Problem 1. Assume that \( f : X \to Y \) is a morphism of projective varieties. Show that \( Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} \) is closed.

Answer: Let \( n = \dim X \), \( m = \dim Y \). We will use induction on \( m \), the case \( m = 0 \) being clear. We may assume \( f \) surjective, else we can replace \( Y \) by the image of \( f \) which is also projective. Let \( d = n - m \geq 0 \). When \( k \leq d \) there is nothing to prove since \( Y_k = Y \) by the theorem of dimension of fibers. Assume \( k > d \). Let \( U \) be an open subset over which \( \dim f^{-1}(y) = d \). The existence of \( U \) is proven in the theorem of dimension of fibers. Let \( Z = Y \setminus U \) is a closed subset of \( Y \), so \( \dim Z < \dim Z = \dim Z \leq m - 1 \). Note that \( Y_k \subset Z \) for \( k > d \). We wish to show that \( Y_k \) is closed in \( Z \), so then it will be closed in \( Y \) as well.

To this end, consider the restriction \( \tilde{f} : W \to Z \), where \( W = f^{-1}(Z) \). We clearly have
\[
Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} = \{ y \in Z : \dim \tilde{f}^{-1}(y) \geq k \}.
\]
We use induction on the dimension of the base to conclude. Indeed, \( \dim Z \leq m - 1 \), and working with each irreducible component of \( Z \) one at a time, we may assume \( Z \) is irreducible. In this case however the domain \( W = f^{-1}(Z) \) may have components \( W_1, \ldots, W_r \). Since \( Z \) is closed, \( W_i \) are closed in \( X \) hence projective in \( X \). Let \( \tilde{f}_i : W_i \to Z \) be the restricted morphism. Then
\[
Y_k = \bigcup Y_k(\tilde{f}_i)
\]
where the sets of the right are calculated with respect to the morphisms \( \tilde{f}_i \). By the induction hypothesis applied to \( \tilde{f}_i \), it follows that \( Y_k(\tilde{f}_i) \) are closed in the image of \( \tilde{f}_i \) which in turn is closed in \( Z \) by projective hypothesis, hence \( Y_k \) is closed in \( Z \) as well. \( \square \)

Problem 2. (i) Let \( d > 2n - 3 \). Show that a general degree \( d \) hypersurface in \( \mathbb{P}^n \) contains no lines.

(ii) Show that any cubic surface in \( \mathbb{P}^3 \) contains at least one line. We will see later that a smooth cubic surface contains exactly 27 lines.

(iii) Let \( f \) be a degree 4 homogeneous polynomial in 4 variables and let \( Z_f \) be the quartic surface \( f = 0 \) in \( \mathbb{P}^3 \). Show that there is a single polynomial \( \Phi \) in the coefficients of \( f \) which vanishes if and only if the quartic surface \( Z_f \subset \mathbb{P}^3 \) contains a line.

Answer: (i) We think of a hypersurface \( X = Z(f) \) as a point in projective space \( \mathbb{P}^N \) for \( N = \binom{n+d}{d} - 1 \), by means of the coefficients \( a_I \) of its defining equation
\[
f = \sum_{I} a_I X^I.
\]
We form the incidence correspondence
\[ J = \{(L, X) : L \subset X\} \subset \mathbb{G}(1, n) \times \mathbb{P}^N \]
and we let
\[ p : J \to \mathbb{G}(1, n), \quad q : J \to \mathbb{P}^N \]
be the two projections.

