Math 203A, Solution Set 8.

Problem 1. Consider the Cremona birational automorphism of \( \mathbb{P}^2 \) given by
\[
C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_2].
\]
Let \( \tilde{\mathbb{P}}^2 \) be the blowup of \( \mathbb{P}^2 \) at the three points \( P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0] \) and \( P_3 = [0 : 0 : 1] \) where \( C \) is undefined. Show that

(i) Show that \( C \) extends to an isomorphism
\[
\tilde{C} : \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2.
\]

(ii) Let \( E_1, E_2, E_3 \) be the exceptional lines for the blowup, and let \( L_{ij} \) be strict transform of the line through \( P_i \) and \( P_j \). Draw the incidence graph of the configuration of lines. What happens to the 6 lines under \( \tilde{C} \)?

Answer: (i) The ideal of \( P_1, P_2, P_3 \) is \( I = (x_1x_2, x_0x_2, x_0x_1) \), where \( x_0, x_1, x_2 \) are the homogeneous coordinates on \( \mathbb{P}^2 \). The blowup
\[
\tilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2
\]
has as equations \([y_0 : y_1 : y_2] = [x_1x_2 : x_0x_2 : x_0x_2] \) whenever the latter is defined. Here \( y_0, y_1, y_2 \) are the coordinates on the second \( \mathbb{P}^2 \). We let \( i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2 \) be the natural involution
\[
([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \to ([y_0 : y_1 : y_2], [x_0 : x_1 : x_2]).
\]
We claim that \( i \) restricts to \( C \) on the open set \( U = \mathbb{P}^2 \setminus \{P_1, P_2, P_3\} \) where \( C \) is defined. Since \( U \) is dense in \( \tilde{\mathbb{P}}^2 \), this implies that \( i(\tilde{\mathbb{P}}^2) \subset \tilde{\mathbb{P}}^2 \). Since \( i \) is an involution, the similar statements for the inverse are automatic, completing the proof.

Consider the inclusion \( i_1 : U \to \tilde{\mathbb{P}}^2 \) into the first blowup given by the graph of the ideal \( I \):
\[
[x_0 : x_1 : x_2] \to ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2] = C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1]).
\]
Similarly, the inclusion \( i_2 : U \to \tilde{\mathbb{P}}^2 \) into the second blowup is given by
\[
[y_0 : y_1 : y_2] \to ([y_0 : y_1 : y_2], [y_1y_2 : yo2y : y0y1]).
\]
Thus along \( U \) we have
\[
i(i_1([x_0 : x_1 : x_2])) = i_2(C[x_0 : x_1 : x_2]).
\]
This clearly shows that on \( U \), the involution \( i \) equals \( C \).

(ii) For distinct indices \( i, j, k \), the line \( L_{ij} \) intersects \( E_i \) and \( E_j \), but does not intersect \( E_k \). The Cremona transformation sends \( L_{ij} \) to \( E_k \).
These statements can be checked explicitly as follows. The equations of $\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ are

(1) \[ x_0 y_0 = x_1 y_1 = x_2 y_2. \]

Indeed, along the graph of $C$ restricted to $U$, we have

(2) \[ [y_0 : y_1 : y_2] = [x_1 x_2 : x_0 x_2 : x_0 x_1] \]

wherever these points are defined, hence (1) is satisfied on $U$ and also on the closure. Since the set

\[ x_0 y_0 = x_1 y_1 = x_2 y_2 \]

is irreducible, it must therefore coincide with the closure. In order to check irreducibility, it suffices to restrict to the affine patch $x_0 = 1, y_0 = 1$. If we show irreducibility in each affine patch, since these patches are easily seen to pairwise intersect, we can deduce irreducibility using Problem Set 1, Problem 6. Now, we must show that the set

\[ X = \{ x_1 y_1 = x_2 y_2 = 1 \} \]

is irreducible. But this is clear, as this last set is the image of the morphism

\[ A^1 \setminus \{ 0 \} \times A^1 \setminus \{ 0 \} \ni (x_1, x_2) \to (x_1 x_2^{-1}, x_2, x_2^{-1}) = (x_1, y_1, x_2, y_2) \in X \]

whose source is irreducible.

