série infinie

\[ a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_mx^m + \ldots \]

est égale à zéro pour toutes les valeurs de \( x \) entre deux limites réelles \( \alpha \) et \( \beta \); on aura nécessairement

\[ a_0 = 0, \ a_1 = 0, \ a_2 = 0, \ldots \ a_m = 0, \ldots \]

en vertu de ce que la somme de la série s'évanouira pour une valeur quelconque de \( x \).

Identity theorem: Caille's Journal 1827, page 286
Midterm - Friday Nov 6

- closed book, closed notes
- honor code - no Zoom proctoring
- available Friday 3PM, due Friday 4 PM
- upload answers in GradeScope
- time zone issues - email me
- buffer: 10 minutes to upload solutions, 4:10 PM
- if questions arise, please email.
Last time we looked at zeros of holomorphic functions.

The following result guarantees existence.

**Lemma** \( f : U \rightarrow \mathbb{C} \) holomorphic, \( \Omega : (a, R) \subset U \)

Assume \( \min |f(z_2)| > |f(a)| \). Then \( f \) has a zero in \( U \).

**Proof** Assume \( f \neq 0 \), set \( g = \frac{1}{f} \).

Note \( |g(a)| = \frac{1}{|f(a)|} > \frac{1}{\min |f(z_2)|} = \max |g(z_2)|, \quad z \in \Omega \).

This contradicts the \( k=0 \) case of Cauchy’s estimate

\[ |g(a)| \leq \max |g(z_2)|, \quad z \in \Omega \]

Thus \( f \) has a zero in \( U \).
**Main Theorems**

I. **Identity Principle**

II. **Open Mapping Theorem**

III. **Maximum Modulus Principle**

---

**II. Open Mapping Theorem.**

Recall: \( f: X \to Y \) is open map if \( \forall U \subseteq X \) open, \( f(U) \) is open.

\[ f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2 \text{ is not open, } U = (-1, 1), \quad f(U) = [0, 1) \]

\[ f: \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^2 \text{ is open. This is because:} \]

---

**Theorem** \( f: U \to \mathbb{C} \text{ not constant holomorphic } \Rightarrow \)

\( \Rightarrow f \) is open.

---

**Proof** Sufficient to show \( f(U) \) is open. Else if \( \forall v \in U, \text{ work with } f(v): v \to \mathbb{C} \). This is not constant because of identity principle.

Let \( a \in U \). We may assume \( f(a) = 0 \).
Claim: There exists $r$ such that $\Delta(0, r) \subseteq f(U)$. This would show $f(U)$ contains a neighborhood of $f(a) = 0 \Rightarrow f(U)$ open.

Proof: Since $U$ open $\Rightarrow \exists \bar{\Delta}(a) \subseteq U$, we may assume $f/\bar{\Delta}(a)$ has no zeros. (Argue by contradiction.) This would give a sequence of accumulating zeros for $f$, contradicting identity principle.

Let $r = \frac{1}{2} \min_{2 \in \bar{\Delta}(a)} |f(2)| > 0$.

Let $w \in \Delta(0, r)$. We need to show $\exists z \in U$, $f(z) = w$.

Apply the lemma to $f-w$ to guarantee $\exists \text{zero } z \text{ for } f-w$.

We need

$$\min_{2 \in \bar{\Delta}(a)} |f(2) - w| > |f(a) - w| = |0 - w| = |w|$$

Indeed,

$$|f(2) - w| \geq |f(2) - w| > 2r - |w| > |w|$$

since $|w| < r$. This completes the proof.
**Example**  
\[ f: \mathbb{R} \to \mathbb{R}, \ P \in \mathbb{R}[x, y] \text{ not constant} \]

\[ P \left( \text{Ref}, \text{Im} f \right) = 0 \implies f \text{ constant.} \]

\[ P = a \cdot x + b \cdot y - c. \]

\[ a \cdot \text{Ref} + b \cdot \text{Im} f = c \implies f \text{ constant.} \]

\[ P = x^{2020} + y^{2020} - 1. \]

\[ (\text{Ref})^{2020} + (\text{Im} f)^{2020} = 1. \implies f \text{ constant.} \]

**Proof**  
By OMT, \( f(u) \) is open so it contains a disc \( \Delta \).

Since \( P \left( \text{Ref}, \text{Im} f \right) = 0 \implies f(u) \subseteq \{(x, y) : P(x, y) = 0\} \),

\[ \implies \Delta \subseteq \{(x, y) : P(x, y) = 0\} \text{ This cannot happen.} \]

Indeed, write \( P(x, y) = \sum_{k=0}^{\infty} a_k(x) y^k \).

Fix \( x \) such that \( a_n(x) \neq 0 \). (finitely many roots). For such \( x \), \( y \) takes on at most \( N \) values for which \( P(x, y) = 0 \).

But if \( \Delta \subseteq \{(x, y) : P(x, y) = 0\} \), for each \( x \) there would be \( \infty \) many \( y \)'s. contradiction.
Example \( f: U \rightarrow V \) bijective, holomorphic & \( f'(z) \neq 0 \) at \( a \in U \). Then \( f^{-1} \) holomorphic.

Proof We show \( f^{-1} \) continuous. This is the OMT.

\[
(f^{-1})'(w) = f'(w) = \text{open}, \quad \forall \ w \in V \text{ open}.
\]

We show \( f^{-1} \) is differentiable. Use the definition.

\[
1 = \lim_{h \to 0} \frac{f(f^{-1}(z+h)) - f(f^{-1}(z))}{h} = \\
= \lim_{h \to 0} \frac{f^{-1}(z+h) - f^{-1}(z)}{h}.
\]

The first limit exists since \( f \) is holomorphic & \( f^{-1} \) is continuous. It equals \( f'(f^{-1}(z)) \). The second limit must exist as well, giving the derivative \( (f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))} \).

Remark We assumed \( f'(a) \neq 0 \) at \( a \). This is automatic (see later).
**Theorem** \( f: U \to \mathbb{C} \) holomorphic, non constant \( \implies \) \( f \) cannot have local maxima.

**Proof** Assume that \( f \) achieves a local maximum at \( a \).

\( \exists v \exists a, v \in U, f \) has a maximum at \( a \).

By OMT, \( f(v) \) is open. \( \implies \) \( f \) disc \( \Delta \) centered at \( f(a) \)

\( \Delta \subseteq f(v) \). Note that \( f \) measures distance from the origin. The disc \( \Delta \) has points farther from \( 0 \) than \( f(a) \)

contradicting the assumption \( f \) has maximum at \( a \), \( \forall v \).
Remarks

Minimum modulus principle

\[ f : U \to \mathbb{C} \text{ holomorphic, not constant, } f \text{ has no zeros in } U. \]

\[ \Rightarrow |f| \text{ has no local minimum} \]

Proof

Let \( g = \frac{1}{f} : U \to \mathbb{C} \). holomorphic. Apply the maximum modulus to the function \( g \) & conclude.

\[ \text{If } I \text{ bounded, } f : \overline{U} \to \mathbb{C} \text{ continuous, holomorphic in } U \]

\[ \Rightarrow \max |f| = \max |f| \text{ in } \overline{U}. \]

Proof

Since \( U \) bounded \( \Rightarrow \overline{U}, \text{ compact so } f \text{ achieves maxima on these sets. Let } f \text{ achieve maximum in } \overline{U} \text{ at } a \in \overline{U}. \]

If \( a \in U \Rightarrow f/|u| \text{ has a maximum at } a \Rightarrow \]

\[ f = \text{constant } \Rightarrow \text{ there's nothing to prove.} \]

Otherwise \( a \in \partial U \) proving \((\star)\).
Laurent Series & Functions in annular regions (Conway V.1)

We have seen $f : \Delta(a, r) \to \mathbb{C}$ then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k \quad - \text{Taylor series}$$

We consider Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k$$

Convergence of Laurent series

$$f^+(z) = \sum_{k=0}^{\infty} a_k (z - a)^k$$
$$f^-(z) = \sum_{k=-\infty}^{0} a_k (z - a)^k = \sum_{k=1}^{\infty} a_{-k} (z - a)^{-k}$$

$$f(z) = f^+(z) + f^-(z)$$

Def $f$ converges absolutely & uniformly provided $f^+, f^-$ do so.

Remark

$f^+$ converges if $|z - a| < R$.

$f^-$ converges if $|z - a| > r' \iff |z - a| > r$.  

radius of convergence
For power series, convergence is absolute & uniform on compact subsets.

\[ \Delta = \text{fin} \quad \Delta (a; r, R) = \{ z : \rho < 1 - a < R \} \quad 0 \leq r < R < \infty. \]

**Theorem**

Let \( f : \Delta (a; r, R) \rightarrow \mathbb{C} \) holomorphic. Then \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k \) can be expanded into a Laurent series, converging absolutely & uniformly on compact sets in \( \Delta (a; r, R) \). Furthermore,

\[ a_k = \frac{1}{2\pi i} \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{k+1}} \, dw, \quad r < \rho < R. \]

**Remark**

An important case is \( r = 0 \). Then

\[ \Delta^* (a, R) = \Delta (a, R) \setminus \{ a \} = \text{punctured disc.} \]

If \( f : \Delta^* (a, R) \rightarrow \mathbb{C} \) holomorphic, then

\[ f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k \]

Compare this to Taylor expansion.
The original work on Laurent series was not published.

Cauchy writes (C.R. Acad. Sci. Paris, 1843, page 938)

L'extension donnée par M. Laurent nous paraît
digne de remarque

(After Remmert, Complex Analysis, page 350)
Proof (of Laurent expansion) \( A = \Delta (a; r, R). \)

\[ \text{wlog } a = 0, \text{ else work with } f(z + a). \]

\[ A = \frac{1}{2\pi i} \int_{\gamma_p} \frac{f(w)}{w^{k+1}} \, dw \]

The expression is independent of \( p \). Indeed

\[ A \sim A_p, \text{ and use Cauchy–Homotopy Theorem.} \]

Indeed, convergence of \( f \) \( \iff \) convergence of \( f^+ \) & \( f^- \)

in \( r < |z| < R. \)

But then \( f^+ \) converges in \( |z| < R \) (power series have discs of convergence) & we remarked convergence is absolute & uniform on compacts. Same for \( f^- \).
Pointwise convergence

Let \( r < s < 121 < S < R \)

Let \( S \) be a segment joining \( Y_s \), \( Y_s \)

avoiding \( Z \).

Let

\[
\gamma = Y_s + S + Y_a + (-S).
\]

Note \( \gamma \sim 0 \). This can be seen by

continuously shrinking \( S \to 0 \).

Also \( n(\gamma, 2) = 1 \) since \( n(\gamma, 2) = 0 \) as \( Z \) is outside and

\( n(\gamma, 2) = 1 \) as \( Z \) is interior to \( Y_s \).

\[ n(\gamma, 2) = 1 \]

CIF:

\[
(+) \quad f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z} \, dw.
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z} \, dw - \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw
\]

(cancelling the contribution of \( S, -S \)).

The two terms will give the positive / negative parts

of Laurent series.
Key expansions (Remember them) \( \alpha < 1 \) \( \frac{1}{1 - \frac{z}{w}} \):

\[
\frac{1}{w - 2} = \frac{1}{w} \cdot \frac{1}{1 - \frac{2}{w}} = \sum_{k=0}^{\infty} \frac{1}{w} \left( \frac{2}{w} \right)^k
\]

\[
= \sum_{k=0}^{\infty} \frac{2^k}{w^{k+1}}. \tag{1}
\]

The convergence is uniform in \( w \) since \( \left| \frac{2}{w} \right| = \frac{12}{S} < 1 \). We can define \( M_k = \frac{|2|^k}{S^{k+1}} \), \( f_k(z) = \frac{2^k}{w^{k+1}} \) and invoke Weierstrass M-test to conclude uniform convergence.

We can multiply by \( f(w) \). Uniform convergence still holds. (Use \( M_k = \frac{|2|^k}{S^{k+1}} \cdot \sup \{f(z)\} \).

We can then integrate term by term. (Rudin). Thus:

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - 2} \, dw = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{k+1}} \, dw = 2^k
\]

\[
= \sum_{k=0}^{\infty} a_k 2^k. \tag{\star}
\]

II) over \( \gamma_s \), we use a different expansion

\[
\frac{1}{w - 2} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{2}{w}} = \sum_{k=0}^{\infty} -\frac{1}{2} \left( \frac{w}{2} \right)^k
\]

\[
= \sum_{k=0}^{\infty} -\frac{w^k}{2^{k+1}}. \tag{2}
\]
Here \( \left| \frac{w}{x} \right| = \frac{3}{3} < 1 \). By the same arguments

\[
- \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w - z} \, dw = \sum_{k=0}^{w} \frac{1}{2\pi i} \int_{\gamma_2} f(w) w^k \, dw \cdot z^{-k-1}
\]

\[
= \sum_{k=0}^{w} a_{-k-1} z^{-k-1}
\]

\[
= \sum_{k=-w}^{w} a_k z^k. \quad (**)
\]

(†), (**), (***). imply the Theorem.
Math 220 A - Lecture 13

November 9, 2020
Logistics

Wed, Nov 11 - holiday - no lecture

Office Hour - Wed 4-5PM (discuss homework, midterm)

Questions about the midterm?
Last time - Laurent expansion

**Theorem** Let $f : \Delta(a; r, R) \to \mathbb{C}$ be holomorphic. Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k$$

is a Laurent series, converging absolutely and uniformly on compact subsets of $\Delta(a; r, R)$.

Today - classification of singularities

- Conway V. 1.

- Characterization of singularities.
Types of singularities

\[ f: \Delta^* (a, R) \to \mathbb{C}, \text{ holomorphic.} \]

\[ f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \]  
Laurent series.

Terminology

The coefficient of \((z-a)^{-1}\):

\[ a_{-1} = \text{Res}_f \]

\(2=a\)

\[ \sum_{n=-\infty}^{-1} a_n (z-a)^n = \text{principal part.} \]

Three cases

\[ a_k = 0 \quad \forall \ k < 0 \quad \implies \quad \text{Taylor expansion} \]

\(\iff f \text{ extends holomorphically across } a\)

Removable singularity

\[ a_k = 0 \quad \forall \ k < -N, \quad a_{-N} \neq 0 \]

Pole of order \(N\),

\[ a_k \neq 0, \ k \neq 0 \text{ happens infinitely often} \]

Essential singularity
Case A: a removable singularity

**Theorem A**

TFAE $f: \Delta^y(a, R) \rightarrow \mathbb{C}$ holomorphic

1. $f$ extends holomorphically across $a$

2. $f$ extends continuously across $a$

3. $f$ bounded near $a$

4. $\lim_{z \to a} f(z) \cdot (z - a) = 0$

**Proof** $1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 4$ is obvious

$4 \Rightarrow 1$. WLOG $a = 0$, else work with $f(z + a)$.