We claim that the fibers of \( p \) have dimension \( N - (d + 1) \). Indeed, fix a line \( L \) and study \( p^{-1}(L) \). Without loss of generality, we may assume \( L \) is given by the equations
\[ x_0 = \ldots = x_{n-2} = 0. \]
If \( X \in p^{-1}(L) \) is given by the polynomial
\[ f = 0, \]
the requirement \( L \subset X \) means
\[ f(0 : \ldots : 0 : s : t) = 0 \]
for all \( s, t \). In particular, the \( d+1 \) coefficients of \( s^i t^{d-1} \) for \( 0 \leq i \leq d \) must vanish:
\[ a_{0\ldots0,i,d-i} = 0, \]
while the other coefficients are arbitrary. Thus \( p^{-1}(L) \) has codimension \( d + 1 \) in \( \mathbb{P}^N \), as claimed. Also the fibers of \( p \) are irreducible so \( J \) is irreducible as well by Problem 3.

With this understood, we conclude by looking at the fibers of \( p \) that
\[ \dim J = \dim \mathbb{G}(1, n) + N - (d + 1) = (2n - 2) + N - (d + 1) < N. \]
Therefore, the morphism \( q \) is not surjective. In particular, the image \( q(J) \) is a proper subvariety of \( \mathbb{P}^N \). For hypersurfaces \( X \) belonging to the complement \( \mathbb{P}^n \setminus q(J) \), the preimage \( q^{-1}(X) \) is therefore empty, or in other words, for there are no lines lying on such hypersurfaces.

(ii) When \( d = n = 3 \), we have \( N = 19 \). For \( J = \{(L, X) : L \subset X\} \), we have \( \dim J = N = 19 \). We seek to show that
\[ q : J \to \mathbb{P}^{19}, \quad (L, X) \mapsto L \]
is surjective. Assume not, then \( q(J) \) is closed in \( \mathbb{P}^{19} \) hence its dimension is \( \leq 18 \). Applying the theorem of fiber dimensions, we have that all nonempty fibers of \( q \) have dimension at least \( \dim J - \dim q(J) \geq 19 - 18 = 1 \). To obtain a contradiction, we exhibit a cubic surface with finitely many lines. For instance, we can let \( X \) to be the surface
\[ x^3 + y^3 + z^3 + w^3 = 0. \]
By symmetry, we may search for lines of the form $x = az + bw, y = cz + dw$ and substituting we find
\[
(az + bw)^3 + (cz + dw)^3 + z^3 + w^3 = 0.
\]
This gives
\[
a^3 + c^3 + 1 = b^3 + d^3 + 1 = 0, \quad a^2b + c^2d = 0, \quad ab^3 + cd^3 = 0.
\]
We claim that there are finitely many solutions for $a, b, c, d$. If $a = 0$ then it is easy to conclude that $d = 0$ and $b, c$ have to satisfy $b^3 = c^3 = -1$, and the solution set is finite. Assume now that neither $a, b, c, d$ is zero. Then
\[
a^2b = -c^2d, \quad ab^2 = -cd^2 \implies a/b = c/d = t.
\]
We find $a = bt, c = dt$ so
\[
0 = a^2b + c^2d = t(b^3 + d^3) = -t
\]
or $a = c = 0$. Again, the set of solutions is finite.

(iii) In this case, we have $d = 4, n = 3, N = 34$. Let
\[
J = \{(L, X) : L \subset X\}.
\]
In this case, the above computation show that $\dim J = N - 1$. We claim that the image $q(J)$ is a codimension 1 subvariety of $\mathbb{P}^N$. We complete the proof letting $\Phi$ be a polynomial cutting out $q(J)$.

