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- functions on affine sets → sheaves
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Example: \( \mathcal{F} \to X \) sheaf, \( U \subset X \) open, then the restriction \( \mathcal{F}|_U \) is a sheaf given by

\[
\mathcal{F}|_U(V) = \mathcal{F}(V)
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for \( V \subset U \) open.
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The stalks are also rings e.g.

\[
[(U, s)] + [(U', s')] = [(U \cap U', s|_{U \cap U'} + s'|_{U \cap U'})].
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Lemma

Let $X \subset \mathbb{A}^n$ be an affine variety. The stalk of the sheaf $\mathcal{O}_X$ at $p$ is the local ring

$$\mathcal{O}_{X,p} = \left\{ \frac{f}{g} : f, g \in A(X), g(p) \neq 0 \right\} = A(X)_m$$

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(e.g. $m =$ functions vanishing at $p$).
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- This is independent of choices since if

\[
\phi = \frac{f}{g} \text{ in } U, \quad \phi = \frac{f'}{g'} \text{ in } V,
\]

then \( \frac{f}{g} = \frac{f'}{g'} \) in \( U \cap V \), hence also in \( K(X) \) (last time).
Ringed spaces

Definition

A pair \((X, \mathcal{O}_X)\) consisting of a topological space \(X\) together with a sheaf of rings \(\mathcal{O}_X\) is called a ringed space.
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Definition

Let \((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\) be ringed spaces endowed with sheaves of \(k\)-valued functions.

Remark: There is an induced pullback on stalks (check) \(f^*: \mathcal{O}_Y, p \rightarrow \mathcal{O}_X, p\).
Definition

Let \((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\) be ringed spaces endowed with sheaves of \(k\)-valued functions.

A morphism \(f : X \to Y\) is a set-theoretic map such that

- \(f\) is continuous,
- for a regular section \(\phi \in \mathcal{O}_Y(U)\), the pullback \(f^*\phi = \phi \circ f : f^{-1}(U) \to k\) is also regular:

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