Math 203A

September 26, 2022
Last time

- We want to establish a geometry-algebra dictionary:

  \[
  \text{algebraic sets in } \mathbb{A}^n \leftrightarrow \text{ideals in } k[x_1, \ldots, x_n]
  \]

  via $Z$ and $I$. 
Last time

- We want to establish a geometry-algebra dictionary:

\[
\text{algebraic sets in } \mathbb{A}^n \leftrightarrow \text{ideals in } k[x_1, ..., x_n]
\]

via \(Z\) and \(I\).

- We know

\[
ZI(X) = X.
\]
Last time

- We want to establish a geometry-algebra dictionary:

  algebraic sets in $\mathbb{A}^n \leftrightarrow$ ideals in $k[x_1, \ldots, x_n]$

  via $Z$ and $I$.

- We know

  $$ZI(X) = X.$$
Question: Is $I$ truly the inverse of $Z$:

$$I(Z(i)) = i?$$
Question: Is $I$ truly the inverse of $Z$:

$$I(Z(i)) = i?$$

Example: Let

$$f = (x - a)\alpha.$$
Question: Is \( I \) truly the inverse of \( Z \):

\[
I(Z(i)) = i?
\]

Example: Let

\[
f = (x - a)^\alpha.
\]

Then

\[
Z(f) = \{a\} \implies IZ(f) = \langle x - a \rangle \neq (f).
\]
Definition
The radical of an ideal $i$ is

$$\sqrt{i} = \{ f : f^r \in i \text{ for some } r > 0 \}$$
Definition
The radical of an ideal \( i \) is

\[ \sqrt{i} = \{ f : f^r \in i \text{ for some } r > 0 \} \]

Definition
An ideal \( i \) is radical if \( \sqrt{i} = i \).
Definition
The radical of an ideal $i$ is

$$\sqrt{i} = \{ f : f^r \in i \text{ for some } r > 0 \}$$

Definition
An ideal $i$ is \textbf{radical} if $\sqrt{i} = i$. 
Nullstellensatz

Theorem (Strong Nullstellensatz)

\[ I_Z(i) = \sqrt{i} \]
Nullstellensatz

Theorem (Strong Nullstellensatz)

\[ IZ(i) = \sqrt{i} \]

Proof uses the weak Nullstellensatz

\[ Z(\alpha) = \emptyset \iff \alpha = (1). \]
Nullstellensatz

Theorem (Strong Nullstellensatz)

\[ I_Z(i) = \sqrt{i} \]

Proof uses the weak Nullstellensatz

\[ Z(a) = \emptyset \iff a = (1). \]

Corollary

There is a 1–1 inclusion reversing correspondence between
Nullstellensatz

Theorem (Strong Nullstellensatz)

$$IZ(i) = \sqrt{i}$$

Proof uses the weak Nullstellensatz

$$Z(a) = \emptyset \iff a = (1).$$

Corollary

There is a 1 – 1 inclusion reversing correspondence between

- **algebraic sets in** \( \mathbb{A}^n \)
Nullstellensatz

Theorem (Strong Nullstellensatz)

\[ I_Z(i) = \sqrt{i} \]

Proof uses the weak Nullstellensatz

\[ Z(a) = \emptyset \iff a = (1). \]

Corollary

There is a 1–1 inclusion reversing correspondence between

- algebraic sets in \( \mathbb{A}^n \)
- radical ideals in \( k[x_1, \ldots, x_n] \).
Nullstellensatz

Theorem (Strong Nullstellensatz)

\[ \mathcal{I} \mathcal{Z}(i) = \sqrt{i} \]

Proof uses the weak Nullstellensatz

\[ \mathcal{Z}(a) = \emptyset \iff a = (1). \]

Corollary

There is a 1 − 1 inclusion reversing correspondence between

- algebraic sets in \( \mathbb{A}^n \)
- radical ideals in \( k[x_1, \ldots, x_n] \).
Figure: David Hilbert
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subset I_Z(i)$. 

Note that $Z(j) = \emptyset$. Indeed, if $(p, t) \in Z(j)$ then $p \in Z(i)$, $tf(p) - 1 = 0 \Rightarrow f(p) \neq 0$ contradiction.
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subseteq I_Z(i)$.

- Let $f \in I_Z(i)$. We show $f^N \in i$. 

