Math 203A

October 7, 2022
Recap: We defined two algebraic invariants of affine varieties
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- coordinate ring \( A(X) \)
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- fraction field $K(X)$
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We have seen that

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- What does $K(X)$ classify?
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We have seen that

- $A(X)$ classifies affine varieties up to isomorphism
- What does $K(X)$ classify?
Rational functions on affine varieties

- $\phi \in K(X)$ yields a partially defined function

Examples

1. For $X = \{y^2 = x^3\}$, take $\phi = \frac{y}{x} \Rightarrow \text{Dom} \phi = X \setminus \{x = y = 0\}$.

2. For $X = \{xw = yz\}$, take $\phi = \frac{x}{y} \Rightarrow \text{Dom} \phi = X \setminus \{y = w = 0\}$.
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- $\phi \in K(X)$ yields a partially defined function

$$\phi : X \rightarrow k.$$
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- $\text{Dom } \phi$ is the largest set where $\phi$ is regular.
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Alternatively:

- A rational function
  
  \[ \phi : X \to k \]

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\[ \phi : X \longrightarrow k \]

is an equivalence class of pairs

- \( (U, \phi), \ U \subset X \) open, \( \phi : U \rightarrow k \) regular

This definition recovers the stalk \( O_{X, X} \) from HWK2. Thus \( O_{X, X} = A(X)(0) = K(X) \) is the old field of rational functions.

Bonus: \( K(X) \) now makes sense even if \( X \) is quasiaffine.
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\[ (U, \phi) \sim (V, \psi) \iff \phi\big|_W = \psi\big|_W, \quad W \subset U \cap U' \text{ open.} \]
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Rational maps of affine varieties

Definition

- A rational map \( f : X \to \mathbb{A}^m \) is a partially defined map

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f = (f_1, \ldots, f_m) \quad \text{where} \quad f_i \in K(X).
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- A rational map $f : X \rightarrow Y$ is a rational map
  \[ f : X \rightarrow \mathbb{A}^m, \text{ and } f(\text{Dom } f) \subseteq Y. \]
Rational maps of affine varieties

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- A rational map $f : X \dashrightarrow \mathbb{A}^m$ is a partially defined map
  
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- We can also think in terms of equivalence classes of pairs
  
  $$(f, U), \quad f : U \rightarrow Y \text{ morphism, } \quad U \subset X \text{ open.}$$
Remark
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\( f : X \to Y \) is dominant if \( \text{Im}(\text{Dom } f) \) is dense in \( Y \).
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Lemma

If $f : X \to Y$ is a dominant map, then

$$f^* : K(Y) \to K(X), \quad \phi \mapsto f^* \phi = \phi \circ f$$

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Proof: If \( F : K(Y) \to K(X) \) let \( f_i = F(y_i) \). Let \( f = (f_1, \ldots, f_m) \).
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- As in the proof for coordinate rings, show \( f : X \rightarrow Y \).
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- As in the proof for coordinate rings, show \( f : X \rightarrow Y \).
- \( f \) is dominant since else \( f(\text{Dom } f) \subset W \) with \( W \subset Y \) closed.
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Proof: If $F : K(Y) \to K(X)$ let $f_i = F(y_i)$. Let $f = (f_1, \ldots, f_m)$.

- As in the proof for coordinate rings, show $f : X \to Y$.
- $f$ is dominant since else $f(\text{Dom } f) \subset W$ with $W \subset Y$ closed.
- Let $h$ be an equation for $W$. Then $h$ is invertible in $K(Y)$ but
  $F(h) = f^*h = 0$

  not invertible. Contradiction.
Definition

A birational isomorphism

\[ f : X \to Y \]

is a dominant map with a dominant rational inverse \( g : Y \to X \)

\[ f \circ g = 1, \quad g \circ f = 1 \]

as rational maps.
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Corollary

$X, Y$ are \textit{birational} iff $K(X) \cong K(Y)$ as $k$-algebras.
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Remark

Note that

\[ X \cong Y \iff A(X) \cong A(Y). \]
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Remark

\begin{itemize}
  \item $X$ and $Y$ are \textit{birational} iff there exist open
  $$U \subseteq X, \ V \subseteq Y, \ U \cong V.$$ 
\end{itemize}
Corollary

$X$, $Y$ are birational iff $K(X) \cong K(Y)$ as $k$-algebras.