To prove $q(J)$ is of dimension 33, assume otherwise, namely that the dimension is 32 or lower. By the theorem of dimension of fibers, for all $[X] \in q(J)$, the fiber $q^{-1}([X])$ has dimension at least $33 - 32 = 1$. In other words all quartics that contain at least one line in fact contain infinitely many lines. One counterexample however is the quartic
\[
x^4 + y^4 + z^4 + w^4 = 0.
\]
By symmetry, we may search for lines of the form $x = az + bw, y = cz + dw$ and substituting we find
\[
(az + bw)^4 + (cz + dw)^4 + z^4 + w^4 = 0.
\]
This gives
\[
a^4 + c^4 + 1 = b^4 + d^4 + 1 = 0, a^2b^2 + c^2d^2 = 0, a^3b + c^3d = 0, ab^3 + cd^3 = 0.
\]
We claim that there are finitely many solutions for $a, b, c, d$. If $a = 0$ then it is easy to conclude that $d = 0$ and $b, c$ have to satisfy $b^4 = c^4 = -1$, and the solution set is finite (and nonempty). Assume now that neither $a, b, c, d$ is zero. Then
\[
a^3b = -c^3d, \quad ab^3 = -cd^3 \implies (a/b)^2 = (c/d)^2
\]
and in addition \((ab)^2 = -(cd)^2\) so multiplying we find \(a^4 = -c^4\) which contradicts \(a^4 + c^4 = -1\). 

\[\square\]

**Problem 3.** Assume that \(f : X \to Y\) is a surjective morphism of projective algebraic sets such that \(Y\) is irreducible and all fibers of \(f\) are irreducible of the same dimension. Show that \(X\) is irreducible as well.

**Answer:** Write \(n\) for the common dimension of the fibers. Write \(X_1, \ldots, X_r\) for the irreducible components of \(X\). Then

\[\bigcup f(X_i) = Y\]

and \(f(X_i)\) are closed in \(Y\) since \(f\) is a morphism of projective sets, hence a closed map. Since \(Y\) is irreducible, there exists \(i\) such that \(f(X_i) = Y\). Assume that \(X_1, \ldots, X_s\) are chosen so that

\[f(X_1) = \ldots = f(X_s) = Y\]

but \(f(X_j) \neq Y\) for \(j > s\). Construct \(U_1, \ldots, U_s\) nonempty open sets in \(Y\) such that

\[y \in U_i, 1 \leq i \leq s \implies \dim(f|_{X_i})^{-1}(y) = n_i = \dim X_i - \dim Y.\]

In fact, even for \(j > s\) we can define \(U_j = Y \setminus f(X_j)\) and for \(y \in U_j\) we have

\[(f|_{X_j})^{-1}(y) = \emptyset.\]

Write

\[U = \bigcap_{i=1}^{r} U_i,\]

which is open and nonempty. For \(y \in U\), \(f^{-1}(y)\) is irreducible and nonempty, and is covered by \(X_1, \ldots, X_r\) so it must exist \(i_0\) such that

\[f^{-1}(y) \subset X_{i_0} \]

It is clear from the choice of \(i_0\) that the entire fiber over \(y\) can be computed in \(X_{i_0}\) so that

\[(f|_{X_{i_0}})^{-1}(y) = f^{-1}(y) \neq \emptyset\]

so

\[f|_{X_{i_0}} : X_{i_0} \to Y\]

must be surjective by the definition of \(U_{i_0}\), and \(i_0 \leq s\). Furthermore \(n = n_{i_0}\) is the common dimension of the fibers since the fiber dimension can be calculated at \(y\) and \((f|_{X_{i_0}})^{-1}(y) = f^{-1}(y)\). If \(z \in Y\), then

\[(f|_{X_{i_0}})^{-1}(z) \subset f^{-1}(z)\]
and the left hand side is at least of dimension $n_{i_0} = \dim X_{i_0} - \dim Y$ by the theorem on dimension of fibers. But $f^{-1}(z)$ is irreducible and $n = n_{i_0}$ dimensional, so must have equality. Thus

$$f^{-1}(z) \subseteq X_{i_0}$$

for all $z \in Y$. This shows that there are no components in $X$ other than $X_{i_0}$, so $X$ is irreducible.

\[\square\]

**Problem 4.** Consider the singular plane curves $Z$ and $W$ given by the equations

$$y^2 - x^2(x + 1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain briefly why $Z$ and $W$ are not isomorphic. Explain that $(0, 0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0, 0)$ for $Z$ and $W$? Sketch (the real points of) $Z$ and $W$. Do $Z$ and $W$ look alike near the origin?