**Note:** $Z(j) = \emptyset$. Indeed, if $(p, t) \in Z(j)$ then $p \in Z(i)$, $tf(p) - 1 = 0$ $\Rightarrow$ $f(p) \neq 0$ contradiction.
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subset I_Z(i)$.

- Let $f \in I_Z(i)$. We show $f^N \in i$.

- Let

  \[ j = i + (tf - 1) \subset k[x_1, \ldots, x_n, t]. \]
Proof of strong Nullstellensatz

► It is easy to see $\sqrt{i} \subset IZ(i)$.

► Let $f \in IZ(i)$. We show $f^N \in i$.

► Let

$$j = i + (tf - 1) \subset k[x_1, \ldots, x_n, t].$$

► Note that $Z(j) = \emptyset$. 
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subseteq I_Z(i)$.

- Let $f \in I_Z(i)$. We show $f^N \in i$.

- Let $j = i + (tf - 1) \subseteq k[x_1, \ldots, x_n, t]$.

- Note that $Z(j) = \emptyset$. Indeed, if $(p, t) \in Z(j)$ then $p \in Z(i), \ tf(p) - 1 = 0$.
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subset IZ(i)$.

- Let $f \in IZ(i)$. We show $f^N \in i$.

- Let
  \[ j = i + (tf - 1) \subset k[x_1, \ldots, x_n, t]. \]

- Note that $Z(j) = \emptyset$. Indeed, if $(p, t) \in Z(j)$ then
  \[ p \in Z(i), \quad tf(p) - 1 = 0 \implies f(p) \neq 0 \]
  contradiction.
Proof of strong Nullstellensatz

- It is easy to see $\sqrt{i} \subset IZ(i)$.

- Let $f \in IZ(i)$. We show $f^N \in i$.

- Let $j = i + (tf - 1) \subset k[x_1, \ldots, x_n, t]$.

- Note that $Z(j) = \emptyset$. Indeed, if $(p, t) \in Z(j)$ then

  $$p \in Z(i), \quad tf(p) - 1 = 0 \implies f(p) \neq 0$$

  contradiction.
Proof

- Hence $j = (1)$:
Proof

Hence $j = (1)$:

$$1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)$$

with $f_i \in i$. 

Let $t \mapsto ft - 1$ and $t \mapsto f_1$.
Proof

- Hence $j = (1)$:
  
  $$1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)$$

  with $f_i \in i$.

- Let
  
  $$R[t] \rightarrow R_f, \ t \mapsto \frac{1}{f}$$
Proof

- Hence $j = (1)$:

\[
1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)
\]

with $f_i \in i$.

- Let

\[
R[t] \rightarrow R_f, t \mapsto \frac{1}{f}
\]

- \[
1 = f_1 \cdot g_1(x_1, \ldots, x_n, 1/f) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, 1/f).
\]
Proof

▶ Hence $j = (1)$:

$$1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)$$

with $f_i \in i$.

▶ Let

$$R[t] \rightarrow R_f, \ t \mapsto \frac{1}{f}$$

▶

$$1 = f_1 \cdot g_1(x_1, \ldots, x_n, \frac{1}{f}) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, \frac{1}{f}).$$

▶ Let $t^N$ the highest power of $t$ occurring in the $g_i$
Proof

- Hence $j = (1)$:

$$1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots$$

$$+ f_m \cdot g_m(x_1, \ldots, x_n, t)$$

with $f_i \in i$.

- Let

$$R[t] \to R_f, t \mapsto \frac{1}{f}$$

- 

$$1 = f_1 \cdot g_1(x_1, \ldots, x_n, 1/f) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, 1/f).$$

- Let $t^N$ the highest power of $t$ occurring in the $g_i$

- 

$$f^N = f_1 \cdot G_1(x_1, \ldots, x_n) + \ldots + f_m \cdot G_m(x_1, \ldots, x_n).$$
Proof

Hence \( j = (1) \):

\[
1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)
\]

with \( f_i \in i \).

Let

\[
R[t] \to R_f, \ t \mapsto \frac{1}{f}
\]

Let \( t^N \) the highest power of \( t \) occurring in the \( g_i \)

\[
f^N = f_1 \cdot G_1(x_1, \ldots, x_n) + \ldots + f_m \cdot G_m(x_1, \ldots, x_n).
\]

\[
f_i \in i \implies f^N \in i.
\]
Proof

- Hence \( j = (1) \):

\[
1 = (ft - 1) \cdot g_0(x_1, \ldots, x_n, t) + f_1 \cdot g_1(x_1, \ldots, x_n, t) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, t)
\]

with \( f_i \in i \).