We show $a_k = 0 + k \leq 0$. Fix $\varepsilon > 0$. Since

$$\lim_{z \to 0} z f(z) = 0 \Rightarrow |f(z)| < \frac{\varepsilon}{12}, \text{ if } |z| < \delta.$$ 

We have for $0 < r < \delta < R$:

$$|a_k| = \left| \frac{1}{2\pi i} \int_{|z| = r} \frac{f(z)}{z^{k+1}} \, dz \right| \leq \frac{1}{2\pi} \cdot \varepsilon \cdot \frac{1}{r} \cdot \frac{1}{r^{k+1}} \cdot 2\pi r$$

$$= \frac{\varepsilon}{r^{k+1}}.$$
If \( k = -1 \): \( |a_{-1}| < \varepsilon \forall \varepsilon > 0 \Rightarrow a_{-1} = 0 \).

If \( k < -1 \), take \( \varepsilon = 1 \), \( |a_k| < \frac{1}{r^{k+1}} \). Make \( r \to 0 \) to obtain \( a_k = 0 \). Since \( k < -1 \).

**Example**

\( f: \mathbb{U} \to \mathbb{C} \) holomorphic

\[
g(z) = \begin{cases} 
\frac{f(z) - f(a)}{z - a}, & z \neq a \\
\lim_{z \to a} f'(a) & z = a
\end{cases}
\]

is holomorphic.

Indeed \( \frac{f(z) - f(a)}{z - a} \) has a removable singularity at \( a \).

by item \( \text{[IV]} \). \( g \) is the continuous holomorphic extension across \( a \).
Case B: a pole of order N.

\[ f(z) = (z - a)^{-N} g(z) \]

holomorphic

\[ g(z) = \sum_{k=0}^{\infty} a_k (z - a)^k, \quad g(a) = a_{-N} \neq 0. \]

Zeros versus Poles

\[ \frac{1}{f(z)} = (z - a)^N, \quad \frac{1}{g(z)}, \quad \frac{1}{f} \] holomorphic near a

Zeros versus Poles

f pole of order N at a \( \iff \frac{1}{f} \) zero of order N at a

Lemma B:

\[ f: \Delta^\circ (a, r) \to \mathbb{C} \] holomorphic TFAE

\[ \begin{align*}
\text{[I]} & \quad \text{a is a pole} \\
\text{[II]} & \quad \text{lim}_{z \to a} f(z) = 0.
\end{align*} \]

Proof

\[ \text{[II]} \Rightarrow \text{[I]} \]

Write \( f(z) = (z - a)^{-N} g(z) \).

Since \( g(a) \neq 0 \) \( \Rightarrow \) \( |g(z)| > M > 0 \) in \( 12 - a/\delta \)

\[ \frac{|f(z)|}{|z - a|^N} \geq \frac{M}{|z - a|^N}. \] Make \( z \to a \) to
conclude \( \lim_{z \to a} f(z) = \infty. \)

\[ [11] \implies [12] \quad \text{Note} \quad \lim_{z \to a} f(z) = \infty \implies \lim_{z \to a} \frac{1}{f(z)} = 0 \]

\( \implies \frac{1}{f} \) bounded near \( a \) \( \implies \frac{1}{f} \) can be extended across \( a \) holomorphically. \text{Note} the extension vanishes at \( a \), say of order \( N \) \( \implies f \) has a pole at \( a \) of order \( N. \)

\text{Definition} \quad S \subseteq U \textit{ discrete}. A function \( f \) holomorphic in \( U \setminus S \), with at worst poles at \( S \) is called \textit{meromorphic}.

\text{Example} \quad \text{polynomials}

\[ \begin{align*}
[11] \quad f(z) &= \frac{P(z)}{Q(z)}, \quad U = \mathbb{C} \quad \textit{meromorphic}. \\
[12] \quad f(z) &= \frac{1}{\sin \frac{1}{2}}, \quad U = \mathbb{C}.
\end{align*} \]

\text{Check} \ z = \frac{1}{n\pi}, \quad n \in \mathbb{Z}^\times \text{ are poles. These do not}
Accumulate in $u = \mathbb{C} \setminus \{0\}$. Thus $f$ meromorphic in $U = \mathbb{C}^*$. 

**Case C**  
$f : \mathbb{D}^* (\alpha, R) \rightarrow \mathbb{C}$ holomorphic

an essential singularity \(\text{e.g.} \quad f(z) = e^{1/z} \).

**Example**  
\[ f(z) = \frac{1}{\sqrt{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{2^k}. \quad \Rightarrow \]

\[ \Rightarrow \alpha = 0 \text{ is essential singularity.} \]

**Remark**  
\[ f \text{ cannot be bounded or go to } \infty. \]

(see cases A & B)

**Question**  
How does $f$ behave near $\alpha$?
**Theorem (Big Picard Theorem)** - Let $C$. 

$\forall \Delta^* (a, \varepsilon) \subseteq \Delta^* (a, R), \ f (\Delta^* (a, \varepsilon)) = \mathbb{C}$ or $\mathbb{C} \setminus \{\text{point}\}$.

**Example** 

$f (z) = e^z, \ a = 0$.

**Claim** 

$f (\Delta^* (a, \varepsilon)) = \mathbb{C} \setminus \{0\}, \ \forall \varepsilon > 0$.

**Proof** 

$y \neq 0: \ |y| = e^{\frac{1}{z}}, \ z \in \Delta^* (0, \varepsilon)$. 

$\! \! \Longleftrightarrow \! \! \frac{1}{z} = \log y + 2\pi i \cdot n$ for any choice of $\log$.

$\! \! \Longleftrightarrow \! \! z = \frac{1}{\log y + 2\pi i \cdot n} \in \Delta^* (0, \varepsilon)$ if $n > 0$.

**Theorem c (Casorati - Weierstrass)** 

$f: \Delta^* (a, R) \to \mathbb{C}$

- $f$ has **essential singularity** at $a$.
- $\forall \Delta^* (a, \varepsilon) \subseteq \Delta^* (a, R), \ f (\Delta^* (a, \varepsilon))$ is dense in $\mathbb{C}$. 
Proof. \( \Rightarrow \) Assume for some \( \varepsilon > 0 \), the set \( f(\Delta^*(a, \varepsilon)) \) is not dense in \( \mathbb{C} \). Then \( \exists \Delta(\gamma, \rho) \)

\[ f(\Delta^*(a, \varepsilon)) \cap \Delta(\gamma, \rho) = \emptyset. \]

Define \( g = \frac{1}{f - a} \) in \( \Delta^*(a, \varepsilon) \). By \((*)\) we know:

\[ |f - a| \geq \rho \text{ in } \Delta^*(a, \varepsilon) \Rightarrow \left| \frac{1}{g} \right| \leq \frac{1}{\rho} \text{ in } \Delta^*(a, \varepsilon) \]

Thus \( a \) is removable singularity for \( g \). But

\[ f = \lambda + \frac{1}{g}. \] \((+)\)

If \( a \) is not a zero for \( g \) \Rightarrow \( \frac{1}{g} \) holomorphic \Rightarrow

\( f \) extends holomorphically across \( a \) \Rightarrow removable singularity.

If \( a \) is a zero for \( g \) \Rightarrow \( \frac{1}{g} \) has a pole at \( a \) \Rightarrow

\( \Rightarrow f \) has pole at \( a \).

Both cases are impossible.
Assume a removable singularity.

Theorem

$f$ bounded near $a \Rightarrow \exists M > 0, \varepsilon > 0$ with

$$|f(z)| < M \quad \text{in} \quad \Delta^*(a, \varepsilon)$$

$\Rightarrow f(\Delta^*(a, \varepsilon))$ cannot be dense.

Lemma B

Assume a pole $\Rightarrow \lim_{z \to a} f(z) = \infty$ $\Rightarrow$

$\Rightarrow \exists \varepsilon > 0, \quad |f(z)| \geq 1 \quad \text{in} \quad \Delta^*(a, \varepsilon) \Rightarrow$

$\Rightarrow f(\Delta^*(a, \varepsilon))$ cannot be dense.

Thus $a$ is essential singularity.
Felice Casorati
1835 - 1890

Karl Weierstrass
1815 - 1897

Émile Picard
1856 - 1941
1. Residues (Conway IV. 2)

A singularity for \( f \)

\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - a)^k
\]

Laurent series

\[
a_{-1} = \text{Res} (f, a) = \text{residue}
\]

Problem: Compute \( \text{Res} (f, a) \)

Method 0 Laurent expansion

Example \( f(z) = \frac{2}{\sin^4 z} \), \( \text{Res} (f, 0) = ? \)

\[
\sin z = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots = 2 \left( 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \right)
\]

\[
\sin^4 z = 2^4 \left( 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots \right)^4
\]

\[
= 2^4 \left( 1 - \frac{4 z^2}{6} + \cdots \right)
\]

\[
f(z) = \frac{2}{2^4 \left( 1 - \frac{4 z^2}{6} + \cdots \right)} = \frac{1}{2^3} \cdot \left( 1 + \frac{4 z^2}{6} + \cdots \right)
\]

\[
= \frac{1}{2^3} + \frac{4}{6} \cdot \frac{1}{z^2} + \cdots
\]
\[ \Rightarrow \text{Res} (f, a) = \frac{1}{3}. \]

**Method**

\[ f(z) = \frac{g(z)}{h(z)}, \quad g, h \text{ holomorphic} \]

Assume a simple zero for \( h \Rightarrow \) a simple pole for \( f \).

\[ \text{Res} (f, a) = \lim_{z \to a} (z - a) f(z) \]

\[ = \lim_{z \to a} (z - a) \frac{g(z)}{h(z) - h(a)} \]

\[ = \lim_{z \to a} \frac{g(z)}{h(z) - h(a)} = \frac{g(a)}{h'(a)} \]

**Conclusion:** \[ \text{Res} (f, a) = \frac{g(a)}{h'(a)}. \]
Example: \( f(z) = \frac{2 - \sin z}{z^2 \sin z} \)

- \( \text{poles} \): \( z = 0, \ z = n\pi \), \( n \neq 0, \ n \in \mathbb{Z} \)

\( \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \) \implies \( \frac{\sin z}{z^2} \to 1 \) as \( z \to 0 \)

- \( z = 0 \) is removable since \( \implies \frac{2 - \sin z}{z^3} \to \frac{1}{3!} \) as \( z \to 0 \)

\[
\lim_{z \to 0} \frac{2 - \sin z}{z^2 \sin z} = \lim_{z \to 0} \frac{2 - \sin z}{z^3} \cdot \frac{z^3}{z^2 \sin z} = \frac{\frac{1}{6}}{1} = \frac{1}{6}.
\]

Since \( z = 0 \) is removable \( \implies \text{Res} (f, 0) = 0 \).

- \( z = n\pi \), \( n \neq 0 \). Take \( g(z) = \frac{2 - \sin z}{z^2} \)

\( h(z) = \sin z \)

\( \implies g(n\pi) = \frac{1}{n\pi} \), \( h'(n\pi) = \cos \pi = (-1)^{\pi} \)

\( \implies \text{Res} (f, n\pi) = \frac{g(n\pi)}{h'(n\pi)} = \frac{1}{n\pi} \cdot (-1)^{n} \).
Method 2

\[ f(z) = \frac{g(z)}{(z-a)^k} \implies \text{Res} \left( f, a \right) = \frac{g^{(k-1)}(a)}{(k-1)!} \]

Write

\[ g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!} (z-a)^n \]

coeff. of \((z-a)^{-1}\) in \(f \iff \) coeff. of \((z-a)^{k-1}\) in \(g\).

This equals \( g^{(k-1)}(a) \)

\( \text{Example} \)

\[ f(z) = \frac{z}{(2^2+i)^2} \implies \text{Res} \left( f, i \right) = ? \]

\[ f(z) = \frac{5(z)}{(2-i)^2}, \quad g(z) = \frac{z}{(2+i)^2} \implies g'(i) = 0 \quad \text{(check)} \]

\[ \text{Res} \left( f, i \right) = g'(i) = 0. \]
2. **Residue Theorem** (Conway, V. 2)

**Toy Example**

\[ f: \Delta^* (a, R) \rightarrow \mathbb{C}, \text{ holomorphic.} \]

\[ \Rightarrow \int_{\gamma_s} f(z) \, dz = 2\pi i \, \text{Res} (f, a) \text{, where } \gamma_s = \partial \Delta (a, s). \]

**Proof**

Write\[ f(z) = \sum_{k = -\infty}^{\infty} a_k \, (z-a)^k. \]

This converges uniformly on compact sets, so we can integrate\[ \Rightarrow \int_{\gamma_s} f(z) \, dz = \sum_{k = -\infty}^{\infty} a_k \int_{\gamma_s} (z-a)^k \, dz \]

\[ = 2\pi i \, a_{-1} = 2\pi i \, \text{Res} (f, a). \]

\[ k \neq -1: \quad (z-a)^k \text{ admits a primitive } \frac{(z-a)^{k+1}}{k+1} \Rightarrow \text{zero integral.} \]

\[ k = -1: \quad \int_{\gamma_s} \frac{dz}{z-a} = 2\pi i \, \eta (\gamma_s, a) = 2\pi i. \]
**Residue Theorem**

\[ \text{Let } U \subseteq \mathbb{C} \text{ open and connected, } S \text{ discrete, } \]

\[ u \gg 0 , \{ \gamma \} \subseteq U \setminus S. \]

\[ \text{Then } f \text{ holomorphic in } U \setminus S, \text{ singularities at } S. \]

\[ \frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{s \in S} \text{Res}(f, s) \cdot n(\gamma, s). \]

**Remarks**

1. \[ S = \emptyset \Rightarrow \int_{\gamma} f \, dz = 0 \Rightarrow \text{Cauchy's Theorem} \]

2. \[ S = \{ a \}, \, \gamma = \gamma_r = \text{small circle near } a \Rightarrow \]

recovers the toy example.