Now, on the preimage $E_1$ of $P_1$ must have

\[ x_1 = x_2 = 0 \]

which implies from (1) that $E_1$ has the equation

\[ y_0 = 0, x_1 = x_2 = 0. \]

We find the strict transform of the line through $P_2$ and $P_3$. This line has equation

\[ x_0 = 0. \]

Its preimage in $\mathbb{P}^2$ has equations

\[ x_1 y_1 = x_2 y_2 = 0. \]

This is the union of the exceptional line $E_2$ where $y_1 = 0, x_0 = x_2 = 0$, the exceptional line $E_3$ where $y_2 = 0, x_0 = x_1 = 0$ and of the strict transform $L_{23}$ with equation

\[ y_1 = y_2 = 0, x_0 = 0. \]

This shows that $\iota$ exchanges $L_{23}$ and $E_1$.

The incidence graph looks like a hexagon with two inner triangles joined together by edges. If we represent the vertices in the order $E_1, L_{12}, E_2, L_{23}, E_3, L_{31}$,
then the Cremona transformation reflects the hexagon such that the vertices become $L_{23}, E_3, L_{13}, E_1, L_{12}, E_3$.

\[ \square \]

**Problem 2.** Show that the Segre embedding \[ \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1} \] has degree \( \binom{n+m}{n} \).

**Answer:** Let \( \Sigma_{m,n} \) be the image of the Segre embedding. Degree \( \ell \) homogeneous polynomials on \( \mathbb{P}^{(n+1)(m+1)-1} \) restrict to \( \Sigma_{m,n} \) as polynomials in the variables \( x_i \) on \( \mathbb{P}^n \) and \( y_j \) on \( \mathbb{P}^m \), bihomogeneous of degree \( \ell \). The dimension of \( S(\Sigma_{m,n})^{(\ell)} \) then equals \[ \binom{\ell+n}{n} \binom{\ell+m}{m}. \]

Expanding, we have \[ \binom{\ell+n}{n} \binom{\ell+m}{m} = \frac{1}{n!m!} \ell^{n+m} + \text{L.o.t.} \]
which shows that the degree of \( \Sigma_{m,n} \) equals \[ \frac{(n+m)!}{n!m!} = \binom{n+m}{n}. \]

\[ \square \]

**Problem 3.** Let \( X \subset \mathbb{P}^n \) be a projective scheme with Hilbert polynomial \( \chi_X \). Define the arithmetic genus of \( X \) to be \[ p_a(X) = (-1)^{\dim X} \chi_X(0) - 1. \]

(i) Show that the genus of \( \mathbb{P}^n \) is zero.

(ii) If \( X \) is a hypersurface of degree \( d \) in \( \mathbb{P}^n \), show that \( p_a(X) = \binom{d-1}{n} \). In particular, a cubic in \( \mathbb{P}^2 \) has genus 1.

(iii) If \( X \) is a complete intersection of two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) then \[ p_a(X) = \frac{1}{2} ab(a+b-4) + 1. \]

In particular, intersection of two quadrics in \( \mathbb{P}^3 \) has genus 1.

**Answer:** (i) In lecture, we calculated the Hilbert polynomial of \( \mathbb{P}^n \) to be \( \chi(\ell) = \binom{\ell+n}{n} \). This yields immediately that \( p_a(\mathbb{P}^n) = 0 \).

(ii) In lecture, we calculated the Hilbert polynomial of a degree \( d \) hypersurface to be \[ \chi(\ell) = \binom{n+\ell}{n} - \binom{n+\ell-d}{n}. \]
This yields \[ p_a(X) = (-1)^{n-1} \left( 1 - \binom{n-d}{n} - 1 \right) = (-1)^n \binom{n-d}{n} = \binom{d-1}{n}. \]
(iii) We claim that the Hilbert polynomial of the complete intersection equals
\[ \chi(\ell) = \binom{\ell + 3}{3} - \binom{\ell + 3 - a}{3} - \binom{\ell + 3 - b}{b} + \binom{\ell + 3 - a - b}{3}. \]

Then, we find
\[ \chi(0) = 1 - \binom{3 - a}{a} - \binom{3 - b}{b} + \binom{3 - a - b}{3} \]
which yields the answer.

The claim about the Hilbert polynomial is justified as follows. Let \( f \) and \( g \) be the equations of the two surfaces of degree \( a \) and \( b \) in \( \mathbb{P}^3 \) whose intersection is \( X \). There is an exact sequence
\[ 0 \rightarrow S(\mathbb{P}^3)^{(\ell - a - b)} \rightarrow S(\mathbb{P}^3)^{(\ell - a)} \oplus S(\mathbb{P}^3)^{(\ell - b)} \rightarrow S(\mathbb{P}^3)^{(\ell)} \rightarrow S(X)^{(\ell)} \rightarrow 0 \]
where the first two maps are given by
\[ P \mapsto (gP, fP) \]
and
\[ (P, Q) \mapsto fP - gQ \]
and the last map is the restriction. We conclude by considering dimensions.

\[ \square \]

**Problem 4.** Given four general lines in \( \mathbb{P}^3 \), show that there are exactly 2 lines which intersect all four of them.