(ii) Show that there are formal power series

$$\tilde{x} = f_1 + f_2 + f_3 + \ldots \text{ and } \tilde{y} = g_1 + g_2 + g_3 \ldots$$

in the variables $x$ and $y$ such that the equation of $Z$ becomes

$$\tilde{x}\tilde{y} = 0.$$  

(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

**Answer:**

(i) First, $Z$ and $W$ are not isomorphic because $Z$ is irreducible and $W$ is reducible.

Now, $Z$ is defined by $y^2 - x^2(x + 1) = 0$. The tangent lines can be found by considering the lowest degree terms $y^2 - x^2$. This factors as $(y - x)(y + x)$. So the 2 tangent lines are $y - x = 0$ and $y + x = 0$.

Similarly, $W = (xy = 0)$ is the union of two lines. The tangent lines are just these two lines $x = 0$ and $y = 0$.

(ii) We wish to find $f_i$ and $g_i$ such that

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).$$

First, compare the degree 2 terms, then

$$f_1 g_1 = y^2 - x^2.$$ 

Hence, we can take

$$f_1 = y - x \text{ and } g_1 = y + x.$$
Comparing the degree 3 terms we have

$$-x^3 = (y - x)g_2 + (y + x)f_2.$$  

The polynomials $g_2 = x^2/2$ and $f_2 = -x^2/2$ will work.

Suppose we have found $f_i$ and $g_i$ for $1 \leq i \leq d - 1$ and

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1)$$

up to degree $d$. Comparing degree $d + 1$ terms, we have:

$$f_1g_d + f_2g_{d-1} + \cdots + f_dg_1 = 0.$$  

Now only $f_d$ and $g_d$ are unknown, others are fixed, we can rearrange the equation:

$$(y - x)g_d + (y + x)f_d = -(f_2g_{d-1} + \cdots + f_{d-1}g_2).$$

Notice that

$$f_2g_{d-1} + \cdots + f_{d-1}g_2$$

is a homogeneous polynomial of degree $d + 1$. Let

$$-(f_2g_{d-1} + \cdots + f_{d-1}g_2) = ax^d + yR(x, y).$$

Isolating $x^{d+1}$ and dividing the remaining term by $y$ to obtain $R$, then we need

$$x(f_d - g_d) + y(f_d + g_d) = ax^{d+1} + yR(x, y)$$

This is possible by letting

$$f_d = \frac{1}{2} \left( \frac{a}{2} x^d + R(x, y) \right) \quad \text{and} \quad g_d = \frac{1}{2} \left( -\frac{a}{2} x^d + R(x, y) \right)$$

Therefore we can find $f_d$ and $g_d$ and

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1)$$

up to degree $d + 1$. Inductively, there exist

$$\bar{x} = f_1 + f_2 + f_3 + \ldots$$

$$\bar{y} = g_1 + g_2 + g_3 + \ldots$$

such that

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).$$

(iii) Suppose $C = (H = 0)$ is a curve which has an ordinary double point, we can change coordinates to assume that the singularity is at the origin. Because $C$ has a double point at the origin,

$$H(x, y) = H_2(x, y) + H_3(x, y) + \cdots \quad \deg H_i = k$$

where $H_2$ is a homogeneous polynomial of degree 2, with distinct factors

$$H_2 = (ax + by)(cx + dy).$$
Change coordinates so that
\[ x' = ax + by, \ y' = cx + dy. \]

In the new coordinates,
\[ H = x'y' + H'_3 + H'_4 + \ldots. \]

By the same method as in (ii), we inductively find
\[ \tilde{x} = x' + f_2 + f_3 + \ldots \]
\[ \tilde{y} = y' + g_2 + g_3 + \ldots \]

such that
\[ H = \tilde{x}\tilde{y}. \]

\[ \square \]