3. \[ S = \{ a \}, \, f(z) = \frac{g(z)}{(z-a)^{k+1}}, g \text{ holomorphic, } u \gg 0 \]

\[ \frac{1}{2\pi i} \int_{\gamma} f \, dz = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{(z-a)^{k+1}} \, dz = \text{Res}(g, a) \cdot n(\gamma, a) \]

\[ = \frac{g^{(k)}(a)}{k!} \cdot n(\gamma, a) \text{ by Method 2.} \]

This recovers CIF for derivatives.
The sum in RHS is finite

Claim \( \{ z \in S, \eta(z,s) \neq 0 \} \) finite

Proof \( W = \{ z \in \mathbb{C} \setminus \gamma : \eta(z,s) \neq 0 \} \).

- \( W = \) union of components of \( \mathbb{C} \setminus \gamma = \) open \( \text{ (Lecture 7)} \)
- \( W \) bounded \( \text{ (Lecture 7)} \)
- \( W \subseteq U \). Indeed, if \( z \in W, z \notin U \implies \eta(z,s) \neq 0 \). But

\[
\eta(z,s) = \frac{1}{2\pi i} \int_{\gamma} \frac{ds}{s-z} = 0 \text{ by Cauchy}
\]

using \( s \rightarrow \frac{1}{s-z} \) holomorphic in \( U, \gamma \cup U \).

\( K = W \cup \{ \gamma \} \subseteq U \) closed \& bounded

\( \Rightarrow K \) compact in \( U, \gamma \) discrete in \( U \)

\( \Rightarrow K \cap \gamma = \text{finite} \).
Let \( s = \{ a_1, \ldots, a_k \} \).

Let \( c_i = \partial \Delta_i \) be circles centered at \( a_i \), \( \Delta_i \subseteq \gamma \).

Let \( \overline{\Delta}_i \subseteq \Delta_i \). Let \( u' = U \setminus \bigcup_{i=1}^{k} \overline{\Delta}_i \) = open.

Let \( \gamma = \sum_{i=1}^{k} a_c_i \). Assume we could show \( \gamma \cap \gamma' \) and \( \pi (\gamma, a_i) = 1 \).

Then by Cauchy's applied to \( f/\gamma' \), we'd have

\[
\int_{\gamma} f \, dz = \int_{\gamma} f \, dz = \sum_{i=1}^{k} \int_{c_i} f \, dz
\]

\[
= 2\pi i \sum_{i=1}^{k} \text{Res} (f, a_i) \quad \text{(toy example)}.
\]

\[
= 2\pi i \sum_{i=1}^{k} \text{Res} (f, a_i) \pi (\gamma, a_i).
\]
Issues:

1. \( \gamma \) is not a path, but a chain.

2. \( u' \gamma \sim \gamma \) and \( \eta (\gamma, q) = 1 \) need proofs.

3. How about more complicated curves?

The proof of the residue theorem requires new ideas.

(next time)
Last time we wish to prove:

**Residue Theorem**  
\( u \subseteq \text{an open connected, } S \text{ discrete} \)

1. \( \gamma \sim 0, \{ \gamma \} \subseteq u \setminus S. \)
2. \( f \) holomorphic in \( u \setminus S \), singularities at \( S. \)

Then

\[
\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{s \in S} \text{Res}(f, s) \cdot n(\gamma, s).
\]
Example

\[ \int \frac{2z^2 + 1}{2z^2(2z-1)} \, dz \]

\( |z| = 3 \)

Take \( U = \Delta (0,4), \ S = \{0, 1\}, \ f(z) = \frac{2z^2 + 1}{2z^2(2z-1)} \).

- \( \text{Res} (f, 0) = \text{Res} \left( \frac{2z^2 + 1}{2z^2(2z-1)} \right)_{z=0} = \left( \frac{2z^2 + 1}{2z^2(2z-1)} \right)'_{z=0} = -2 \)

  by Method 2 of computing residues last time

- \( \text{Res} (f, 1) = \text{Res} \left( \frac{2z^2 + 1}{2z^2(2z-1)} \right)_{z=1} = \frac{2z^2 + 1}{2z^2(2z-1)} = \frac{2}{1} = 2 \)

  by Method 1 of computing residues last time.

Thus \( \int f \, dz = 2\pi i \left( \text{Res} (f, 0) + \text{Res} (f, 1) \right) \)

\( |z| = 3 \)

\[ = 2\pi i \left( -2 + 2 \right) = 0. \]
1. Proof of the Residue Theorem

**Terminology**  
\[ U^* \subseteq \sigma, \quad \gamma^* = \sum_{i=1}^{m} m_i \cdot \gamma_i \cdot C' - \text{chain} \]

1. \[ \int f \, dz = \sum_{i=1}^{m} m_i \cdot \int f \, dz \]

2. \[ n (\gamma^*, a) = \sum_{i=1}^{m} m_i \cdot n (\gamma_i, a) \]

**Definition**  
\[ \gamma^* \sim 0 \, \text{ if } \, n (\gamma^*, a) = 0 \quad \forall \, a \notin U^* \]  
(we say \( \gamma^* \) is null homologous in \( U^* \)).

**Remark**  
\[ \gamma^* \quad \text{loop in } \, U^*. \]  
Then  
\[ \gamma^* \sim 0 \quad \implies \quad \gamma^* \sim 0. \]

Indeed, if \( a \notin U^* \), then  
\[ n (\gamma^*, a) = \frac{1}{2\pi i} \int_{\gamma^*} \frac{dw}{w-a} = 0 \]
by homotopy form of Cauchy, applied to \( \gamma^* \sim 0 \) and to the holomorphic function \( \frac{1}{w-a} \) in \( U^* \) (\( a \notin U^* \))
[a] the converse is false \( U^* = \mathcal{C} \setminus \{a, b\} \)

Check \( \gamma^* \cong \mathbb{Z} \). Indeed \( n(\gamma^*, a) = n(\gamma^*, b) = 0 \). To see this,

find two subloops of \( \gamma^* \) going clockwise &

counter-clockwise once around \( a \) (same for \( b \)).

However \( \gamma^* \not\cong \mathbb{Z} \).

Remark* In algebraic topology, one learns that 1st homology is the abelianization of \( \pi_1 \) (which is defined via homotopy).
Enhanced Cauchy's Theorem

We seek to prove a "homology" version of Cauchy:

\[ \text{Theorem } \quad f : U \to \mathbb{C} \text{ holomorphic}, \quad \gamma \circ \eta \Rightarrow \int_{\gamma} f \, dz = 0. \]

Of course, this implies the homotopy version of the theorem, proved in previous lectures.

Remark: We show next

Enhanced Cauchy Theorem \Rightarrow \text{Residue Theorem.}
Proof of residue theorem

We let \( f \) holomorphic in \( U \setminus S \), \( \gamma \circ \partial U \). We want

\[
\frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{s \in S} \text{Res} (f, s) \cdot n (\gamma, s).
\]

Last time we saw RHS is finite since

\[
\{ s \in S : n (\gamma, s) \neq 0 \} \text{ is finite.}
\]

Enumerate this set to be \( \{ a_1, \ldots, a_k \} \), \( m_i = n (\gamma, a_i) \neq 0 \).

Let \( \Delta_i \) be small disjoint discs near \( a_i \). \( \Delta_i \subseteq U \).

Define \( U^* = U \setminus S \).

\[ u^* = u \setminus S \]

\[ v^* = \gamma + \sum_{i=1}^{k} (-m_i) C_i \] where \( C_i = \partial \Delta_i \) (positive orientation).

Claim \( \gamma^* \sim 0 \)

Enhanced Cauchy for \( (u^*, \gamma^*) \) \( \Rightarrow \int_{\gamma^*} f \, dz = 0 \)

\Rightarrow \frac{1}{2\pi i} \int_{\gamma} f \, dz = \sum_{i=1}^{k} m_i \cdot \frac{1}{2\pi i} \int_{C_i} f \, dz

= \sum_{i=1}^{k} m_i \cdot \text{Res} (f, a_i) \text{ by toy example last time. QED.}
Proof of the claim

Want \( n(\gamma^*, a) = 0 \) if \( a \not\in U^* \).

\[ \text{if } a \not\in U. \] Note \( \gamma \sim \alpha \Rightarrow \gamma \not\sim \beta \Rightarrow n(\gamma, a) = 0. \)

Also \( a \not\in \Delta_i \Rightarrow n(c_i, a) = 0 \). Then

\[
n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) n(c_i, a) = 0.
\]

\[ \text{if } a \in S. \] Note that \( n(c_i, a) = \begin{cases} 
0 & \text{if } a \neq a_i; \\
1 & \text{if } a = a_i.
\end{cases} \)

If \( a = a_i \): \( \Rightarrow n(\gamma^*, a) = n(\gamma, a) + (-m_i) n(c_i, a) = m_i + (-m_i) = 0. \)

If \( a \neq a_i \): \( \Rightarrow n(\gamma^*, a) = 0 \) by definition of the \( a_i \)'s

\[ \Rightarrow n(\gamma^*, a) = n(\gamma, a) + \sum (-m_i) n(c_i, a) = 0. \]
Remarks

11 Proof of residue thm only requires $\gamma \neq 0$, not $\gamma^u$.

$\gamma \neq 0$. ~ improvement of hypothesis.

\[ \text{Residue Theorem } \Rightarrow \text{Enhanced CIF for derivates.} \]

Let $\gamma \neq 0$. Apply the residue theorem:

\[ s = \{ a \} \]

\[ \frac{1}{2\pi i} \int \frac{f(z)}{y \cdot (z - a)^{k+1}} \, dz = \operatorname{Res}_{\gamma=a} \frac{f(z)}{(z-a)^{k+1}} \]

\[ = n(\gamma, a) \cdot \frac{f^{(k)}(a)}{k!} \]

(using Method 2 from last time)
2. Proof of Enhanced Cauchy's Theorem

- change notation $u \leftrightarrow u^*$, $\gamma \leftrightarrow \gamma^*$
- modify statement slightly

**Theorem (enhanced CIF)**

$$\gamma \approx 0, \ f: u \rightarrow a \text{ holomorphic}, \ a \in U \setminus \{a\}$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} \, dz = n(\gamma, a) \cdot f(a).$$

**Remark** Using the above, for $f^{(2)}(a) = f'(2), (2-a), f^{(2)}(a) = 0$

we obtain $\gamma \approx 0 \Rightarrow \int f \, dz = 0$. This is **Enhanced Cauchy**.

**Remark** TFAE:

- Enhanced CIF $\Rightarrow$ Enhanced Cauchy's Theorem above
- $\Rightarrow$ Residue Theorem pages 8
- $\Rightarrow$ Enhanced CIF for derivatives pages 9
The proof of Residue thm / Enhanced Cauchy requires:

**Theorem (enhanced CIF / Conway IV. 5)**

\[ \gamma \ni z, \quad f: u \rightarrow a \text{ holomorphic, } a \in U \setminus \{z\} \]

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \, dw = \eta(\gamma, a) \cdot f(a). \]

Rewriting:

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \, dw = \frac{1}{2\pi i} \int_{\gamma} \frac{f(a)}{w-a} \, dw \]

\[ \Leftrightarrow \int_{\gamma} \frac{f(w) - f(a)}{w-a} \, dw = 0 \]

\[ \gamma(\eta, w) \]
Auxiliary function \( \varphi : U \times U \to \mathbb{C} \)

\[
\varphi(z, w) = \begin{cases} 
\frac{f(z) - f(w)}{z - w}, & z \neq w, \\
 f'(z), & z = w.
\end{cases}
\]

Want: \( \int_{\gamma} \varphi(z, w) \, dw = 0 \quad \forall \gamma \in U \) \( \text{(*)} \)

Apply \( \text{(*)} \) to \( \gamma = \partial U \setminus \{y\} \) to conclude \text{enhanced CIF}.

Claims

1. \( \varphi \) continuous in \( U \times U \)

2. \( z \mapsto \varphi(z, w) \) holomorphic \( \forall w \in U \) fixed.

Proof of 2. This was explained in Lecture 13 as an application of \text{Removable Singularity Theorem}. 

Proof of 1. This was explained in Lecture 13 as an application of \text{Removable Singularity Theorem}. 

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Proof of 1. This was explained in Lecture 13 as an application of \text{Removable S
Proof of II. \( \gamma \) contnuous in \( U \times U \). Recall

\[
\gamma(z, w) = \begin{cases} 
\frac{f(z) - f(w)}{z - w}, & z \neq w, \\
f'(z), & z = w.
\end{cases}
\]

Continuity is clear at points where \( z \neq w \).

We show continuity at \( (a, a) \). We have

\[
\left| \gamma(z, w) - \gamma(a, a) \right| = \left| \frac{1}{w - z} \int_2^w f'(t) \, dt - f'(a) \right|
\]

\[
= \frac{1}{|w - z|} \left| \int_2^w f'(t) - f'(a) \, dt \right|
\]

\[
\leq \sup_{t \in [2, w]} |f'(t) - f'(a)| < \varepsilon
\]

if \( z, w \in \Delta(a, \delta) \).

This holds in \( \Delta(a, \delta) \) for some \( \delta \), because \( f' \) is continuous (in fact holomorphic).
**Proof of (x)**  \[
\text{Want } \int_{\gamma} y(2,w) \, dw = 0 \text{ if } \gamma \approx u.
\]

**Question**: How do we make use of \( \gamma \approx u \)?

**Answer**: Define

\[
V = \left\{ z \in \mathcal{C} \setminus \gamma, \; n(y, z) = 0 \right\}.
\]

- \( U \cup V = \mathcal{C} \). (this is the only place where \( \gamma \approx u \) is used).

Indeed if \( z \notin U \Rightarrow n(y, z) = 0 \) since \( \gamma \approx u \). Also \( z \in \mathcal{C} \setminus \gamma \).

- \( V \) open. Indeed, by Lecture 7, \( V \) is union of components of \( \mathcal{C} \setminus \{ y \} = \text{open} \Rightarrow V \text{ open} \).

- \( V \) unbounded. In fact, by Lecture 7, \( \exists R > 0 \) with \( \{ |z| > R \} \subseteq V \).
Define \( h : \mathbb{C} \rightarrow \mathbb{C} \)

\[
h(z) = \begin{cases} 
\int \varphi(z, w) \, dw, & z \in U \\
\int \frac{f(w)}{w - z} \, dw, & z \in V
\end{cases}
\]

Claim \( \square \) \( h \) well-defined

Claim \( \square \) \( h \) bounded, \( \lim_{x \to \infty} h(z) = 0 \)

Claim \( \square \) \( h \) entire

Conclusion By Liouville \( \Rightarrow h \) constant \( \Rightarrow h \equiv 0 \).

Thus if \( z \in U \Rightarrow h(z) = \int \varphi(z, w) \, dw = 0 \Rightarrow (*) \). Q.E.D.