**Answer:** Recall that the space of lines in \( \mathbb{P}^3 \) is parametrized by the Grassmannian \( G = G(1, 3) \) which can be realized as a quadric in \( \mathbb{P}^5 \) via the Plucker embedding. For each line \( L_i \) define
\[ X_i = \{ M \text{ line in } \mathbb{P}^3 : M \cap L_i \neq \emptyset \} \subset G(1, 3) \subset \mathbb{P}^5. \]

We claim that
\[ X_i = H_i \cap G \]
for a hyperplane \( H_i \) in \( \mathbb{P}^5 \). Indeed, working in Plucker coordinates, assume that \( L = L_i \) has coordinates \( l_{ij} \) and \( M \) has Plucker coordinates \( m_{kl} \). If these are calculated with respect to points \( A, B \) over \( L \) and points \( C, D \) on \( M \) then we have
\[ a \wedge b = \sum_{ij} l_{ij} e_i \wedge e_j \]
\[ c \wedge d = \sum_{kl} m_{kl} e_k \wedge e_l. \]
The requirement that \( L \) and \( M \) meet is equivalent to
\[ a \wedge b \wedge c \wedge d = 0 \]
since the vector space spanned by \(a, b, c, d\) is 3 dimensional in this case. This gives
\[
\left( \sum_{ij} l_{ij} e_i \wedge e_j \right) \wedge \left( \sum_{kl} m_{kl} e_k \wedge e_l \right) = 0
\]
which gives
\[
l_{12}m_{34} - l_{13}m_{24} + l_{14}m_{23} + l_{23}m_{14} - l_{24}m_{13} + l_{34}m_{12} = 0.
\]
This is clearly a linear equation in the variables \(m_{kl}\) for each fixed \(l_{ij}\).

Now, the lines \(M\) that intersect \(L_1, L_2, L_3, L_4\) are found as the intersection points
\[
X_1 \cap X_2 \cap X_3 \cap X_4 \subset G(1, 3).
\]
In other words, these points correspond to
\[
H_1 \cap H_2 \cap H_3 \cap H_4 \cap G(1, 3) \subset \mathbb{P}^5.
\]
We claim that this intersection consists of 2 points in general.

We claim first that the intersection \(H_1 \cap H_2 \cap H_3 \cap H_4\) is a line \(\ell\) in \(\mathbb{P}^5\) in general. In any case, the intersection is given as the null space of the \(4 \times 6\) matrix of coefficients describing the hyperplanes \(H_i\). In general, this null space is 2 dimensional, so the intersection is a line, but it can also be that the null space has dimension 3 or higher. This condition is described as the rank of the matrix being 3 or less – in turn this is given by the vanishing of the \(4 \times 4\) minors, so it is a closed subset \(Z\) in the space \(G \times G \times G \times G\). We assume \((L_1, L_2, L_3, L_4)\) are chosen away from \(Z\).

Next, if \(\ell\) is the intersection line, we claim it intersects the quadric \(G\) in \(\mathbb{P}^5\) in 2 points. Indeed, we may assume that after a change of coordinates, this line is given by \(x_2 = x_3 = x_4 = x_5 = 0\). The quadric \(G\) will be given by \(\sum a_{ij} x_i x_j\) and the intersection of the line \(\ell\) is obtained by solving
\[
 a_{00}x_0^2 + a_{11}x_1^2 + a_{01}x_0x_1 = 0
\]
which has exactly two solutions. The only exceptions correspond to
\[
a_{01}^2 = 4a_{00}a_{11}
\]
which corresponds to one solution, or the case \(a_{00} = a_{01} = a_{11} = 0\) which corresponds to infinitely many solutions. These are closed conditions determining a closed set \(W\) as one can check by tracing back this condition under the inverse change of coordinates that took \(\ell\) to the special line and recalling that \(\ell\) is given as the null vector of the matrix constructed out of the Plucker coordinates of the lines \(L_1, L_2, L_3, L_4\).

