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**Claim:** each $Y \subset X$ closed irreducible is a prevariety.

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**Affine case:** regular functions on $U \subset Y$ are locally $f = \phi \psi$.

These extend locally to $g = \phi \psi$ on the subset $V = \{x \in X: \psi(x) \neq 0\}$ of $X$. 
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In general,

- for $U \subset Y$ open, $\mathcal{O}_Y(U) = $ functions $f : U \to k$ which are locally restrictions of regular functions on open subsets of $X$.

$X$ can be covered by affine opens $U_i$ so $Y$ can be covered by opens $U_i \cap Y$ is closed in $U_i$ hence affine.
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- for $U \subset Y$ open, $\mathcal{O}_Y(U)$ = functions $f : U \to k$ which are locally restrictions of regular functions on open subsets of $X$

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Morphisms of prevarieties

Morphisms of prevarieties are morphisms of underlying ringed spaces.
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**Morphisms** of prevarieties are morphisms of underlying **ringed spaces**.

We can **glue** prevarieties.

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**Lemma (Gluing of morphisms)**

Let $f: X \to Y$ be a set-theoretic map of prevarieties. Let $\{U_i\}$ be an open cover of $X$. Assume $f_i = f|_{U_i}: U_i \to Y$. Then $f$ is a morphism iff $f_i$ are morphisms.
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If $f_i$ are morphisms, $f_i$ continuous, so $f$ is continuous.
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Lemma

Any morphism

\[ f : \mathbb{P}^1 \to \mathbb{A}^1 \] is constant.

Proof: Let \( \mathbb{P}^1 = X_1 \cup X_2 \) be the standard charts.

\[ f \mid_{X_1} : X_1 \to \mathbb{A}^1 \text{ is morphism } f(x) = p(x) \text{ for } x \in X_1 = \mathbb{A}^1. \]

\[ f \mid_{X_2} : X_2 \to \mathbb{A}^1 \text{ is morphism } f(y) = q(y) \text{ for } y \in X_2 = \mathbb{A}^1. \]

\[ \text{over overlap } y = 1 \text{ and } p(x) = q(y) = q(1x). \]

\[ p, q \text{ are constant } \Rightarrow f \text{ is constant.} \]
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Lemma
Let $X$ be a prevariety, $Y$ affine variety. Each $f : X \to Y$ induces

$$F : A(Y) \to \mathcal{O}_X(X)$$

and conversely, all $k$-algebra homomorphisms $F$ arise this way.
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- hence $f : X \to Y$. 
Roadmap:
Additional constructions: products

**Goal:** if $X$, $Y$ are prevarieties, we define $X \times Y$ as a prevariety
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WANT: $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ so we can’t use product topology.
Universal property of products

- there are natural morphisms

\[ p : X \times Y \to X, \quad q : X \times Y \to Y \]
Universal property of products

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\[ f : Z \to X, \quad g : Z \to Y, \]
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Note that $X \times Y$ is unique up to unique isomorphism, if it exists.
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(i) $X, Y$ affine
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$(i) + (iii) \implies (iv)$,

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