Proof of \( \square \) \( h \) well-defined. Take \( z \in U \cap V \). We show

\[
\int \varphi(z, w) \, dw = \int \frac{f(w)}{w - z} \, dw.
\]

\(<\Rightarrow>\) \( \int \frac{f(z)}{w - z} \, dw = 0 <\Rightarrow> f(z) \cap (z, \infty) = 0 \) which is true since \( \cap (z, \infty) = 0 \) for \( z \in V \).
Proof of IBI

Let \( K > 0 \) such that \( \{ \gamma \} \subseteq \Delta (0, K) \) by compactness.

We have \( |w - 2| \geq 1 - 1 |w| \geq 1 - K \) if \( w \in \{ \gamma \} \).

If \( R > K \), \( |z| \leq R \Rightarrow 2 > e \). Then

\[
|z| = \int \frac{f(w)}{w - z} \, dw \leq \text{length}(\gamma) \cdot \sup_{\{ \gamma \}} \left| \frac{1}{w - z} \right| \frac{1}{12 - K}
\]

\[
\text{constants as } z \to \infty.
\]

Since \( h \) is continuous by \( \square \) \( \Rightarrow h \) bounded.

Why?

\[
\lim_{z \to \infty} h(z) = 0 \Rightarrow 3 \alpha, \quad 1 |h(z)| \leq 1 \quad \text{if } 12 \geq \alpha
\]

\[
|h| \text{ continuous } \Rightarrow 3M, \quad 1 |h(z)| \leq M \quad \text{if } 12 \leq \alpha
\]

\[
\Rightarrow 1 |h| \leq \max (1, M).
\]
Proof of \( \mathbb{C} \) to entire

Recall Conway Exercise IV. 2.2. HWK 3 #7.

**Key statement** \( \psi: U \times \{z\} \to \mathbb{C} \)

. \( \psi \) continuous

. \( z \to \psi(z,w) \) holomorphic \( \forall w \in \{z\} \).

Then \( g(z) = \int \psi(z,w) \, dw \) holomorphic.

**Proof** See Solution Set 3.

Alternatively, let \( R \subseteq U \). Then

\[
\int_R g \, dz = \int_R \int_y \psi(z,w) \, dw \, dz
\]

\( \xrightarrow{\text{Fubini's theorem}} \)

\( \xrightarrow{\psi \text{ continuous}} \)

\( \xrightarrow{\text{Goursat's lemma or Cauchy}} \)

\( = \int_y 0 \, dw = 0 \)

\( \implies g \) admits a primitive in any disc \( \Delta \subseteq U \), \( g = c' \)

\( \implies g \) holomorphic (\( c = \) holomorphic \( = \infty \) many times differentiable)
Back to [C]. Apply Key Statement to

- the set $U$, for $w = \phi \Rightarrow h$ holomorphic in $U$

- the set $V$, for $w(z, w) = \frac{f(z)}{w - z} : V \times \{z\} \rightarrow \mathbb{C}$

$\Rightarrow h$ holomorphic in $V$.

Thus $h$ is entire. QED.
2. Applications of the Residue Theorem to real analysis

\[ \frac{1}{2\pi i} \int \frac{f}{z} \, dz = \sum \text{Res} (f, s) \cdot n (\gamma, s), \quad \gamma \approx 0. \]

Applications

1. Trigonometric functions
2. Rational functions
3. Fourier integrals
4. Logarithmic integrals
5. Mellin transforms

Poisson: "Je n'ai remarqué aucune intégrale qui ne fût pas déjà connue"
Example \( a > 1, \ \text{a} \in \mathbb{R} \). \( I = \int_0^{2\pi} \frac{dt}{a + \sin t} \).

\[ z = e^{it} \Rightarrow \frac{dz}{it} = dt \]

\[ \sin t = \frac{2 - e^{-z}}{2i} \]

By substitution, we find

\[ I = \int \frac{2 \, dz}{z^2 - 2 + 2ai} \]

poles \( z^2 - 2 + 2ai = 0 \)

\[ \Rightarrow z = \frac{1}{a} \pm i \sqrt{a^2 - 1} \]

Note \( |z^+| < 1, \ |z^-| > 1 \). Thus

Method 1

\[ I = 2\pi i \quad \text{Res} \left( f, z^+ \right) = 2\pi i \cdot \frac{2}{(z^2 - 2 + 2ai)' z = z^+} \]

\[ = 2\pi i \cdot \frac{2}{2z + 2ai} \bigg|_{z = z^+} = \frac{2\pi}{\sqrt{a^2 - 1}} \]
Office hour next week: Tuesday 2-3:30 PM.

Applications of the Residue Theorem to real analysis

1. Trigonometric functions
2. Rational functions
3. Fourier integrals
4. Logarithmic integrals
5. Mellin transforms
Rational functions 

\[ I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx. \]

**Require:**
1. \( Q \) has no zeroes on the real axis.
2. \( \deg P - \deg Q \leq -2. \)

**Claim:** The integral converges absolutely.

Write \( f(x) = \frac{P(x)}{Q(x)} \).

By (2) \( \lim_{|x| \to \infty} x^2 f(x) = a \) \( \Rightarrow \exists R > 0 \) with \( f(x) < \frac{a+1}{x^2} \) for \( |x| > R \). \( * \)

By the comparison test \( \Rightarrow \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \). QED.

**Conclusion** \( I = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, dx. \)
**Strategy**

\[ f(z) = \frac{P(z)}{Q(z)} \]

\[ \gamma_R = [-R, R] \cup S_R. \]

**Residue Theorem**

\[ \int_{-R}^{R} f(x) \, dx + \int_{S_R} f \, dz = \int_{\gamma_R} f \, dz = 2\pi i \sum_{a_j \in \mathbb{Z}^+} \text{Res} \left( \frac{f}{a_j}, a_j \right). \]

**Make** \( R \to \infty. \) **Show** \( \lim_{R \to \infty} \int_{S_R} f \, dz = 0. \)

\[ \left| \int_{S_R} f \, dz \right| \leq \pi R \cdot \frac{2 + 1}{R^2} \to 0 \text{ as } R \to \infty, \text{ using } (\ast). \]

From \((\ast)), \text{ we obtain} \]

**Conclusion**

\[ \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx = 2\pi i \sum_{a_j \in \mathbb{Z}^+} \text{Res} \left( \frac{P}{Q}, a_j \right). \]
Example

\[ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} \]

Poles at \( z^4 + 1 = 0 \)

\[ 2^k = e^{\frac{\pi i}{4}(2k + 1)}, \quad k = 0, 1, 2, 3. \]

Only \( e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}} \in \mathbb{P}^{+} \).

By Method 1,

\[ \text{Res } \frac{1}{z^4 + 1} = \frac{1}{4} 2^3 / 2 = 2_k, \quad 2 = 2_k. \]

Thus

\[ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = 2\pi i \left( \text{Res } \frac{1}{z^4 + 1} + \text{Res } \frac{1}{z^4 + 1} \right) \]

\[ = 2\pi i \left( -\frac{1}{4} e^{\frac{\pi i}{4}} - \frac{1}{4} e^{\frac{3\pi i}{4}} \right) \]

\[ = \frac{\pi i}{\sqrt{2}}. \]
\[ I = \int_{-\infty}^{\infty} f(x) \ e^{ix} \, dx \quad \text{(use upper half plane)} \]

\[ I = \int_{-\infty}^{\infty} f(x) \ e^{-ix} \, dx \quad \text{(use lower half plane)} \]

**Require**

1. \( f \) extends meromorphically to \( \mathbb{C}^+ \)
2. No poles on the real axis.
3. \( \int_{-\infty}^{\infty} |f(x)| \, dx < \infty \)

**Convergence:** By (3), \( \int_{-\infty}^{\infty} f(x) \ e^{ix} \, dx \) converges absolutely.

Thus

\[ I = \lim_{R \to \infty} \int_{-R}^{R} f(x) \ e^{ix} \, dx. \]
Strategy: Use the same contour

\[ \gamma_R = [-R, R] \cup S_R. \]

By the residue theorem

\[ \int_{-R}^{R} f(x) e^{ix} \, dx + \int_{S_R} f(z) \, dz = \int_{S_R} f(z) \, dz = 2\pi i \sum \text{Res} (f(z) e^{iz}) \]

Make \( R \to \infty \). Assume moreover

\[ \lim_{z \to \infty} f(z) = 0. \]

The next lemma shows \( \lim_{R \to \infty} \int_{S_R} f(z) \, dz = 0. \)

Conclusion

\[ \int_{-\infty}^{\infty} f(x) e^{ix} \, dx = 2\pi i \sum_{q_j \in \mathbb{Z}} \text{Res} (f(z) e^{iz}) \]

\[ f(x) \in \mathbb{R}. \]
Lemma \[ f \left( \lim_{z \to 0} \frac{f(z)}{z^3} \right) = 0 \] then
\[
\lim_{R \to \infty} \int_{S_R} f(z) e^{iz} \, dz = 0
\]

Proof

Write \( \theta = R \cos t \), \( 0 \leq t \leq \pi \).

\[
M_R = \sup_{z \in S_R} |f(z)|, \quad M_R \to 0 \quad \text{as} \quad R \to \infty
\]

\[
\left| \int_{S_R} f(z) e^{iz} \, dz \right| = \left| \int_{0}^{\pi} f(Re^{it}) e^{iRe^{it}} \, dt \right|
\]

\[
\leq \int_{0}^{\pi} M_R \left| e^{iRe^{it}} \right| \cdot R \, dt
\]

\[
= \int_{0}^{\pi} R M_R \left| e^{iR \cos t} - Re^{iR \sin t} \right| \, dt
\]

\[
= \int_{0}^{\pi} R M_R \left| e^{-Re^{iR \sin t}} \right| \, dt
\]

\[
= 2 \int_{0}^{\pi/2} R M_R \left| e^{-Re^{iR \sin t}} \right| \, dt
\]

Claim

\[
\leq 2 \int_{0}^{\pi/2} R M_R e^{-R \cdot \frac{2}{\pi} t} = \frac{\pi}{2} M_R (1 - e^{-R}) \to 0
\]

as \( R \to \infty \).
Claim \[ \frac{2}{\pi} \leq \frac{\sin t}{t} \quad \forall \ t \in \left(0, \frac{\pi}{2}\right) \]

Proof \[ f(t) = \frac{\sin t}{t} \]
\[ f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \]

We show \( f \) is decreasing. Then \[ f(t) \geq f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \Rightarrow \]
\[ \Rightarrow \frac{\sin t}{t} \geq \frac{2}{\pi} \]

To this end, compute \( f'(t) = \frac{t \cos t - \sin t}{t^2} \leq 0 \)
\[ \iff t \cos t \leq \sin t \]
\[ \iff \tan t - t \geq 0. \]

Let \[ g(t) = \tan t - t, \quad g(0) = 0 \]

We compute \( g'(t) = \frac{1}{\cos^2 t} - 1 \geq 0 \Rightarrow g \uparrow \Rightarrow \)
\[ \Rightarrow g(t) \geq g(0) = 0 \text{ as needed. } \text{Q.E.D} \]
Example: \[ \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx = \text{Re} \, I. = \frac{\pi}{e}. \]

Let \( f(z) = \frac{1}{1 + z^2} \), \( z = i \) is the only pole in \( \mathbb{C} \).

\[ I = \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx = 2\pi i \cdot \text{Res} \left( \frac{e^{iz}}{1 + z^2} \right) \]

\[ = 2\pi i \cdot \frac{e^{i^2}}{i^2} \Bigg|_{z = i}. \]

\[ = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}. \]
Example $\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$

- Issues at $0$ & $\infty$.

$I = \lim_{R \to \infty} \int_0^R \frac{\sin x}{x} \, dx$

- $Y_R = S_R + [-R, -r] + [-3r] + [r, R]

$0 = \int f \, d\lambda = \int f \, d\lambda - \int f \, d\lambda + \int f \, d\lambda$

$= \int f \, d\lambda - \int f \, d\lambda + \int f \, d\lambda$

$= \int f \, d\lambda - \int f \, d\lambda + \int f \, d\lambda$

Make $r \to 0$, $R \to \infty$. By the claim:

$0 = 0 - i\pi + 2i \int_0^\infty \frac{\sin x}{x} \, dx$ $\Rightarrow$ $\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}$. 
Claim (a) follows from the previous Lemma applied to

\[ f(x) = \frac{1}{2} \]

Claim (b) requires a proof. We will go over the proof next time.
Example \( I = \int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2} \)

\[
\gamma = S_R + [-R, -r] + (-S_r) + [r, R]
\]

\[
\oint_{S_R} f(z) \, dz = \int_{S_R} f(z) \, dz + \int_{S_R} f(z) \, dz + \int_{S_R} f(z) \, dz = \int_{S_R} f(z) \, dz = 0
\]

\[
\text{as } r \to 0, R \to \infty.
\]

Claim 1: \[
\lim_{R \to \infty} \int_{S_R} \frac{e^{iz}}{z^2} \, dz = 0
\]

\[
\Rightarrow I = \frac{\pi}{2}.
\]

Claim 2: \[
\lim_{r \to 0} \int_{S_r} \frac{e^{iz}}{z^2} \, dz = i\pi
\]
Part \( \text{IAI} \) is a consequence of Lemma last time:

For \( g(z) = \frac{1}{z} \):

**Lemma**

If \( \lim_{z \to \infty} \left| g(z) \right| = 0 \) then

\[
\lim_{R \to \infty} \int_{S_R} g(z) e^{iz} \, dz = 0
\]

Part \( \text{II} \) uses the next lemma for \( g(z) = \frac{1}{z} \):

**Lemma**

Let \( g \) have simple pole at \( 0 \). Then

\[
\lim_{r \to 0} \int_{S_r} g(z) e^{iz} \, dz = \pi i \cdot \text{Res}(g, 0).
\]
Proof. Since $g$ has a simple pole at 0, write

$$g(z) = \frac{\alpha}{z} + G(z)$$

Taylor series

$$\alpha = \text{Res}_0 (g, 0), \ G \ \text{holomorphic near } 0$$

Taylor

$$e^{i \alpha} = 1 + \mathbb{R} F(\alpha), \ F \ \text{holomorphic near } 0$$

$$e^{i \alpha} g(z) = \left( \frac{\alpha}{z} + G \right) \left( 1 + \mathbb{R} F \right) = \frac{\alpha}{z} + H,$$

$$H = G + \mathbb{R} F G + \alpha F \ \text{holomorphic near } 0$$

$$\Rightarrow H \ \text{bounded near } 0 . \Rightarrow \exists M, \delta : |H(z)| \leq M \ \text{if } |z| \leq \delta .$$

Compute

$$\int_{S_r} e^{i \alpha} g(z) \, dz = \int_{S_r} \frac{\alpha}{z} + H \, dz.$$ 

Note

$$\alpha \int_{S_r} \frac{dz}{z} = \alpha \int_0^\pi \frac{d(r e^{i \theta})}{r e^{i \theta}} = \alpha \int_0^\pi i \, d\theta = \pi i \cdot \alpha$$

$$\left| \int \frac{H}{S_r} \, dz \right| \leq M, \ \pi r \to 0 \ \text{as } r \to 0.$$

Thus

$$\int_{S_r} e^{i \alpha} g(z) \, dz \to \pi i \cdot \alpha \ \text{as } r \to 0 , \ \text{as claimed.}$$
Applications of the Residue Theorem to real analysis

- Trigonometric functions
- Rational functions
- Fourier integrals
- Logarithmic integrals
- Mellin transforms
Logarithmic integrals

\[ \int_0^\infty R(x) \log x \, dx \]

\( R = \text{even rational function, without real poles} \)

**Example**

\[ R(x) = \frac{1}{1 + x^2} \implies \int_0^\infty \frac{\log x}{1 + x^2} \, dx = 0. \]

**HWK**

\[ R(x) = \frac{1}{(1 + x^2)^2} \implies \int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx. \]

**Issues:**
- logarithm undefined at 0 (use circle \( S_r \))
- holomorphic extension for logarithm

Define for \( z = r e^{i\theta} \)

\[ \log(z) = \log r + i\theta \]

\[ -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \]
\[
\gamma = S_R + [-R, -r] + (-r, r) + [r, R]
\]

\[
f(z) = \frac{l(z)}{1+z^2}
\]

has poles at \(i\) and \(-i\).

