Math 203A, Solution Set 8.

Problem 1.  
(i) Show that any singular irreducible cubic in \( \mathbb{P}^2 \) is isomorphic to either the nodal or the cuspidal cubics:
\[
y^2z = x^2(x + z) \text{ or } y^2z = x^3.
\]
(ii) Using (i), show that irreducible cubics in \( \mathbb{P}^2 \) can have at most 1 singular point. Exhibit a cubic in \( \mathbb{P}^2 \) with 3 singular points.

Answer:  
(i) Assume the singularity is at \([0 : 0 : 1]\) and let \( f \) be the polynomial giving the cubic. Then since
\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0,
\]
f may not contain an \( xz^2 \), \( yz^2 \) or \( z^3 \) terms. That is, \( z \) does not appear to a power higher than 1 in \( f \), and so we may write the cubic as
\[
(\text{quadratic polynomial in } x, y) \cdot z = Q(x, y)
\]
where \( Q \) is a cubic polynomial in \( x, y \).

The quadratic polynomial is either the square of a linear term or the product of two distinct linear terms in \( x \) and \( y \). In the first case, let \( y^{\text{new}} \) be that linear term, or in the second let \( x^{\text{new}} \) and \( y^{\text{new}} \) be the two linear terms. We obtain
\[
y^2z = Q(x, y) \quad \text{or} \quad xyz = Q(x, y)
\]

Consider the first case, \( y^2z = Q(x, y) \). We note first that \( Q \) must contain an \( x^3 \) term, as otherwise both sides are divisible by \( y \) contradicting that the conic is irreducible. First make a scaling change of coordinates of \( x \) so that this \( x^3 \) term has coefficient 1. We seek to "complete the cube" on the right-hand side. Given the coefficient of \( x^2y \) in \( Q \), there are specific coefficients for \( xy^2 \) and \( y^3 \) such that the right-hand side is a perfect cube. By making the change of coordinates
\[
z \mapsto \lambda x + \mu y + z,
\]
with the appropriate choices of \( \lambda \) and \( \mu \), we produce the terms \( \lambda xy^2 \) and \( \mu y^3 \), which can be used to complete the cube. This produces
\[
y^2z = (x + by)^3.
\]

With the final change of coordinates \( x^{\text{new}} = x + by \), we end with
\[
y^2z = x^3
\]
as desired.
Now consider the second case, \( xyz = Q(x, y) \). As before, \( Q \) must contain an \( x^3 \) and a \( y^3 \) term, or else it would violate irreducibility. First make scaling change of coordinates of \( x \) and \( y \) so that the coefficients of \( x^3 \) and \( y^3 \) are 1. Then by making the change of coordinates \( z \mapsto \lambda x + \mu y + z \), we once again complete the cube on the right-hand side (this time filling in the terms \( x^2y \) and \( xy^2 \)). This produces

\[
xyz = (x + y)^3.
\]

Now make the change of coordinates

\[
\begin{align*}
x' &= x + y \\
y' &= x - y \\
z' &= -z/4
\end{align*}
\]

with inverse

\[
\begin{align*}
x &= (x' + y')/2 \\
y &= (x' - y')/2 \\
z &= -4z'
\end{align*}
\]

Under these change of coordinates, the equation becomes

\[
(y^2 - x^2)z = x^3,
\]

or equivalently \( y^2z = x^2(x + z) \)

as desired.

(ii) Consider the case of the nodal cubic

\[
f = x^2(x + z) - y^2z = 0.
\]

Then

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 + 2xz \\
\frac{\partial f}{\partial y} &= -2yz \\
\frac{\partial f}{\partial z} &= x^2 - y^2.
\end{align*}
\]

If the second is to vanish, then \( y = 0 \) or \( z = 0 \). If \( y = 0 \), then from the \( z \)-partial we get \( x = 0 \), so we find the point \([0 : 0 : 1]\), which is indeed singular. If \( z = 0 \), then from the \( x \)-partial we get \( x = 0 \), then from the \( z \)-partial we get \( z = 0 \). But then \( x = y = z = 0 \), which is impossible. Therefore, the only singularity is \([0 : 0 : 1]\).

For the cuspidal cubic

\[
f = x^3 - y^2z = 0,
\]

the partials are

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3x^2 \\
\frac{\partial f}{\partial y} &= -2yz \\
\frac{\partial f}{\partial z} &= -y^2.
\end{align*}
\]

If all are to vanish, then we must have \( x = y = 0 \), so we get the unique singular point \([0 : 0 : 1]\).