- Let

\[
R[t] \to R_f, t \mapsto \frac{1}{f}
\]

- \( 1 = f_1 \cdot g_1(x_1, \ldots, x_n, 1/f) + \ldots + f_m \cdot g_m(x_1, \ldots, x_n, 1/f) \).

- Let \( t^N \) the highest power of \( t \) occurring in the \( g_i \)

\[
f^N = f_1 \cdot G_1(x_1, \ldots, x_n) + \ldots + f_m \cdot G_m(x_1, \ldots, x_n).
\]

- \( f_i \in i \implies f^N \in i \).
Irreducibility

We study affine algebraic sets by breaking them into smaller pieces.

Example: \( X = \mathbb{Z}(xy) \subset \mathbb{A}^2 \) is the union of the two coordinate axes: \( X = X_1 \cup X_2 \) where \( X_1 = \mathbb{Z}(y) \), \( X_2 = \mathbb{Z}(x) \). The set \( X \) is said to be reducible, and \( X_1 \) and \( X_2 \) are its irreducible components.

Definition A topological space \( X \) is reducible if \( X = X_1 \cup X_2 \) for two proper closed subsets \( X_1 \) and \( X_2 \).
Irreducibility

We study affine algebraic sets by breaking them into smaller pieces.

Example: $X = Z(xy) \subset \mathbb{A}^2$ is the union of the two coordinate axes:
Irreducibility

We study affine algebraic sets by breaking them into smaller pieces.

Example: \( X = Z(xy) \subset \mathbb{A}^2 \) is the union of the two coordinate axes:

\[ X = X_1 \cup X_2 \]

where \( X_1 = Z(y) \), \( X_2 = Z(x) \).
Irreducibility

We study affine algebraic sets by breaking them into smaller pieces.

Example: \( X = Z(xy) \subseteq \mathbb{A}^2 \) is the union of the two coordinate axes:

\[
X = X_1 \cup X_2
\]

where \( X_1 = Z(y), X_2 = Z(x) \).

The set \( X \) is said to be reducible, and \( X_1 \) and \( X_2 \) are its irreducible components.
Irreducibility

We study affine algebraic sets by breaking them into smaller pieces.

Example: $X = Z(xy) \subset \mathbb{A}^2$ is the union of the two coordinate axes:

$$X = X_1 \cup X_2$$

where $X_1 = Z(y)$, $X_2 = Z(x)$.

The set $X$ is said to be reducible, and $X_1$ and $X_2$ are its irreducible components.

Definition

A topological space $X$ is reducible if $X = X_1 \cup X_2$ for two proper closed subsets $X_1$ and $X_2$. 
Irreducibility

Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.
Irreducibility

Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.

A disconnected set is reducible.
Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.

A disconnected set is reducible.

Remark: The affine line $\mathbb{A}^1$ is irreducible in the Zariski topology (cofinite topology).
Irreducibility

**Remark:** If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.

A disconnected set is reducible.

**Remark:** The affine line $\mathbb{A}^1$ is irreducible in the Zariski topology (cofinite topology).

When $k = \mathbb{C}$, $\mathbb{A}^1$ is reducible in the usual topology.
Irreducibility

Remark: If \( X_1 \) and \( X_2 \) are required disjoint, \( X \) is said to be disconnected.

A disconnected set is reducible.

Remark: The affine line \( \mathbb{A}^1 \) is irreducible in the Zariski topology (cofinite topology).

When \( k = \mathbb{C} \), \( \mathbb{A}^1 \) is reducible in the usual topology. Set

\[
X_1 = \{ z \in \mathbb{C} : |z| \geq 1 \}, \quad X_2 = \{ z \in \mathbb{C} : |z| \leq 1 \}.
\]
Irreducibility

Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.

A disconnected set it reducible.

Remark: The affine line $\mathbb{A}^1$ is irreducible in the Zariski topology (cofinite topology).

When $k = \mathbb{C}$, $\mathbb{A}^1$ is reducible in the usual topology. Set

$$X_1 = \{z \in \mathbb{C} : |z| \geq 1\}, \quad X_2 = \{z \in \mathbb{C} : |z| \leq 1\}.$$ 

However, $\mathbb{A}^1$ is connected.
Irreducibility

Remark: If $X_1$ and $X_2$ are required disjoint, $X$ is said to be disconnected.