**Residue Theorem**

\[
\text{Res}(f, i) = \frac{l(2)}{2} \left| \frac{z-i}{z+i} \right|_{z=i} = \frac{i\pi/2}{2i} = \frac{\pi}{4}.
\]

\[
\text{Residue thm: } \int_{\gamma} f(z) \, dz = 2\pi i \text{ Res}(f, i) \quad (\star).
\]

\[
\int_{S_R} f(z) \, dz = \int_{S_r} f(z) \, dz + \int_{r}^{\infty} f(x) \, dx + \int_{-r}^{r} f(x) \, dx.
\]

We make \(r \to 0, R \to \infty\).
Segment integrals

\[ \int_{-R}^{R} \frac{l(x)}{1 + x^2} \, dx + \int_{-R}^{-r} \frac{l(x)}{1 + x^2} \, dx = \int_{r}^{R} \frac{\log x}{1 + x^2} \, dx + \int_{-r}^{r} \frac{\log (x) + i \pi}{1 + x^2} \, dx \]

\[ = 2 \int_{r}^{R} \frac{\log x}{1 + x^2} \, dx + i \pi \int_{-r}^{r} \frac{dx}{1 + x^2} \]

\[ \lim_{r \to 0} \left( \lim_{R \to \infty} \right) 2 \int_{r}^{R} \frac{\log x}{1 + x^2} \, dx + i \pi \arctan x \bigg|_{x = 0}^{x = -\pi} \]

\[ = 2 \int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx + i \pi \cdot \frac{\pi}{2} \]

Claim: \[ \lim_{\rho \to 0} \int_{-\rho}^{\rho} \frac{l(x)}{1 + x^2} \, dx = 0. \]

Conclusion: From (4) we get as \( r \to 0, R \to \infty \):

\[ \frac{i \pi^2}{2} = 2 \int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx + i \pi \cdot \frac{\pi}{2} \]

\[ \Rightarrow \int_{0}^{\infty} \frac{\log x}{1 + x^2} \, dx = 0 \]
Proof of the claim \( z = \rho \ e^{it}, \ 0 \leq t \leq \pi \)

\[
\left| \int_{\rho}^{1} \frac{d\rho}{1 + 2^2 \rho^2} \right| = \left| \int_{0}^{\pi} \frac{\log\rho + it}{1 + \rho^2 e^{2it}} \cdot \rho \ e^{it} \ dt \right|
\]

\[
\leq \int_{0}^{\pi} \frac{1 \log\rho + \pi}{1 + \rho^2 e^{2it}} \cdot \rho \ dt
\]

\[
\leq \int_{0}^{\pi} \frac{1 \log\rho + \pi}{1 + \rho^2} \cdot \rho \ dt
\]

\[
= \pi \cdot \frac{\rho \log\rho}{\rho^2 - 1} + \pi^2 \cdot \frac{\rho}{\rho^2 - 1} \rightarrow 0 .
\]

As \( \rho \rightarrow \infty \), \( \frac{\rho \log\rho}{\rho^2 - 1} \) and \( \frac{\rho}{\rho^2 - 1} \) \( \rightarrow 0 \).

As \( \rho \rightarrow 0 \), the same is true.

The only term that requires justification is

\[
\rho \log\rho = -\frac{\nu}{e^\nu} \rightarrow 0 \ \text{as} \ \nu \rightarrow \infty, \ \text{where} \ \rho = e^{-\nu}, \ \rho \rightarrow 0 .
\]
Applications of the Residue Theorem to real analysis

- trigonometric functions
- rational functions
- Fourier integrals
- logarithmic integrals
- Mellin transforms
Mellin transforms

\[ \int_0^\infty \frac{R(x)}{x^\alpha} \, dx \quad , \quad 0 < \alpha < 1 \]

\( R = \text{rational function, no poles on positive real axis} \)

Useful in prime counting.

**Example**

\[ R(x) = \frac{1}{x+1} \Rightarrow \int_0^\infty \frac{1}{x^\alpha (x+1)} \, dx = \frac{\pi}{\sin \pi \alpha} \]

(next time)

**Homework**

\[ R(x) = \frac{1}{x^n+1} \Rightarrow \int_0^\infty \frac{1}{x^\alpha (x^n+1)} \, dx \]
Remark

**Fourier transform**

\[ f \rightarrow \hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} \, dx \]

**Laplace transform**

\[ f \rightarrow \mathcal{L}[f](s) = \int_{0}^{\infty} f(x) e^{-s x} \, dx \]

**Mellin transform**

\[ f \rightarrow M[f](s) = \int_{0}^{\infty} f(x) x^{s-1} \, dx \]

Remark (will not use)

The Mellin transform of \( f(x) = e^{-x} \) is known as the \( r \)-function.

\[ r(s) = \int_{0}^{\infty} x^{-s} e^{-x} \, dx \]
Hjalmar Mellin (1854 – 1933)

Finnish mathematician
Applications of the Residue Theorem to real analysis

- Trigonometric functions
- Rational functions
- Fourier integrals
- Logarithmic integrals
- Mellin transforms

Mellin transforms: \[ \int_0^\infty \frac{R(x)}{x^a} \, dx \]

\( R = \text{rational function, no poles on positive real axis} \)
Example \( R(x) = \frac{1}{x^\alpha + 1} \) \implies \int_0^\infty \frac{dx}{x^\alpha (x+1)} = \frac{\pi}{\sin \pi \alpha}

for \( 0 < \alpha < 1 \)

Homework \( R(x) = \frac{1}{x^\alpha + 1} \) \implies \int_0^\infty \frac{dx}{x^\alpha (x+1)}

Convergence uses \( 0 < \alpha < 1 \).

\( 0 < \alpha < 1 \) : \int_0^1 \frac{dx}{x^\alpha (x+1)} < \int_0^1 \frac{dx}{x^\alpha} = \frac{x}{-\alpha} \Big|_0^1 < \infty

\( 1 < \alpha < \infty \) : \int_1^\infty \frac{dx}{x^\alpha (x+1)} < \int_1^\infty \frac{dx}{x^\alpha} = \frac{x}{-\alpha} \Big|_1^\infty < \infty

\( I = \int_0^\infty \frac{dx}{x^\alpha (x+1)} = \lim_{r \to 0} \int_r^\infty \frac{dx}{x^\alpha (x+1)} \).
\[ I = \int_0^\infty \frac{dx}{x^\alpha (x+1)} \cdot \]

**Question:**
- What function?
- What contour?

**Issues**
- Extend \( x^\alpha \) holomorphically

\[ Z^\alpha = \exp (\alpha \ell(z)) \to \text{branch cut along } [0, \infty). \]

For \( z = r e^{i\theta} \), \( \ell(z) = \log r + i\theta \), \( 0 < \theta < 2\pi \)

- Pole at 0 — use \( C_r \) to isolate the pole

**Remark**
It is precisely the fact that we cut along \([0, \infty)\). 

\( (= \text{domain of integration}) \) that allows us to calculate \( I \).

Before, we were cutting away from domain of integration.
Solutions

(a) \( f(z) = \frac{1}{z^2(2+z)} \)

(b) \( \gamma = \text{key-hole contour} \)

\[ \gamma = C_R + (-L^-) + (-C_R) + L^+ \]
**Residue Theorem**

\[ f(z) = \frac{1}{z^2(z+1)} \]

*Pole of -1.*

\[ \text{Res} (f, -1) = \frac{1}{\text{Res} \frac{1}{z^2(z+1)} \bigg|_{z=-1}} = \frac{1}{(-1)^2} = \frac{1}{e^{-\pi i \alpha}} = e^{-\pi i \alpha} \]

\[ \int_{\gamma} f(z) \, dz = 2\pi i \quad \text{Res} (f, -1) = 2\pi i \cdot \exp(-\alpha \pi i). \]

\[ \int_{c_R} f(z) \, dz = \int_{c_R} f(z) \, dz + \int_{L^+} f(z) \, dz - \int_{L^-} f(z) \, dz \]

*Make* \( r \to 0, \quad R \to \infty, \quad \delta \to 0. \)
Claim 1 \[ \lim_{\rho \to 0} \int_{\mathbb{R}} \frac{d\rho}{2^\rho (2^\rho + 1)} = 0 \]

Claim 2 \[ \lim_{\gamma \to 0} \int_{L^+} \frac{d\gamma}{2^\gamma (2^\gamma + 1)} = R \]

Claim 3 \[ \lim_{\gamma \to 0} \int_{L^-} \frac{d\gamma}{2^\gamma (2^\gamma + 1)} = C - 2\pi i \alpha \]

Conclude \( \text{In (R)} \) make \( \delta \to 0, r \to 0, R \to \infty \):

\[ 0 - 0 + 1 - e^{-2\pi i \alpha} = 1 - e^{-\pi i \alpha} - 2\pi i \]

\[ I = \frac{2\pi i e^{-\pi i \alpha}}{1 - e^{-2\pi i \alpha}} = \frac{2\pi i}{e^{\pi i \alpha} - e^{-\pi i \alpha}} = \frac{\pi}{\sin \pi \alpha} \]
Proof of (10)

\[ \int_{L} \frac{d^2}{2 \pi (2+1)} \leq 2 \pi \rho \cdot \frac{1}{\rho_{r+1/\rho-1}} \rightarrow 0 \]

as \( \rho \rightarrow 0 \) or \( \rho \rightarrow \infty \), because \( 0 < \alpha < 1 \).

Proof of (15)

\[ g(z) = \frac{1}{z}, \quad L^+ = \{ t+i\delta : r \leq t \leq R \} \]

\[ \lim_{\delta \rightarrow 0} \int_{L^+} \frac{g(z)}{z+1} dz = \int_{R} \frac{t-\alpha}{t+1} dt \rightarrow I. \]

as \( t \rightarrow 0 \) \( R \rightarrow \infty \).

Why (17)?

\[ \int_{L^+} \frac{g(z)}{z+1} dz = \int_{R} \frac{g(t+i\delta)}{1+t+i\delta} dt \rightarrow 0, \quad \delta \rightarrow 0. \]

Because

\[ \phi(t, \delta) = \begin{cases} \frac{g(t+i\delta)}{1+t+i\delta} - \frac{t-\alpha}{1+t}, & \delta \neq 0, \\ 0, & \delta = 0. \end{cases} \]

\( r \leq t \leq R, \quad 0 \leq \delta \leq 1. \)
Given any $E, F > 0$ such that if

$$|\delta - 0| < \varepsilon, \quad |t - t_0| < \varepsilon \Rightarrow |G(t, \delta) - G(t_0, 0)| < \varepsilon$$

and

$$|G(t, \delta)| < \varepsilon, \quad \Rightarrow \int_{R}^{\infty} \left| G(t, \delta) \right| dt \leq (R-r) \varepsilon$$

as $181 < 2$.

**Remark** If we wish to parameterize $L^+$ by $r \leq t \leq R$, we’d need to use circles $C_r, C_\varepsilon$ of radii:

$$R^* = \sqrt{R^2 + \delta^2}, \quad r^* = \sqrt{r^2 + \varepsilon^2}.$$

The argument in [14] still applies since $r^* \to 0, R^* \to \infty$.

As $r \to 0, \delta \to 0, R \to \infty$. 
Proof of (2)

Difference

\[ g(t - i \delta) \rightarrow t^{-\alpha} e^{-2\pi i \delta}. \]

The rest of the proof is the same as [16].

Indeed

\[ g(t - i \delta) = (t - i \delta)^{-\alpha} = \exp(-\alpha \log(t - i \delta)). \]

\[ \delta \rightarrow \exp(-\alpha \log t - 2\pi i \delta) = t^{-\alpha} e^{-2\pi i \delta}. \]

This explains the extra factor \( e^{-2\pi i \delta} \) in the answer to [16].
Residues at $\infty$ and Shadows of Riemann Surfaces

[1A]: Topology on $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$

- basic neighborhood of $\infty$
  
  \[ U = \{ \infty \} \cup \{ \{ z \in \mathbb{R} \mid |z| > R \} \} \text{ for some } R. \]

- $\hat{\mathbb{C}}$ is a topological space

- $\hat{\mathbb{C}}$ compact

Remark $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $z \to \frac{1}{z}$, $f(0) = \infty$, $f(\infty) = 0$

(punctured) neighborhoods of $0$, $1/2 < r$

(punctured) neighborhoods of $\infty$

\[ \left| \frac{1}{z^2} \right| > \frac{1}{r} \]
Singularities \& Residues at $w$

Recall Conway V. 1.13 - Post 5 / Problem 6

If $f: \{ |z| > R \} \to \mathbb{C}$ holomorphic

$\Rightarrow$ $w$ is isolated singularity

Types

- removable

- pole $\Rightarrow g(z) = f \left( \frac{1}{z} \right)$.