The reducible cubic given by

\[
xyz = 0
\]

has the three singularities \([1 : 0 : 0]\), \([0 : 1 : 0]\), and \([0 : 0 : 1]\).
Problem 2. Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x, y, z]$. Consider the morphism

$$\Phi : C \to \mathbb{P}^2, p \mapsto \left[ \frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p) \right].$$

The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to $C$.

(i) Why is $\Phi$ a well-defined morphism? Find a geometric description of $\Phi$, independent of coordinate choices.

(ii) If $C$ is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic $C$ is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$. How does the morphism $\Phi$ change when we change coordinates?

(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

Answer: (i) If $p$ is a nonsingular point of $C$, $\frac{\partial f}{\partial x}(p)$, $\frac{\partial f}{\partial y}(p)$, $\frac{\partial f}{\partial z}(p)$ cannot be zero simultaneously by the projective Jacobi criterion. Thus $\Phi$ is well-defined. The line $L_p$ with equation

$$\frac{\partial f}{\partial x}(p)x + \frac{\partial f}{\partial y}(p)y + \frac{\partial f}{\partial z}(p)z = 0$$

is the tangent line to $C$ at $p$. Under the identification of $\mathbb{P}^2$ with the dual projective space $(\mathbb{P}^2)^\vee$, $\Phi$ associates to each $p \in C$, the tangent line at $p$.

(ii) From previous homeworks, we’ve learned to use linear coordinate changes to make a irreducible conic into the form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.$$

If one of the $\lambda_i = 0$, the conic is reducible. So we can assume $\lambda_i \neq 0$ for all $i$. Then

$$\Phi : C \to \mathbb{P}^2, [x : y : z] \mapsto [\lambda_1 x : \lambda_2 y : \lambda_3 z].$$

Therefore, for points in the image $\Phi(C)$ we have

$$X = \lambda_1 x, Y = \lambda_2 y, Z = \lambda_3 z,$$

with $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$.

Thus $\Phi(C)$ is the conic

$$\frac{1}{\lambda_1} X^2 + \frac{1}{\lambda_2} Y^2 + \frac{1}{\lambda_3} Z^2 = 0.$$

Taking into account the changes of coordinates, we see that $\Phi(C)$ is always a conic.
(iii) Let \( C \) be an arbitrary conic. We claim that the dual of the dual of \( C \) is \( C \).
Indeed, since the description of \( \Phi \) is independent of coordinates, we may first assume that \( C \) is of the form
\[
\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0.
\]
We have seen above that the dual conic becomes
\[
\frac{1}{\lambda_1} X^2 + \frac{1}{\lambda_2} Y^2 + \frac{1}{\lambda_3} Z^2 = 0.
\]
Dualizing once more, we recover the original conic \( C \).

Suppose \( L_i \) are 5 lines \( a_i x + b_i + c_i z = 0 \) such that the 5 points \([a_i, b_i, c_i] \in \mathbb{P}^2\) are in general position, i.e. no three of this points are collinear (or equivalently no three of the lines are concurrent). Let \( D \) be the unique conic passing through the 5 points \([a_i, b_i, c_i]\). Thus the dual conic of \( D \) is still a conic. By definition, the dual conic of \( D \) in \( \mathbb{P}^2 \) is tangent to the five lines \( L_i \).

\[ \square \]

**Problem 3.** Resolve the singularities of the following curve by subsequent blow-ups
\[
y^2 - x^{k+1} = 0.
\]

**Answer:** Assume first \( k = 2n \) is even. Let \( C_n \) be the curve \( y^2 - x^{2n+1} = 0 \). The only singularity for \( C_n \) is \((0,0)\). Let \( \pi \) be the blowup of \( \mathbb{A}^2 \) at the origin and let \( \overline{C}_n \) be the proper transform. The blowup has two charts. In one of the charts the blow down map is
\[
(u, v) \rightarrow (u, uv).
\]
In this chart, we compute the proper transform \( \overline{C}_n \). The curve \( \pi^{-1}(C_n) \) is given by
\[
f(u, uv) = (uv)^2 - u^{2n+1} = u^2(u^2 - v^{2n+1})
\]
is the union of \( \ell = \{u = 0\} \) and \( C_{n-1} \). The curve \( C_{n-1} \) is still singular at the origin.