A disconnected set it reducible.

Remark: The affine line $\mathbb{A}^1$ is irreducible in the Zariski topology (cofinite topology).

When $k = \mathbb{C}$, $\mathbb{A}^1$ is reducible in the usual topology. Set

$$X_1 = \{z \in \mathbb{C} : |z| \geq 1\}, \quad X_2 = \{z \in \mathbb{C} : |z| \leq 1\}.$$  

However, $\mathbb{A}^1$ is connected.
Lemma

Let $X$ be irreducible.

- $U$ and $V$ are nonempty open subsets of $X$, then $U \cap V \neq \emptyset$. 

Definition

An irreducible affine algebraic set is called an affine variety.
Irreducibility

Lemma

Let $X$ be irreducible.

- $U$ and $V$ are nonempty open subsets of $X$, then $U \cap V \neq \emptyset$.
- Nonempty open sets $U$ of $X$ are dense.
Irreducibility

Lemma
Let $X$ be irreducible.

- $U$ and $V$ are nonempty open subsets of $X$, then $U \cap V \neq \emptyset$.
- Nonempty open sets $U$ of $X$ are dense.

Proof: Assume otherwise. Then the closed sets $\overline{U}$ and $X \setminus U$ would cover $X$. 
Lemma

Let $X$ be irreducible.

- $U$ and $V$ are nonempty open subsets of $X$, then $U \cap V \neq \emptyset$.
- Nonempty open sets $U$ of $X$ are dense.

Proof: Assume otherwise. Then the closed sets $\overline{U}$ and $X \setminus U$ would cover $X$.

Definition

An irreducible affine algebraic set is called an affine variety.
Prime ideals

Affine algebraic sets are in 1 – 1 correspondence with radical ideals.
Prime ideals

Affine algebraic sets are in 1 – 1 correspondence with radical ideals.

How about affine varieties?
Prime ideals

Affine algebraic sets are in 1 – 1 correspondence with radical ideals.

How about affine varieties?

Theorem

An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$
Prime ideals

Affine algebraic sets are in 1 − 1 correspondence with radical ideals.

How about affine varieties?

Theorem
An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$

Proof: We prove $X$ is reducible $\iff I(X)$ is not prime.
Prime ideals

Affine algebraic sets are in 1 – 1 correspondence with radical ideals.

How about affine varieties?

Theorem

An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$

Proof: We prove $X$ is reducible $\iff I(X)$ is not prime.

Assume $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets.
Prime ideals

Affine algebraic sets are in 1 − 1 correspondence with radical ideals.

How about affine varieties?

Theorem

An affine algebraic set \( X \subset \mathbb{A}^n \) is irreducible \( \iff \) \( I(X) \) is a prime ideal of \( k[X_1, \ldots, X_n] \)

Proof: We prove \( X \) is reducible \( \iff \) \( I(X) \) is not prime.

Assume \( X = X_1 \cup X_2 \), with \( X_1, X_2 \neq X \) proper closed subsets.

Find \( f \in I(X_1) \setminus I(X) \) and \( g \in I(X_2) \setminus I(X) \).
Prime ideals

Affine algebraic sets are in 1−1 correspondence with radical ideals.

How about affine varieties?

**Theorem**

An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$

**Proof:** We prove $X$ is reducible $\iff I(X)$ is not prime.

Assume $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets.

Find $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Since $f = 0$ on $X_1$ and $g = 0$ on $X_2$, the product $fg = 0$ on $X_1 \cup X_2 = X$. 
Prime ideals

Affine algebraic sets are in $1-1$ correspondence with radical ideals.

How about affine varieties?

**Theorem**

An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$

**Proof:** We prove $X$ is reducible $\iff I(X)$ is not prime.

Assume $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets.

Find $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Since $f = 0$ on $X_1$ and $g = 0$ on $X_2$, the product $fg = 0$ on $X_1 \cup X_2 = X$.

Therefore $fg \in I(X)$, while $f, g \notin I(X)$. 
Prime ideals

Affine algebraic sets are in 1−1 correspondence with radical ideals.

How about affine varieties?

Theorem

An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$

Proof: We prove $X$ is reducible $\iff I(X)$ is not prime.

Assume $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets.