Setting \(U = G \setminus (Z \cup W)\) we obtain that for \((L_1, L_2, L_3, L_4)\) in \(U\), there are exactly 2 lines intersecting \(L_i\). Then \(U\) is dense in \(G \times G \times G \times G\) if nonempty. To show nonemptyness, we can pick 4 lines
\[
L_1 = \{x_0 = x_1 = 0\}, L_2 = \{x_0 = x_2 = 0\}, L_3 = \{x_0 + x_1 = x_2 + x_3 = 0\}
\]
\[ L_4 = \{x_0 + 2x_1 = x_2 + 2x_3 = 0\}. \]

We claim this quadruple lies in \( U \). Indeed, one can easily run the argument above to find the equations of the hyperplanes \( H_1, H_2, H_3, H_4 \) above in terms of the Plucker coordinates. We obtain

\[ m_{01} = 0, m_{02} = 0, m_{13} + m_{03} + m_{12} + m_{02} = 0, 4m_{13} + 2m_{03} + 2m_{12} + m_{02} = 0. \]

We also have

\[ m_{01}m_{23} - m_{02}m_{13} + m_{03}m_{12} = 0 \]

for the equation of the quadric. These equations only have 2 common solutions as one checks immediately. \( \square \)

**Problem 5.** Let \( X \) be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree \( d \) and codimension \( c \) in \( \mathbb{P}^n \).

(i) (Intersecting \( X \) with hyperplanes to cut down the dimension), show inductively that

\[ d \geq c + 1. \]

(iii) Show that equality holds for rational normal curves in \( \mathbb{P}^n \), and for the image \( v(\mathbb{P}^2) \) of the Veronese embedding

\[ v : \mathbb{P}^2 \to \mathbb{P}^5. \]

(iii) Can you classify the varieties of degree 2?

**Answer:** (i) We prove the statement by induction on \( n \), allowing for \( X \) to be not necessarily irreducible (but non degenerate). Consider a general hyperplane \( H \) and consider the intersection \( X \cap H \). By Bezout this has the same degree \( d \) as \( X \), and it has codimension \( c \) in \( H \), hence the inequality to prove for \( X \) is equivalent to the inequality to prove for \( X \cap H \) (which is still nondegenerate, see below). By induction, we reduce to the case when \( X \) consists of \( d \) distinct points in \( \mathbb{P}^n \). In this case, we need to show \( d \geq n + 1 \) which is clear since if \( d \leq n \), then \( X \) would be degenerate, as any \( n \) points are contained in a hyperplane \( H \). This last statement can be seen by representing the \( n \) points by \( n \) vectors of size \( n + 1 \) spanning a vector subspace \( W \), and picking a vector \((a_0, \ldots, a_n)\) in the orthogonal complement \( W^\perp \). The hyperplane is

\[ a_0x_0 + \ldots + a_nx_n = 0. \]

To see \( Y = X \cap H \) is nondegenerate, assume it is contained in a hyperplane \( W \) of \( H \), and let \( p \in X \setminus W \) which is possible since \( X \) is nondegenerate. Let \( U \) be the hyperplane in \( \mathbb{P}^n \) spanned by \( p \) and \( W \). Then

\[ \deg X = \deg(X \cap U) \]
and $X \cap U$ is of dimension 1 less than $X$ because $X$ is not contained in $U$ being nondegenerate. The intersection $X \cap U$ contains $X \cap W = (X \cap H) \cap W = Y \cap W = Y$ and another component $Z$ in which $p \not\in Y$ lies. There could be additional components. But then
\[ \deg X = \deg(X \cap U) \geq \deg Z + \deg Y > \deg Y = \deg(X \cap H) = \deg X \]
which is a contradiction.

(ii) Both cases are particular examples of the Veronese embedding whose degree we calculate below.

Consider Veronese embedding
\[ v_d : \mathbb{P}^n \to \mathbb{P}^N \]
constructed from degree $d$ monomials. We claim that the image $V_d$ has degree $d^n$. Indeed, degree $\ell$ polynomials in $N + 1$ variables become, after restricting to $V_d$, polynomials of degree $d\ell$ on $\mathbb{P}^n$. Hence the Hilbert function of $V_d$ equals
\[ \chi(\ell) = \binom{d\ell + n}{n} = d^n \frac{\ell^n}{n!} + \text{l.o.t.} \]
confirming the claim.

Now, it is easy to see that
\[ d^n = \text{codim } V_d + 1 = \binom{d + n}{n} - n \]
holds for $n = 1$ or for $d = n = 2$.

(iii) After passing to a smaller projective space, we may assume that $X$ is nondegenerate. Degree $d = 2$ forces $c = 1$ hence $X$ is isomorphic to a projective quadric.

\[ \square \]