The other chart of the blowup is similar, the blow down map being \((u, v) \rightarrow (uv, u)\). A similar argument shows that there are no singularities of the proper transform in this chart. Thus \( \pi : \overline{C}_n \rightarrow C_n \) and \( \overline{C}_n \) has one singular point, such that \( \overline{C}_n \) in an affine chart is isomorphic to \( C_{n-1} \subset \mathbb{A}^2 \). Continuing the process until \( n = 1 \), \( C_1 \) is given by \( y^2 - x = 0 \) is nonsingular, thus the singularities of \( C_n \) are resolved after a chain of \( n \) blow-ups. The exceptional set is a chain of lines.

The case \( k \) odd is entirely similar. \( \square \)

**Problem 4.** Let \( X \subset \mathbb{A}^n \) be an affine variety and let \( p \in X \). Let \( \mathfrak{m} \) is the maximal ideal of \( \mathcal{O}_{X,p} \). Show that there exists an isomorphism
\[
k[x_1, \ldots, x_n]/\mathfrak{m}^n \rightarrow \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}.
\]
Remark: If $i^{in}$ is radical, then the coordinate ring $A(C_{X,p})$ of the tangent cone of $X$ at $p$ is isomorphic to the graded algebra $\bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$, so the tangent cone is intrinsic. However, the tangent cone is better defined as a “scheme”

$$\text{Spec} \left(k[x_1, \ldots, x_n]/i^{in}\right) = \text{Spec} \left(\bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}\right).$$

Answer: Let $i \subset k[x_1, \ldots, x_n]$ be the ideal of $X$, and assume $p = 0$. Let $\mathfrak{m}$ denote the ideal of $p$ in $A(X)$. Since $C_{X,p}$ was defined as the vanishing locus of the initial ideal $i^{in}$, we first explain the isomorphism

$$\phi : k[x_1, \ldots, x_n]/i^{in} \rightarrow \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}.$$

This is defined via

$$\phi : f \mapsto f^{(k)}|_X$$

where $f^{(k)}$ are the homogeneous pieces of a polynomial $f$. This is well defined since if $f = h^{(in)}$ for some $h \in i$ then

$$h = f + \text{terms of order at least } k + 1 \equiv f \mod \mathfrak{m}^{k+1}.$$ 

Restricting to $X$, since $h|_X = 0$ we obtain that

$$\phi(f) = f|_X \in \mathfrak{m}^{k+1}$$

proving that $\phi$ is well-defined. Furthermore, $\phi$ is an algebra homomorphism. Since $\phi$ maps $x_i \rightarrow x_i$ which generate the algebra on the right hand side, $\phi$ is surjective. It is not hard to check that $\phi$ is injective. Indeed, if $\phi(f) = 0$ then

$$f^{(k)}|_X \in \mathfrak{m}^{k+1} \subset (x_1, \ldots, x_n)^{k+1}.$$ 

This implies that for some $h \in i$, we have

$$f^{(k)} - h \in (x_1, \ldots, x_n)^{k+1}.$$ 

In particular, looking at the pieces of degrees $j \in \{0, \ldots, k\}$, we obtain

$$f^{(k)} = h^{(k)}, \quad h^{(j)} = 0 \text{ for } j < k.$$ 

This means $f^{(k)} = h^{in} \in i^{in}$ for all $k$, which gives $f \in i^{in}$. This proves injectivity and establishes the isomorphism

$$k[x_1, \ldots, x_n]/i^{in} \rightarrow \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}.$$ 

To complete the proof, we show

$$\bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1} \cong \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}.$$
Recall that $m$ is the maximal ideal in $\mathcal{O}_{X,p} = A(X)_m$. Thus it suffices to show the following algebraic statement:

**Lemma.** Let $A$ be a ring, $m$ a maximal ideal, $n = mA_m$ the maximal ideal in $A_m$. Then

$$\bigoplus_k \frac{m^k}{m^{k+1}} \cong \bigoplus_k \frac{n^k}{n^{k+1}}$$

is an isomorphism.

**Proof.** It suffices to prove

$$\pi_k : A/m^k \to A_m/n^k, \quad a \mapsto \frac{a}{1}$$

is an isomorphism. Then, using induction on $k$, and the exact sequences

$$0 \to \frac{m^k}{m^{k+1}} \to A/m^{k+1} \to A/m^k \to 0$$

$$0 \to \frac{n^k}{n^{k+1}} \to A_m/n^{k+1} \to A_m/n^k \to 0$$

we conclude the proof of the lemma.