- essential Inspect singularity at $0$.

Example

$f(z) = \frac{z^5 + z}{z - 1} \Rightarrow$ poles at $1, \infty \in \mathbb{C}$.

$g(z) = f \left( \frac{1}{z} \right) = \frac{\frac{1}{z^5} + \frac{1}{z}}{\frac{1}{z} - 1} = \frac{1 + 2z^5}{1 - z} \cdot \frac{1}{z^4}$ pole at $z = 0$

$\Rightarrow f$ has a pole at $0$. 
Residue at $\rho$  \( \text{Res} (f, \rho) = ? \)

**Beware**

\( \text{Res} (f, \infty) \neq \text{Res} (g, 0). \)

**Instead**

\[
\text{Res} (f, \infty) := -\frac{1}{2\pi i} \int_{|z|=\rho} f \, dz \quad \text{where } \rho > R.
\]

By Homotopy Cauchy this does not depend on $\rho$.

**Question**

Why do we care about the residue at $\infty$?

**Homework Example**

\[
\int_{|z|=5} \frac{z^3}{(1 - z)(2 - z)(3 - z)(4 - z)} \, dz = -2\pi i \, \text{Res} \left( \frac{z^3}{(1 - z)(2 - z)(3 - z)(4 - z)}, \infty \right).
\]

This is better than computing 4 different residues.
Next time we will answer the following:

**Question** How do we calculate the residue at \( z_0 \)?
Math 220 A - Lecture 20

November 30, 2020
**Last time** \(- \text{Residue at } \infty \)

If \( f: \{ |z| > R \} \to \mathbb{C} \) holomorphic, \( \infty \) isolated singularity

\[ g: \Delta^*(0, \frac{1}{R}) \to \mathbb{C}, \quad g(z) = f\left(\frac{1}{z}\right), \quad \text{isolated singularity} \]

**Beware**

\[ \text{Res} (f, \infty) \neq \text{Res} (g, 0). \]

Instead define

\[ \text{Res} (f, \infty) : = - \frac{1}{2 \pi i} \int_{|z|=\rho} f(z) \, dz \quad \text{where } \rho > R. \]

By Homotopy Cauchy this does not depend on \( \rho > R \).

**Example**

\[ \int_{|z|=\pi} \frac{z^9}{(z-1) \ldots (z-10)} = -2\pi i \text{ Res} (-, \infty). \]
**Question** How do we compute the residue at \( \infty \)?

**Answer**

\[
\text{Res} (f, \infty) = - \text{Res} \left( g(w), \frac{1}{w^2} \right)
\]

**Proof** Let \( \rho \) be sufficiently large. Then

\[
\text{Res} (f, \infty) = - \frac{1}{2\pi i} \int_{|z| = \rho} f \, dz = d_z = - \frac{dw}{w^2}.
\]

\[
= - \frac{1}{2\pi i} \int_{|w| = 1/\rho} g \frac{-dw}{w^2} \quad \text{(change variables)}
\]

\[
(\text{the change of orientation yields an extra sign}).
\]

\[
= \text{Res} \left( g(w), \frac{1}{w^2} \right)
\]

using the usual residue theorem.
Residue Theorem for $\hat{\mathbb{C}}$

If $f$ has isolated singularities only at $a_1, \ldots, a_n \in \hat{\mathbb{C}}$ and possibly at $\infty$ then

$$\sum_{a \in \hat{\mathbb{C}}} \text{Res} (f, a) = 0.$$ 

Proof

Let $\rho$ be large enough, $\rho > |a_j|$ for all $j$.

$$\text{Res} (f, \infty) = - \frac{1}{2\pi i} \int_{|z|=\rho} f(z) \frac{dz}{z}$$ (definition)

$$= - \sum_{j} \text{Res} (f, a_j)$$ (usual residue theorem)

$$\Rightarrow \sum_{a \in \hat{\mathbb{C}}} \text{Res} (f, a) = 0.$$ 

Remark * This generalizes correctly to other compact Riemann surfaces.
Example \[ f(z) = \frac{2^5 + 2}{z - 1} \].

Last lecture, we saw that \( f \) has pole at \( z = 1 \) and \( z = \infty \).

\[
\text{Res} \left( f, \frac{1}{2} \right) = \frac{2^5 + 2}{(2-1)^'} = 3.
\]

\[
\text{Res} \left( f, \infty \right) = \text{Res} \left( g(w), \frac{1}{w^2} \right)
\]

\[
2^5 = w \Rightarrow g(w) = f \left( \frac{1}{w} \right) = \frac{\frac{1}{w} 5 + 2}{\frac{1}{w} - 1} = \frac{1 + 2 w^5}{1 - w} \cdot \frac{1}{w^4}.
\]

Thus \[
\text{Res} \left( f, \infty \right) = \text{Res} \left( g(\infty), \infty \right) = \text{Coeff} w^5 - \frac{1 + 2 w^5}{1 - w} = -3.
\]

This is consistent with the residue theorem on \( \mathbb{C} \).
Example (Lagrange) 

\[ f(\alpha) = \frac{P(\alpha)}{Q(\alpha)} \]. \hspace{1cm} Assume that 

- \text{deg } P = p, \text{ deg } Q = q, \quad p \leq q - 2 
- Q \text{ has simple roots } \alpha_1, \ldots, \alpha_2 

\( f \) has poles at \( \alpha_1, \ldots, \alpha_2 \) and possibly at \( \infty \).

\textbf{Method} 

- \( \text{Res } (f, \alpha_i) = \frac{P(\alpha_i)}{Q'(\alpha_i)} \) 
- \( \text{Res } (f, \infty) = 0 \) (next page). 

\textbf{Residue Theorem for } \mathbb{C} 

When \( P(z) = z^p, \quad Q(z) = \frac{1}{11} (z - \alpha_i) \), this gives

\[ \sum_{i=1}^{2} \frac{\alpha_i^p}{11 (\alpha_i - \alpha_i)} = 0 \quad \forall \quad p \leq q - 2. \] 
\( \forall \alpha_1, \ldots, \alpha_2 \) distinct
\textbf{Proof} \quad \text{Res} \left( \frac{P}{Q}, \omega \right) = 0 \quad \text{if} \quad p < 2 - 2.

Write \quad P = a_0 2^p + \ldots + a_p, \quad a_0 \neq 0

Q = b_0 2^2 + \ldots + b_2, \quad b_0 \neq 0.

\text{Res} \left( \frac{P}{Q}, \omega \right) = \text{Res} \left( \frac{\frac{a_0}{w^p} + \frac{a_1}{w^{p-1}} + \ldots + a_p}{\frac{b_0}{w^2} + b_1 \frac{1}{w^2} + \ldots + b_2 \frac{1}{w^2}}, \frac{-1}{w^2} \right)_{\omega = 0}

= \text{Res} \left( \frac{w^2}{w^p}, \frac{a_0 + a, w + \ldots + a_p w^p}{b_0 + b, w + \ldots + b_2 w^2}, \frac{-1}{w^2} \right)_{\omega = 0}

= - \text{Res} \left( w^2 - p - 2, \frac{a_0 + a, w + \ldots + a_p w^p}{b_0 + b, w + \ldots + b_2 w^2} \right)_{\omega = 0}

= 0.

\text{Holomorphic near} \quad \omega \quad \text{since} \quad p + 2 \leq 2.
Remark (will not use)

Better to speak about residue of forms versus $f\,dz$ versus $f$

Example $f(z) = \frac{1}{z}$. Clearly $\operatorname{Res} \left( f, 0 \right) = 1$. But if we change coordinates

$$z = \lambda w \Rightarrow f = \frac{1}{\lambda w} \Rightarrow \operatorname{Res} \left( f, 0 \right) = \frac{1}{\lambda}.$$

However, if we work with forms, these issues are absent

$$f\,dz = \frac{d\lambda}{\lambda} = \frac{d\left(\lambda w\right)}{\lambda w} = \frac{dw}{w}.$$

Residues of forms are coordinate-independent!

This can be seen from $\operatorname{Res} \left( f, 0 \right) = \frac{1}{2\pi i} \int f\,dz$ using change of variables formula.
This independence applies to the residue at $\infty$ as well:

\[
\text{Res} \left( \frac{f}{d(z)} \right)_{z = \infty} = \text{Res} \left( \frac{g(w)}{d \left( \frac{1}{w} \right)} \right)_{w = 0} = \frac{1}{w}
\]

This justifies the choice of sign in the definition of the residue at $\infty$. 
2. Applications of the Residue Theorem

Argument Principle

Rouche's Theorem

Conway V.3

Eugene Rouche
(1932 - 1910)
The Argument Principle

Order \( f: U \to \mathbb{C}, \, u \in U, \, a \in U \).

\[
\text{ord} (f, a) = \begin{cases} 
   n, & \text{a zero of order } n \\
   -n, & \text{a pole of order } n \\
   0, & \text{otherwise} 
\end{cases}
\]

Remarks \( \text{ord} (f, a) = n \iff f = (z - a)^n \) when \( g \) holomorphic near \( a \), \( g(a) \neq 0 \).

This follows by inspecting the Taylor/Laurent expansion.

\[
\text{ord} (fg, a) = \text{ord} (f, a) + \text{ord} (g, a)
\]

Indeed, let \( \text{ord} (f, a) = m \), \( \text{ord} (g, a) = n \).

Write \( f = (z - a)^m F \), \( g = (z - a)^n G \), \( F(a), G(a) \neq 0 \)

\[
\implies fg = (z - a)^{m+n} FG \text{ with } FG(a) \neq 0.
\]

\[
\implies \text{ord} (fg, a) = m + n = \text{ord} (f, a) + \text{ord} (g, a).
\]
Math 220 A - Lecture 21

December 2, 2020
Last time: Conway § 3.

\( f : U \rightarrow \mathbb{C} \) meromorphic, \( U \subseteq \mathbb{C} \), \( a \in U \).

\[
\text{Def}
\]

\[
\text{ord} (f, a) = \begin{cases} 
  n, & \text{a zero of order } n \\
  -n, & \text{a pole of order } n \\
  0, & \text{otherwise}
\end{cases}
\]

Remark: \( \text{ord} (f, a) = k \Leftrightarrow f = (z - a)^k g \)

where \( g \) holomorphic near \( a \), \( g(a) \neq 0 \)

This definition treats zeros & poles equally.
**Question**
Find poles & residues of $\frac{f^{'}}{f}$

**Answer**
Poles of $\frac{f^{'}}{f}$ come from zeros or poles of $f$.

Let $a$ be a zero/pole with $\text{ord} (f, a) = k$.

$\Rightarrow f = (z - a)^k g, \ g \text{ holomorphic}, \ g(a) \neq 0.$

$\Rightarrow \frac{f^{'}}{f} = \frac{k (z - a)^{k-1} g + (z - a)^k g^{'}}{(z - a)^k g} = \frac{k}{z - a} + \frac{g^{'}}{g}$

Since $g \neq 0$ near $a \Rightarrow \frac{f^{'}}{f}$ holomorphic near $a$

$\Rightarrow \frac{f^{'}}{f}$ has simple pole and

$\text{Res} \left( \frac{f^{'}}{f}, a \right) = k = \text{ord} (f, a)$
**Theorem**

Given $f$ meromorphic in $U$, $\gamma \sim 0$, avoiding the zeros and poles of $f$, we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f} \, dz = \sum_{a} n(\gamma, a) \\text{ord}(f, a)$$

This follows by the Residue Theorem & above discussion.

**Remarks**

In practice, $\gamma$ is a circle or a simple closed curve with $\text{Int}\, \gamma \subseteq U$. Then

$$n(\gamma, a) = \begin{cases} 1, & a \in \text{Int}\, \gamma \\ 0, & a \in \text{Ext}\, \gamma \end{cases}$$

Thus

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'}{f} \, dz = \# \text{ Zeros in } \text{Int}\, \gamma - \# \text{ Poles in } \text{Int}\, \gamma$$

(counted with multiplicity)
Why is it called "argument principle"?

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} \, dz = \frac{1}{2\pi i} \int_{\gamma} d\log f
\]

\[= \frac{1}{2\pi i} \bigtriangleup \log f\]

\[= \frac{1}{2\pi i} \bigtriangleup \log |f| + iy \arg f\]

\[= \frac{1}{2\pi i} \bigtriangleup \arg f\]

**Enhanced version**

\[g : U \rightarrow \mathbb{C} \text{ holomorphic}\]

\[f : \text{meromorphic in } U, \gamma \not\ni 0 \text{ avoiding the zeros and poles of } f,\]

\[
\frac{1}{2\pi i} \int_{\gamma} g \frac{f'}{f} \, dz = \sum_{a} g(a) \cdot n(\gamma, a) \cdot \text{ord}(f, a)
\]
If \( \gamma \) is simple closed, \( \text{Int } \gamma \subset U \), then

\[
\frac{1}{2\pi i} \oint_{\gamma} g \frac{f'}{f} \, dz = \sum_{\text{poles of } f} g \left( \text{zeros of } f \right) - g \left( \text{poles of } f \right)
\]

appear with multiplicity.

**Proof** We apply the Residue Theorem.

We show \( \text{Res} \left( g \cdot \frac{f'}{f}, a \right) = g(a) \text{ ord} (f; a) \).

Let \( \text{ord} (f; a) = k \). We know from page 2:

\[
\frac{f'}{f} = k \frac{1}{z-a} + \mathcal{O}, \quad F, G \text{ holomorphic near } a
\]

\[
g = g(a) + (2-a) G \quad (\text{Taylor expansion})
\]

\[
\Rightarrow g \cdot \frac{f'}{f} = \left( k \frac{1}{z-a} + \mathcal{O} \right) \left( g(a) + (2-a) G \right)
\]

\[
= \frac{k}{2-a} g(a) + H \quad \text{where } H \text{ holomorphic near } a
\]

\[
\Rightarrow \text{Res} \left( g \cdot \frac{f'}{f}, a \right) = k g(a) = \text{ord} (f; a) \cdot g(a).
\]
Applications (Conway V.3)

Let \( f: U \to \mathbb{C} \) holomorphic, \( \Delta \subseteq U \) such that \( f/\Delta \) injective. Let \( V = f(\Delta) = \text{open} \). Then

\[ f: \Delta \to V \text{ bijection.} \]

Proposition The following integral formula for the inverse function holds

\[ f^{-1}(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f'(z)}{f(z) - z} \, dz \quad \forall \, z \in V. \]

In particular \( f^{-1}: V \to \Delta \) is holomorphic.