Find $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Since $f = 0$ on $X_1$ and $g = 0$ on $X_2$, the product $fg = 0$ on $X_1 \cup X_2 = X$.

Therefore $fg \in I(X)$, while $f, g \notin I(X)$. Thus $I(X)$ is not prime.
Prime ideals

Affine algebraic sets are in $1 - 1$ correspondence with radical ideals.

How about affine varieties?

**Theorem**

*An affine algebraic set $X \subset \mathbb{A}^n$ is irreducible $\iff I(X)$ is a prime ideal of $k[X_1, \ldots, X_n]$*

**Proof**: We prove $X$ is reducible $\iff I(X)$ is not prime.

Assume $X = X_1 \cup X_2$, with $X_1, X_2 \neq X$ proper closed subsets.

Find $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. Since $f = 0$ on $X_1$ and $g = 0$ on $X_2$, the product $fg = 0$ on $X_1 \cup X_2 = X$.

Therefore $fg \in I(X)$, while $f, g \notin I(X)$. Thus $I(X)$ is not prime.
Example: The curve $y^2 - x^3 = 0$ in $\mathbb{A}^2$ is irreducible.
Example: The curve $y^2 - x^3 = 0$ in $\mathbb{A}^2$ is irreducible.

Example: The only proper irreducible closed subsets of $\mathbb{A}^1$ are single points.
Example: The curve $y^2 - x^3 = 0$ in $\mathbb{A}^2$ is irreducible.

Example: The only proper irreducible closed subsets of $\mathbb{A}^1$ are single points.

Lemma

If $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map and $X$ is irreducible in $\mathbb{A}^n$, then $f(X)$ is also irreducible.
Example: The curve \( y^2 - x^3 = 0 \) in \( \mathbb{A}^2 \) is irreducible.

Example: The only proper irreducible closed subsets of \( \mathbb{A}^1 \) are single points.

Lemma

If \( f : \mathbb{A}^n \to \mathbb{A}^m \) is a polynomial map and \( X \) is irreducible in \( \mathbb{A}^n \), then \( f(X) \) is also irreducible.

Proof: This follows from HWK Problem 2(i) and Problem 7.
Example: The curve $y^2 - x^3 = 0$ in $\mathbb{A}^2$ is irreducible.

Example: The only proper irreducible closed subsets of $\mathbb{A}^1$ are single points.

Lemma

If $f : \mathbb{A}^n \rightarrow \mathbb{A}^m$ is a polynomial map and $X$ is irreducible in $\mathbb{A}^n$, then $f(X)$ is also irreducible.

Proof: This follows from HWK Problem 2(i) and Problem 7.
Example: Let $C$ be the curve given parametrically by

$$(t^2, t^4, t^5), \ t \in k.$$
**Example:** Let $C$ be the curve given parametrically by

$$(t^2, t^4, t^5), t \in k.$$ 

We claim $C$ is irreducible.
Example: Let $C$ be the curve given parametrically by
\[(t^2, t^4, t^5), t \in k.\]

We claim $C$ is irreducible.

Proof: This curve is the image of the polynomial map
\[f : \mathbb{A}^1 \to \mathbb{A}^3, t \to (t^2, t^4, t^5).\]
Example: Let $C$ be the curve given parametrically by

$$(t^2, t^4, t^5), t \in k.$$ 

We claim $C$ is irreducible.

Proof: This curve is the image of the polynomial map

$$f : \mathbb{A}^1 \to \mathbb{A}^3, t \to (t^2, t^4, t^5).$$

By the lemma, $C$ is irreducible.
Example: Let $C$ be the curve given parametrically by

$$(t^2, t^4, t^5), t \in k.$$ 

We claim $C$ is irreducible.

Proof: This curve is the image of the polynomial map

$$f : \mathbb{A}^1 \to \mathbb{A}^3, t \to (t^2, t^4, t^5).$$

By the lemma, $C$ is irreducible.

Remark: It is harder to see $C$ is irreducible using the equations of the curve

$$y = x^2, x^5 = z^2.$$
Example: Let $C$ be the curve given parametrically by

$$(t^2, t^4, t^5), t \in k.$$

We claim $C$ is irreducible.

Proof: This curve is the image of the polynomial map

$$f : \mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t^2, t^4, t^5).$$

By the lemma, $C$ is irreducible.

Remark: It is harder to see $C$ is irreducible using the equations of the curve

$$y = x^2, x^5 = z^2.$$