Let $S = A - m$. Then

$$n^k = mA_m = \left\{ \frac{a}{s} : a \in m^k, s \in S \right\}.$$

To show $\pi_k$ is injective, assume

$$\pi_k(a) = 0 \implies \frac{a}{1} \in n^k \implies \frac{a}{1} = \frac{\alpha}{s} \text{ with } \alpha \in m^k, s \in S.$$

Thus, for some $u \in S$, we have $asu = \alpha u$ so $asu \in m^k$ so $asu = 0$ in $A/m^k$. However, $su$ is a unit in $A/m^k$. Indeed, $A/m^k$ has only one maximal ideal $m/m^k$ (since those maximal ideals correspond to maximal ideals in $A$ containing $m$, hence to $m$ itself), and $su$ is not in the maximal ideal by assumption. Thus since $su$ is a unit in $A/m^k$ it follows $a = 0$ in $A/m^k$ as needed.

To show $\pi_k$ is surjective, consider $\frac{a}{s}$ an element in $A_m/n^k$. We seek $a \in A$ such that $\frac{a}{1} = \frac{a}{s} \mod n^k$ in $A_m$. Since $s \in S$, and the only maximal ideal in $A/m^k$ is $m/m^k$ which does not contain $s$, it follows $s$ is a unit in $A/m^k$. Therefore

$$(s) + m^k = A.$$

Thus, there exists $b \in A$, $\beta \in m^k$ such that

$$bs + \beta = 1 \implies s \alpha + \beta \alpha = \alpha \implies \frac{\alpha}{s} = \frac{b\alpha}{1} + n^k.$$

Setting $a = b\alpha$, we have $\pi_k(a) = \frac{a}{s}$ as needed.

□

**Problem 5.** Consider the blowup of the affine variety $X \subset \mathbb{A}^n$ at $p \in X$. Show that the exceptional hypersurface is the projectivization of the tangent cone

$$E \cong \mathbb{P}(C_{X,p}).$$
You may want to generalize the argument we had in class for plane curves.

Answer: Changing coordinates we may assume $p = (0, \ldots, 0)$. Write $X$ as the vanishing locus of $f_1, \ldots, f_r$, let $f_1^{\text{in}}, \ldots, f_r^{\text{in}}$ denote the initial terms, whose degrees are $d_1, \ldots, d_r$. We need to prove

$$E \cong Z_p(f_1^{\text{in}}, \ldots, f_r^{\text{in}}),$$

where $Z_p$ denotes the projective vanishing locus.

The blowup at the origin $\widetilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ can be covered by coordinate patches

$$U_i = \{(x, y) \in \widetilde{\mathbb{A}}^n, y_i \neq 0\},$$

where $x \in \mathbb{A}^n, y \in \mathbb{P}^{n-1}$. Each $U_i$ is isomorphic to $\mathbb{A}^n$. For instance, the isomorphism

$$\mathbb{A}^n \hookrightarrow U_1 \subset \widetilde{\mathbb{A}}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

is given by

$$(x_1, y_2, \ldots, y_n) \mapsto ((x_1, x_1 y_2, \ldots, x_1 y_n), [1 : y_1 : \ldots : y_n]).$$

Consider $V_i = \mathcal{X} \cap U_i$, so that we have a covering

$$\mathcal{X} = \bigcup_{i=1}^n V_i.$$

We will work locally in each affine open $U_i$. Let $i = 1$. Over the subset $V_1 \subset U_1 \cong \mathbb{A}^n$ we must have

$$f_k(x_1, x_1 y_2, \ldots, x_1 y_n) = 0, \quad 1 \leq k \leq r.$$

Set

$$g_k(x_1, y_2, \ldots, y_n) = \frac{f_k(x_1, x_1 y_2, \ldots, x_1 y_n)}{x_1^{d_k}}.$$

Note that for $V_1 \cap \{x_1 \neq 0\} = V_1 \setminus E$, we must have $g_k = 0$, so that same thing must be true over the closure $\overline{V}_1$. Thus $V_1 \subset \mathbb{A}^n$ is cut out by the polynomials $g_k(x_1, y_2, \ldots, y_n)$.

Expanding

$$f_k = f_k^{\text{in}} + \text{h.o.t}$$

we obtain

$$g_k(x_1, y_2, \ldots, y_n) = f_k^{\text{in}}(1, y_2, \ldots, y_n) + \text{terms that involve } x_1.$$ 

Over the exceptional hypersurface $E$, we must have $x_1 = 0$, hence $E \cap V_1$ is given by setting $x_1 = 0$ in $g_k$ yielding

$$f_k^{\text{in}}(1, y_2, \ldots, y_n),$$

which is just the restriction of $f_k^{\text{in}}$ to the affine patch $U_1$. This proves that

$$E \cong Z_p(f_1^{\text{in}}, \ldots, f_r^{\text{in}}).$$

□