Proof Apply the enhanced Argument Principle to \( f - 2 \) and \( g(z) = 2 \). Since \( f \) injective, \( \exists ! \, p \in \Delta \) with

\[ f(p) = 2, \Rightarrow f^{-1}(2) = p. \]

But

\[ \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f'(z)}{f(z) - 2} \, dz = g(p) = p = f^{-1}(2). \]

no zeros on \( \partial \Delta \) since \( 2 \in f(\Delta) = 2 \notin f(3\Delta) \) as \( f/\Delta \) injective.
Recall from Lecture 16

**Key statement** \( \psi : \mathbb{U} \times \{ r \} \rightarrow \mathbb{C} \)

- \( \psi \) continuous
- \( z \rightarrow \psi(z,w) \) holomorphic \( \forall w \in \{ r \} \)

Then \( g(z) = \int_{\gamma} \psi(z,w) \, dw \) holomorphic.

Apply this to \( \psi : \Delta \times \partial \Delta \rightarrow \mathbb{C} \)

\( \psi(g(z),z) = g \cdot \frac{f'(z)}{f(z) - 2} \) continuous \& holomorphic in \( z \), \( \forall z \in \Delta \). Then

\[ f^{-1}(z) = \int_{\partial \Delta} \psi(1,z) \, d\mathbf{\omega} = \text{holomorphic in } z. \]

**Remark**

This extends a result from Lecture 11, concerning holomorphicity of the inverse \( \text{removes } f' \neq 0 \).
Further Applications of the Argument Principle

Elliptic functions

- studied by Abel, Jacobi, Weierstrass

- connected with arclength of ellipse

- elliptic integrals

- elliptic curves

- rich theory

- we will only say a few words about them

(More in Math 220 B & C)
Carl Gustav Jacob Jacobi (1804 - 1851)

Jacobian, Jacobi symbol, Jacobi identity, symbol 2

Karl Weierstraß (1815 - 1897)
**Definition**

Let \( \omega_1, \omega_2 \in \mathbb{C} \setminus \{0\} \), \( \frac{\omega_1}{\omega_2} \notin \mathbb{R} \). Define the lattice

\[
\Lambda = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 = \left\{ m \omega_1 + n \omega_2 : m, n \in \mathbb{Z} \right\}
\]

**Def** An elliptic function \( f \) satisfies

1. \( f \) meromorphic on \( \mathbb{C} \)
2. \( f \) periodic, \( f(z) = f(z + \omega_1) = f(z + \omega_2) \)

Note that in fact for \( \lambda \in \Lambda \), \( f(z) = f(z + \lambda) \)
The best-known elliptic function is

\[ J_0(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \left( \frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right) \]

We will study this function in detail later in 220 B & C.

\[ f \text{ elliptic } \Rightarrow f' \text{ elliptic.} \]

Indeed \( f(x) = f(x + \lambda) \Rightarrow f'(x) = f'(x + \lambda), \forall x \in \Lambda \)
Math 220 A - Lecture 22

December 4, 2020
Last time in real analysis we encounter periodic functions. In complex analysis:

Let \( \omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}, \ \frac{\omega_1}{\omega_2} \notin \mathbb{R} \).

**Def.** An elliptic function \( f \) satisfies

\[
\begin{align*}
\text{meromorphic on } \mathbb{C} \\
\text{doubly periodic:}
\end{align*}
\]

\[
f(z) = f(z + \omega_1) = f(z + \omega_2) \quad \forall \ z
\]

**Remark**

\[
\Lambda = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 = \left\{ m \omega_1 + n \omega_2 : m, n \in \mathbb{Z} \right\}
\]

\((\Lambda): \quad \forall \ z \in \Lambda, \ f(z) = f(z + \lambda)\)
Basic Properties of Elliptic Functions

Note that $\Lambda$ is a subgroup of $\mathbb{C}$.

Define

$$z \equiv w \mod \Lambda \iff z - w \in \Lambda.$$ (w)

$$z \equiv w \mod \Lambda \Rightarrow f(z) = f(w).$$

Remark $f$ is determined by values mod $\Lambda$

We will restrict $f$ to a parallelogram.

$$P_a = \{a + t_1 \omega_1 + t_2 \omega_2 : 0 \leq t_1, t_2 \leq 1\}$$

Each point in $\mathbb{C}$ is congruent to a point in $P_a$. (see next picture)
Claim: \( \exists a \) such that \( \mathcal{E}_a \) contains no zeroes/poles.

Proof: Start with any \( a \). Since \( I_a \) is compact & zeroes/poles are discrete \( \Rightarrow \exists \) finitely many of them in \( I_a \). A suitable translation would ensure \( \mathcal{E}_a \) avoids them.

Write \( P = I_a \) where \( P \) is chosen as above.
Remark

If $f$ holomorphic in $\mathbb{C}$ \Rightarrow $f/p$ continuous

$p$ compact \Rightarrow $f/p$ bounded

periodic \Rightarrow $f$ bounded

Lioville \Rightarrow $f$ constant

Thus in general $f$ will have poles.

Notation

zeros in $\mathbb{C}$: $\alpha_1, \ldots, \alpha_k$ (w/ multiplicity)

poles in $\mathbb{P}$: $\beta_1, \ldots, \beta_k$ (w/ multiplicity)

Theorem

$k = l$: \# zeros ($f$) = \# poles ($f$)

\[ \sum_{i=1}^{k} \varphi_i \equiv \sum_{i=1}^{l} \beta_i \mod \Lambda. \]
Remark: Given \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \) with

\[
\sum_{i=1}^k \alpha_i \equiv \sum_{i=1}^k \beta_i \mod \Lambda
\]

there is an elliptic function with these zeroes/poles.

This is not obvious. \( \rightarrow \) Abel-Jacobi theory

Proof: By the Argument Principle

\[
\frac{1}{2\pi i} \int_{\gamma_P} \frac{f'}{f} \, dz = \# \text{Zeros of } f \left( \gamma_P \right) - \# \text{Poles of } f \left( \gamma_P \right)
\]

We show \( \int_{\gamma_P} \frac{f'}{f} \, dz = 0 \) for \( \gamma_P = L_1 + L_2 + (-L_3) + (-L_4) \)

We show \( \int_{L_1} \frac{f'}{f} \, dz = \int_{L_3} \frac{f'}{f} \, dz \) and \( \int_{L_2} \frac{f'}{f} \, dz = \int_{L_4} \frac{f'}{f} \, dz \).

Both claims follow by periodicity

\[
\int_{L_1} \frac{f'}{f} \, dz = \int_{L_1} \frac{f'}{f} \left( a + \omega_2 \right) \, dz = \int_{L_3} \frac{f'}{f} \, dz
\]

(\( L_3 = L_1 + \omega_2 \)).
We use the Enhanced Argument Principle \((g(1) = 2)\).

\[
\frac{1}{2\pi i} \int_{\partial \Omega} \frac{f'}{f} \, d\zeta = \sum_{\zeta \in \Omega} \alpha_i - \sum_{\zeta \in \Omega} \beta_i.
\]

We show

\[
\frac{1}{2\pi i} \left( \int_{\partial l_1} \frac{f'}{f} \, dz - \int_{\partial l_2} \frac{f'}{f} \, dz \right) \in \Lambda \quad \text{and}
\]

\[
\frac{1}{2\pi i} \left( \int_{\partial l_3} \frac{f'}{f} \, dz - \int_{\partial l_4} \frac{f'}{f} \, dz \right) \in \Lambda.
\]

This will complete the proof.

We only consider 1st expression. \(l_3 = l_1 + \omega_2\)

\[
\frac{1}{2\pi i} \left( \int_{\partial l_1} \frac{f'}{f} \, dz - \int_{\partial l_2} \frac{f'}{f} \, dz \right) = \frac{1}{2\pi i} \left( \int_{\partial l_1} \frac{f'}{f} \, dz - \int_{\partial l_2} \frac{(2 + \omega) f'}{f} \, dz \right)
\]

\[
= - \frac{1}{2\pi i} \int_{\partial l_1} \frac{f'}{f} \, dz \cdot \omega_2 \quad \text{w} = f(c_2).
\]

\[
= - \left( \frac{1}{2\pi i} \int_{f(l_1)} \frac{dw}{w} \right) \cdot \omega_2
\]

\[
= - n \left( f(l_1), 0 \right) \omega_2 \in \Lambda.
\]

Note that \(f(l_1)\) is a loop (by periodicity), not containing 0.
2. **Rouché's Theorem** (Conway 5.3).

**Idea:** \( f, g : U \rightarrow \mathbb{C} \) holomorphic

\[ f = g + \text{lower order terms} \]

\[ \Rightarrow \# \text{zeros}(f) = \# \text{zeros}(g) \quad \text{(w/ multiplicity)}. \]

We can ignore the lower order terms.

**Setup:** \( \gamma \in U \) simple closed curve, \( \text{Int} \gamma \subseteq U \).

*E.g.* \( \gamma = \partial \Delta, \quad \bar{\Delta} \subseteq U \).
Theorem \[ f, g : u \to \mathbb{C} \text{ holomorphic, } \gamma \text{ as above.} \]

If \( |f - g| < |g| \) on \( \gamma \) \( \Rightarrow \)

\[ \# \text{Zeros}(f) = \# \text{Zeros}(g), \text{ in } \text{Int}(\gamma). \]

\text{(w/ multiplicity)}

Note that \( f \neq 0 \) \& \( g \neq 0 \) on \( \gamma \).

Remark \text{\textit{Conway's version is more general but less useful in practice.}}

\text{\textit{Conway.}}

\begin{itemize}
  \item \( f, g \text{ meromorphic} \)
  \item \( |f - g| < |f| + |g| \) on \( \gamma \)
\end{itemize}

\( \Rightarrow \)

\[ \# \text{Zeros}(f) - \# \text{Poles}(f) = \# \text{Zeros}(g) - \# \text{Poles}(g) \]

\text{in } \text{Int}(\gamma).\]
Eugene Rouche'

(1832 - 1910)
Example

\[ f = 2^5 + 24 \cdot 2^3 + 2 \cdot 2^2 + 32 + 1 \]

How many roots in \(|z| < 1\).

Let \( g = 24 \cdot 2^3 \) and \( \gamma = \{ |z| = 1 \} \).

We verify \( |f - g| < |g| \) when \(|z| = 1\).

Note \( |g| = 24 \cdot |z|^3 = 24 \). 

\[
|f - g| = 1 \cdot 2^5 + 2 \cdot 2^2 + 32 + 1 \leq 1 \cdot 2^5 + 2 \cdot 2^2 + 3 \cdot 2 + 1 \\
= 1 + 2 + 3 + 1 = 6.
\]

\[ \Rightarrow |f - g| < |g| \] on \( \gamma \)

\[ \Rightarrow \# \text{Zeroes}(f) = \# \text{Zeroes}(g) = 3 \] in \( \{ |z| < 1 \} \).
Example 111 Fundamental Theorem of Algebra

\[ f = z^n + a_n z^n - 1 + \ldots + a_0 \]

\[ g = 2^n = \text{dominant term when } |z| \text{ large.} \]

\[ f - g = a_n z^n - 1 + \ldots + a_0. \]

When \( |z| = R \),

\[ |f - g| < 1, 1 R^n - 1 + \ldots + |a_n| < R^n = |z|^n = |g|. \]

This happens for \( R \) large as \( \lim_{R \to \infty} \frac{R^n}{|a_n| R^{n-1} + \ldots + |a_0|} = 0. \)

By Rouche:

\[ \# \text{Zeros}(f) = \# \text{Zeros}(g) = n \quad \text{in} \quad \Delta(0, R). \quad R > 0 \]

\[ \Rightarrow \# \text{Zeros}(f) = n. \quad \text{in} \quad \mathbb{C} \]
Math 220 A - Lecture 23

December 7, 2020
Last time - Rouché’s theorem

A simple closed curve, \( \gamma \subset \mathbb{C} \).

\[ \gamma = \partial \Delta, \quad \Delta \subset \mathbb{C} \]

Theorem \( f, g : \mathbb{C} \to \mathbb{C} \) holomorphic, \( \gamma \) as above.

If \( |f - g| < |g| \) on \( \gamma \) \Rightarrow

\[ \# \text{zeros} (f) = \# \text{zeros} (g), \quad \text{in } \text{Int} (\gamma). \]

(w/ multiplicity)

What does the hypothesis mean?

\[ f = g + (f - g) \]

- dominant lower order term terms
Proof (see Conway for a different proof)

Let $h_t = g + t(f-g)$, $0 \leq t \leq 1$.

Want $t \to \# \text{zeros } (h_t)$ is continuous in $t$.

This implies $\# \text{zeros } (h_t) = \text{constant}$.

Since $h_0 = g$, $h_1 = f \Rightarrow \# \text{zeros } (f) = \# \text{zeros } (g)$.

To show continuity, we use the Argument Principle:

$$\# \text{zeros } (h_t) = \frac{1}{2\pi i} \int_{\gamma} \frac{h_t'(z)}{h_t(z)} \, dz.$$  

Note $|h_t| = |g + t(f-g)| \geq |g| - |t| |f-g|$

$$\geq |g| - |f-g| > 0 \text{ on } \gamma.$$

Set $\psi(t, z) = \frac{h_t'(z)}{h_t(z)} ; [0,1] \times \{y\} \to \mathbb{C}$.

Note $\psi$ is continuous.
**Key Fact** \( \psi : [0,1] \times \{\gamma\} \rightarrow \mathbb{R} \) is continuous

\[ \Rightarrow \quad \Phi(t) = \int_{\gamma}^{\psi(t, \omega)} d\omega \text{ is continuous in } t. \]

**Quick proof:** Since \([0,1] \times \{\gamma\}\) is compact, \(\psi\) is uniformly continuous.

Fix \(\varepsilon > 0\). Then \(\exists \delta > 0\) with

\[ |t - t'| < \delta \Rightarrow |\psi(t, \omega) - \psi(t', \omega)| < \varepsilon. \]

\[ \Rightarrow \quad |\Phi(t) - \Phi(t')| = \int_{\gamma}^{\psi(t, \omega) - \psi(t', \omega)} d\omega \]

\[ < \varepsilon \cdot \text{length(}\gamma\text{)} \]

\[ \Rightarrow \quad \Phi \text{ continuous.} \]
Applications

Find the location of zeroes of holomorphic fn.

\[ f = z^5 + 24z^3 + 2z^2 + 3z + 1 \ (\text{last time}) \]

We can also use this for non-polynomial functions.