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- $X$ and $Y$ are birational iff there exist open
  
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- $X$ is rational if $X$ is birational to $\mathbb{A}^n$
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- \( X \) and \( Y \) are birational iff there exist open

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- \( X \) is rational if \( X \) is birational to \( \mathbb{A}^n \)

  \[ K(X) \cong k(t_1, \ldots, t_n). \]
Corollary

$X$, $Y$ are birational iff $K(X) \simeq K(Y)$ as $k$-algebras.

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Remark

- $X$ and $Y$ are birational iff there exist open

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- $X$ is rational if $X$ is birational to $\mathbb{A}^n$

  $$K(X) \cong k(t_1, \ldots, t_n).$$

Thus “most” of $X$ can be “parametrized” by rational functions.
Proof of the remark

- **reverse direction:**

\[ f : U \to V, \quad g : V \to U \]

give birational maps

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▶ forward direction: Let

\[ f : U \to Y, \ \ g : V \to X \quad U \subset X \quad V \subset Y \text{ nonempty open.} \]
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- **reverse direction:**
  
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- **Know:** \( f \circ g = 1 \) on some open set \( W \)
Proof of the remark

- reverse direction:
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- forward direction: Let
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- Know: \( f \circ g = 1 \) on some open set \( W \)
\[ g^{-1}(U) \xrightarrow{g} U \xrightarrow{f} Y \] is the inclusion on \( g^{-1}(U) \cap W \) open,

Also \( f(\{ V \}) \subset g^{-1}(U) \subset V = \Rightarrow g(\{ g^{-1}(U) \}) \subset f^{-1}(\{ V \}) \).

Define the open sets \( U' = f^{-1}(\{ V \}) \), \( V' = g^{-1}(\{ U \}) \).

\( f: U' \to V' \), \( g: V' \to U' \) are well-defined and inverses.

Thus \( U' \cong V' \).
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f(g(g^{-1}(U))) \subset g^{-1}(U) \subset V \implies g(g^{-1}(U)) \subset f^{-1}(V).
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Plane conics

(HWK3) All conics in $\mathbb{A}^2$ are isomorphic to

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$$X = \{y^2 = f(x)\}$$
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- distinct roots
- double root
- triple root
Plane conics

(HWK3) All **conics** in $\mathbb{A}^2$ are **isomorphic** to

- $y - x^2 = 0$
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$$f : \mathbb{A}^1 \rightarrow X, \quad t \mapsto (t^2, t^3).$$
Cusp. Let \( X = \{ y^2 = x^3 \} \). Define

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f: \mathbb{A}^1 \rightarrow X, \quad t \mapsto (t^2, t^3).
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Then \( f \) is a birational map with inverse

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g: X \rightarrow \mathbb{A}^1, \quad (x, y) \mapsto y/x.
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  \( X \) is *rational*. 
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- **Node.** Let \( X = \{ y^2 = x^2(x + 1) \} \).
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- **Node.** Let $X = \{y^2 = x^2(x + 1)\}$. Let

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$f$ is birational with inverse

$$g: X \dashrightarrow \mathbb{A}^1, \ (x, y) \mapsto y/x.$$

$X$ is rational.
Plane cubics

- **Cusp.** Let \( X = \{ y^2 = x^3 \} \). Define
  \[
  f : \mathbb{A}^1 \rightarrow X, \quad t \mapsto (t^2, t^3).
  \]

  Then \( f \) is a birational map with inverse
  \[
  g : X \dasharrow \mathbb{A}^1, \quad (x, y) \mapsto y/x.
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  \( X \) is rational.

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In particular \( X \) is not rational.