Math 203A
Last time

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\dim X = \max_i \dim U_i.
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- For $X = \mathbb{P}^n$ and $U_i = \{x_i \neq 0\} \simeq \mathbb{A}^n$, we have
  \[ \dim \mathbb{P}^n = \dim \mathbb{A}^n \implies \dim \mathbb{A}^n = n. \]
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Proposition (Noether normalization)

Let \( X \subseteq \mathbb{P}^n \) be irreducible, and \( p \notin X, p = [0 : \ldots : 0 : 1] \).
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Let $X \subset \mathbb{P}^n$ be irreducible, and $p \notin X$, $p = [0 : \ldots : 0 : 1]$. Let $f \in k[x_0, \ldots, x_n]$ homogeneous.
Proposition (Noether normalization)

Let \( X \subsetneq \mathbb{P}^n \) be irreducible, and \( p \notin X, p = [0 : \ldots : 0 : 1] \).
Let \( f \in k[x_0, \ldots, x_n] \) homogeneous. There exists \( D > 0 \) and
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a_1, \ldots, a_D \in k[x_0, \ldots, x_{n-1}] \text{ homogeneous}
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such that
\[
f^D + a_1 f^{D-1} + \ldots + a_D = 0 \text{ on } X.
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Remark:

- Take \( f = x_n \).

Then, fibers of \( \pi : X \to \mathbb{P}^{n-1} \) have \( D \) points (counted with multiplicity).
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Proof:

- $d = \text{deg } f$. Define

\[ \Phi : X \to \mathbb{P}^n, \quad \Phi(x) = [x_0^d : \ldots : x_{n-1}^d : f]. \]
Proof:

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\[ \Phi \text{ is well-defined} \]
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- **Claim:**

  $$Z(F_1, \ldots, F_r, y_0, \ldots, y_{n-1}) = \emptyset$$
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Detour: Finite maps - affine case

We already defined quasi-finite maps.
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Algebraically: $A \subset B$ are rings.

$B$ is integral over $A$ if all $x \in B$ satisfy a monic equation $x^n + a_1 x^{n-1} + ... + a_n = 0$, $a_i \in A$.

$B$ is finite over $A$ if $B$ is finitely generated $A$-module.

$\text{finite} \implies \text{integral}$

for finitely generated $k$-algebras $\text{finite} \iff \text{integral}$
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Geometrically: Let $f : X \to Y$ be dominant morphism of affine algebraic sets, and

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- **think** $A(Y) \hookrightarrow A(X)$
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- $f$ is finite if $A(Y) \hookrightarrow A(X)$ is integral.
Algebra is the offer made by the devil to the mathematician.

The devil says: ‘I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.’

M. Atiyah
Intuitively: If $X \subset \mathbb{A}^n$, the coordinate function $t_i \in A(X)$ satisfies

$$t_i^k + a_1 \cdot t_i^{k-1} + \ldots + a_k = 0$$

for $a_i \in A(Y)$. Thus $t_i(x)$ takes on finitely many values.

As $y$ varies, the points in $f^{-1}(y)$ can come together but cannot disappear.
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- for each $y \in Y$, $x \in f^{-1}(y)$,

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\text{finite} \quad \Rightarrow \quad \text{quasi} - \text{finite}
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- As \( y \) varies, the points in \( f^{-1}(y) \) can come together but cannot dissapear. Draw!
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\[ Z = \{ xy - 1 = 0 \} , \quad \pi : Z \to \mathbb{A}^1 , \quad (x, y) \to x \]

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Lemma
For a finite morphism of affine sets $f : X \to Y$

Sketch:

(i) If $y \in Y$, let $m$ be the maximal ideal in $A(Y)$.

There exists a maximal ideal $n$ in $A(X)$ such that $n \cap A(X) = m$.

$n$ corresponds to $x \in X \Rightarrow f(x) = y$.

(ii) Let $Z \subset X$ be closed. Work with $f : Z \to W$, $W = f(Z)$.

By (i), $f$ is surjective, so $W = f(Z)$.

Thus $f(Z)$ is closed.
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(i) If \( y \in Y \), let \( m \) be the maximal ideal in \( A(Y) \). Let \( n \) maximal ideal in \( A(X) \) such that \( n \cap A(X) = m \).

\( n \) corresponds to \( x \in X \implies f(x) = y \).

(ii) Let \( Z \subset X \) be closed. Work with

\[ f : Z \to W, \quad W = \overline{f(Z)}. \]
Lemma

For a finite morphism of affine sets $f : X \to Y$

(i) $f$ is surjective

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If $X$, $Y$ are varieties, $f : X \rightarrow Y$ is finite if there exists an affine open cover $V_i$ of $Y$ s.t. $f^{-1}(V_i)$ is affine.
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Rephrasing of Noether normalization

**Theorem**

Let $p \notin X \subset \mathbb{P}^n$. The *projection* away from $p$

$$\pi : X \to \pi(X)$$

is a *finite* morphism.