Example

\[ f(z) = e^{z^2 - z^3} + 1, \quad \gamma = \{1, 2\} = \{1\} \]

Dominant term \( g(z) = -z^3 \).

Indeed, \( |g| = 5 \) for \( |z| = 1 \).

\[
|f - g| = |e^{z^2} + 1| \leq |e^{z^2}| + 1 = e^{\Re z^2} + 1
\]

\[
\leq e^{1^2} + 1 = e + 1 < 5 = |g|
\]

\[ \Rightarrow \# \text{ zeroes } (f) = \# \text{ zeroes } (g) = 3 \text{ in } \Delta(0, 1). \]
Example

\( h : \mathbb{U} \rightarrow \mathbb{U}, \ \Delta (0, 1) \subseteq \mathbb{U}, \ |h(2)| < 1, \ |2| = 1. \)

\( \Rightarrow h \) has one fixed point in \( \Delta (0, 1). \)

Proof We show \( h(2) = 2 \iff h(2) - 2 = 0 \) has a unique solution in \( \Delta (0, 1). \)

Let \( f(2) = h(2) - 2, \ g(2) = -2, \ \gamma = \{ |z| = 1 \}. \)

Then

\[ |f - g| = |h| < 1 = |g| \quad \text{on } \gamma \]

\( \Rightarrow \# \text{zeros}(f) = \# \text{zeros}(g) = 1. \)

\( \Rightarrow h \) has a unique fixed point in \( \Delta (0, 1). \)

Remark Hurwitz' theorem will be another abstract application of Rouche.
Sequences of holomorphic functions (Conway VIII).

Outline
- notions of convergence
  - Weierstrass' theorem
  - Hurwitz's theorem \( \leq \) Rouché' \( \leq \)

Types of convergence

Question: What is the correct notion of convergence for holomorphic functions?

\[ f_n : U \rightarrow \mathbb{C}, \quad f : U \rightarrow \mathbb{C} \text{ be any functions.} \]

Pointwise convergence

\[ f_n \rightarrow f \iff \forall x \in U, \quad f_n(x) \rightarrow f(x). \]

Uniform convergence

\[ f_n \Rightarrow f \iff \sup_{U} |f_n - f| \rightarrow 0 \text{ as } n \rightarrow \infty. \]
Issues II. Pointwise convergence is not well behaved under differentiation or even integration. (Baby Rudin)

The pointwise limit of continuous functions need not be continuous (Baby Rudin / Math 140B).

Uniform convergence is better. But the motion is strong. For instance, take:

\[ f_n(z) = \frac{z^n}{n}, \quad f(z) = 0, \quad f_n \not\to f \text{ on } |z| < 1. \]

We consider slightly weaker notions.
**Better**

(a) uniform convergence on compact sets

(b) local uniform convergence

1. **Notation**
   \[ f_n \xrightarrow{c} f \text{ or } f_n \xrightarrow{c} f \]

2. **Definition**
   \[ \forall K \subseteq U \text{ compact, } \sup_K |f_n - f| \to 0 \text{ as } n \to \infty. \]

(b) **Notation**
   \[ f_n \xrightarrow{l.u.} f \]

2. **Definition**
   \[ \forall x \in U \exists \Delta(x, r_x) \subseteq U \text{ with } f_n \xrightarrow{c} f \]
   \[ \Delta(x, r_x). \]
   \[ \text{ in } \Delta(x, r_x). \]
   \[ \text{ local uniform converg.} \]

Claim: \[ \text{(a) } \Rightarrow \text{(b)} \]

Thus \[ f_n \xrightarrow{c} f, f_n \xrightarrow{c} f, f_n \xrightarrow{l.u.} f \] mean the same thing.

**Proof** \[ \text{(a) } \Rightarrow \text{(b)} \]. If \[ \text{(a) holds for all } K, \text{ take} \]

\[ K = \Delta(x, r_x) \subseteq U, K \text{ compact. This choice of } K \text{ yields (b).} \]
(b) Ÿ Ď. Let \( f_n \to f \). Take \( K \) compact in \( U \).

For \( x \in K \), \( \exists \Delta(x, r_x) \) with \( f_n \Rightarrow f \) in \( \Delta(x, r_x) \).

Since \( K \subseteq \bigcup_{x \in K} \Delta(x, r_x) \Rightarrow K \subseteq \bigcup_{i=1}^{\infty} \Delta(x_i, r_{x_i}) \) by compactness. Since

\[
\sup_{K} |f_n - f| \leq \max_{1 \leq i \leq N} \left( \sup_{\Delta(x_i, r_{x_i})} |f_n - f| \right) \to 0
\]

\( \Rightarrow f_n \Rightarrow f \) in \( K \) \( \Rightarrow f_n \overset{c}{\to} f \).

Example \( f_n = \frac{z}{n}, \quad f = 0 \). \( f_n \overset{c}{\to} f \) in \( \mathbb{C} \).

Indeed, \( \sup_{K} |f_n - f| = \sup_{z \in \overline{K}} |f_n| < \frac{M}{n} \to 0 \).

so \( f_n \overset{c}{\to} f \). This was the example disallowed before.
Remark (Continuity & Math 140B).

If \( f_n \) is continuous and \( f_n \rightarrow f \) then \( f \) is continuous.

(because continuity is a local concept).

Important Convention

\( \mathcal{C}(U) = \) continuous functions in \( U \)

\( \mathcal{O}(U) = \) holomorphic functions in \( U \)

We will always consider local uniform convergence for both \( \mathcal{O}(U) \) and \( \mathcal{C}(U) \).
Let \( f_n : U \to \mathbb{C} \) be holomorphic, \( f_n \overset{\text{l.u.}}{\to} f \). Then

1. \( f \) is holomorphic

2. \( f_n \overset{\text{l.u.}}{\to} f \)

**Remark**

\( \mathcal{O}(U) \subset \mathcal{O}(U) \) "closed," under local uniform limits.

\( \int f_n \, dz \overset{\text{l.u.}}{\to} \int f \, dz \) since \( \{ \gamma \} \) compact.

The statement fails in real analysis (Baby Rudin or Math 140B for examples).

**The proof will be given next time.**
Let $f_n : u \to a$ be holomorphic, $f_n \to f$. Then

1. $f$ holomorphic
2. $f_n^{(k)} \to f^{(k)}$

Proof

Let $R \subseteq u$ closed rectangle, $\exists R = \text{compact}$.

Since $f_n \to f \Rightarrow \int_{\partial R} f_n \, dz = \int_{\partial R} f \, dz$

Since $f_n$ holomorphic $\Rightarrow \int f_n \, dz = 0$. (Lecture 6)

$\Rightarrow \int f \, dz = 0 \Rightarrow f$ admits a primitive $F$ in any disc in $u$.

$\therefore f = f' = \text{holomorphic in any disc in } f \text{ holomorphic.}$
By induction, suffice to show

\[ f'_n \to f' \quad \text{in} \quad u. \]

Let \( a \in u \), \( \bar{\Delta} (a, r) \subseteq \bar{\Delta} (a, R) \subseteq U \).

For \( r < R \),

\[ \text{suffices } f'_n \to f' \quad \text{in} \quad \bar{\Delta}_r. \]

We use CIF for \( z \in \bar{\Delta}_R \)

\[
\left| f_n'(z) - f'(z) \right| = \left| \frac{1}{2\pi i} \int_{\bar{\Delta}_R} \frac{f_n(w) - f(w)}{(w-z)^2} \, dw \right| = \frac{1}{2\pi} \sup_{\bar{\Delta}_R} |f_n - f| \cdot \frac{1}{(R-r)^2} \cdot \frac{2\pi R}{R} \leq \frac{1}{(R-r)^2} \sup_{\bar{\Delta}_R} |f_n - f| \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus

\[ \sup_{\bar{\Delta}_r} |f_n' - f'| \leq \frac{R}{(R-r)^2} \sup_{\bar{\Delta}_r} |f_n - f| \to 0 \quad \text{as} \quad n \to \infty.
\]

\[ \Rightarrow f'_n \to f'. \]
Series \( f_n : \mathbb{U} \rightarrow \mathbb{C} \) holomorphic. Assume

\[ \forall \mathcal{K} \subseteq \mathbb{U} \text{ compact } \exists M_n(K), \; |f_n| \leq M_n(K). \]

over \( \mathcal{K} \) & \( \sum_{n=1}^{\infty} M_n(K) < \infty. \)

\( M \)-test

\[ \Rightarrow f = \sum_{n=1}^{\infty} f_n \text{ converges absolutely & uniformly on every } \mathcal{K}. \]

Weierstrass

\[ \Rightarrow f \text{ holomorphic & } f' = \sum_{n=1}^{\infty} f_n'. \]

Thm

Remark We have seen a particular case of this for power series. (Lecture 2).
Example \((S\text{-function})\)

\[ S(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \] gives a holomorphic function in \(\text{Res} > 1\).

Take \(f_n(s) = \frac{1}{n^s}\). holomorphic in \(S\).

Take \(K = \{\text{Res} > 1\}\). Since \(\text{Re} : K \to \mathbb{R}\) is continuous,

it achieves a minimum on \(K \Rightarrow \text{Res} \geq 2\alpha + 2 \in K, \alpha > 1\).

\[ |f_n| = \left| \frac{1}{n^s} \right| = \frac{1}{n^\alpha} \] where \(s = \alpha + iy\)

\[ \sum_{n=1}^{\infty} |a_n| < \infty. \text{ by real analysis} \Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ holomorphic in } S \]

\[ \Rightarrow S \text{ holomorphic in } S, \text{ Res} > 1. \]

Remarks \(\text{III} \)

We have seen \(S(2n) = \frac{(-1)^{n-1}}{2} \frac{2^n B_{2n}}{(2n)!} (\text{HWKE})\)

\(\text{III} \) This can be extended holomorphically to \(s \neq 1\)

\((\text{requires work})\)
2.1 Hurwitz Theorem

\( f_n : u \to \mathbb{C} \) holomorphic, \( f_n \to f \), \( \overline{u} \subset u \) compact

If \( f \) has no zeroes,

\[
\# \text{Zeros} (f_n) = \# \text{Zeros} (f) \quad \forall \ n \geq N.
\]

**Proof**

**Most useful case (Conway)**

\( V = \triangle (a, R) \)

Since \( f \) has no zeroes \( \Rightarrow \exists \varepsilon = \min \{ f(x) : x \in V \} > 0 \).

Since \( f_n \to f \) over \( \partial V \) \( \Rightarrow \exists N \) s.t. over \( \partial V \) \( \forall n \geq N \).

\[
\frac{|f_n - f|}{\varepsilon} \leq 1 \Rightarrow 1 \leq \frac{|f_n - f|}{|f|} \quad \text{over} \ \partial V.
\]

\( \Rightarrow \) Rouché: \( \# \text{Zeros} (f) = \# \text{Zeros} (f_n) \quad \text{in} \ \overline{V}. \)
General case

If $V$ compact $\Rightarrow f$ has finitely many zeroes $c_1, \ldots, c_k$ in $\overline{V}$.

Surround $c_j$ by small disjoint discs $\Delta_j$, $W = V \setminus \bigcup_j \Delta_j$.

$\Rightarrow f$ has no zeroes in $\overline{W}$. $\Rightarrow \exists N$ s.t. $\forall n \geq N$, $f_n$ has no zeroes in $\overline{W}$. (If $\varepsilon = \min |f| > 0$ $\Rightarrow \exists N$, s.t. $\forall n \geq N$)

$|f_n - f| < \varepsilon$ in $\overline{W}$ $\Rightarrow f_n \neq 0$ in $\overline{W}$ for $n \geq N$.

$\Rightarrow \# \text{Zeroes (} f \text{)} = \sum_{j=1}^{k} \# \text{Zeroes (} f \text{)}$ for $n$ large by

1st case applied to $f_n$ on $\Delta_j$ using $f_n$ has no zeroes in $\overline{W}$

$= \sum_{j=1}^{k} \# \text{Zeroes (} f_n \text{)}$ for $n \geq N$.
Corollary A \[ f_n \to f, \text{ fn holomorphic in } U, \]

If \( f_n \) is zero free \( \forall n \Rightarrow f \) zero-free or \( f \equiv 0 \).

This fails in real analysis, \( f_n = x^2 + \frac{1}{n} \Rightarrow f = x^2 \).

\[ \text{Proof} \]

Indeed if \( f \neq 0 \), let \( a \) be chosen so that \( f(a) = 0 \). Let \( V = \Delta(a, r) \), \( f|_V \) has no zeroes.

(Argue by contradiction, otherwise zeroes of \( f \) would accumulate).

\[ \text{Hurwitz} \]

\[ \Rightarrow \quad \# \text{ zeroes (} f_n \text{)} = \# \text{ zeroes (} f \text{)} \geq 1 \quad \forall \quad n \geq N. \]

\[ \Rightarrow \quad f \text{ is zero-free} \]

\[ \text{Example} \quad U = \mathbb{C}^* = \mathbb{C} \setminus \{0\} \]

\[ \cdot f_n(z^2) = 2, \quad f(z^2) = 2, \quad f_n \not\leq f, \quad f \text{ zero free.} \]

\[ \cdot f_n(z) = \frac{2}{n}, \quad f(z) = 0, \quad f_n \not\leq f, \quad f \equiv 0. \]

Both possibilities occur.
Corollary B \[ fn \rightarrow f, \quad fn \text{ holomorphic in } U, \]

If \( fn \) are injective \( \forall n \rightarrow f \) injective or \( f \) constant.

\[ \text{Proof.} \quad \text{Assume } f \text{ not injective}, \quad f(a) = f(b), \quad a \neq b. \]

\[ \tilde{f} = f - f(a). \]

Since \( fn(a) \rightarrow f(a) \)

\[ \tilde{fn} \rightarrow f \]

\( fn \text{ injective } \rightarrow \tilde{fn} \text{ zero free on } \tilde{U} = U \setminus \{a\} \).

Corollary A

\[ \Rightarrow \tilde{f} \text{ is zero free on } \tilde{U} \text{ or } \tilde{f} \equiv 0. \text{ on } \tilde{U} \]

Note that \( \tilde{f}(b) = f(b) - f(a) = 0 \Rightarrow \tilde{f} \text{ is not zero free in } \tilde{U}. \text{ Thus } \tilde{f} \equiv 0 \text{ in } \tilde{U} \Rightarrow f \text{ constant.} \]
Adolf Hurwitz (1859 - 1919)

Riemann – Hurwitz formula, Hurwitz automorphism thm