Math 203A, Solution Set 7.

**Problem 1.** Assume that \( f : X \to Y \) is a morphism of projective varieties. Show that \( Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} \) is closed.

**Answer:** Let \( n = \dim X, m = \dim Y \). We will use induction on \( m \), the case \( m = 0 \) being clear. We may assume \( f \) surjective, else we can replace \( Y \) by the image of \( f \) which is also projective. Let \( d = n - m \geq 0 \). When \( k \leq d \) there is nothing to prove since \( Y_k = Y \) by the theorem of dimension of fibers. Assume \( k > d \). Let \( U \) be an open subset over which \( \dim f^{-1}(y) = d \). The existence of \( U \) is proven in the theorem of dimension of fibers. Let \( Z = Y \setminus U \) is a closed subset of \( Y \), so \( \dim Z < \dim Y \implies \dim Z \leq m - 1 \). Note that \( Y_k \subset Z \) for \( k > d \). We wish to show that \( Y_k \) is closed in \( Z \), so then it will be closed in \( Y \) as well.

To this end, consider the restriction \( \tilde{f} : W \to Z \), where \( W = f^{-1}(Z) \). We clearly have
\[
Y_k = \{ y \in Y : \dim f^{-1}(y) \geq k \} = \{ y \in Z : \dim \tilde{f}^{-1}(y) \geq k \}.
\]
We use induction on the dimension of the base to conclude. Indeed, \( \dim Z \leq m - 1 \), and working with each irreducible component of \( Z \) one at a time, we may assume \( Z \) is irreducible. In this case however the domain \( W = f^{-1}(Z) \) may have components \( W_1, \ldots, W_r \). Since \( Z \) is closed, \( W_i \) are closed in \( X \) hence projective in \( X \). Let \( \tilde{f}_i : W_i \to Z \) be the restricted morphism. Then
\[
Y_k = \bigcup Y_k(\tilde{f}_i)
\]
where the sets of the right are calculated with respect to the morphisms \( \tilde{f}_i \). By the induction hypothesis applied to \( \tilde{f}_i \), it follows that \( Y_k(\tilde{f}_i) \) are closed in the image of \( \tilde{f}_i \), which in turn is closed in \( Z \) by projective hypothesis, hence \( Y_k \) is closed in \( Z \) as well. \( \square \)

**Problem 2.**

(i) Let \( d > 2n - 3 \). Show that a general degree \( d \) hypersurface in \( \mathbb{P}^n \) contains no lines.

(ii) Show that any cubic surface in \( \mathbb{P}^3 \) contains at least one line. We will see later that a smooth cubic surface contains exactly 27 lines.

(iii) Let \( f \) be a degree 4 homogeneous polynomial in 4 variables and let \( Z_f \) be the quartic surface \( f = 0 \) in \( \mathbb{P}^3 \). Show that there is a single polynomial \( \Phi \) in the coefficients of \( f \) which vanishes if and only if the quartic surface \( Z_f \subset \mathbb{P}^3 \) contains a line.

**Answer:**

(i) We think of a hypersurface \( X = Z(f) \) as a point in projective space \( \mathbb{P}^N \) for \( N = \binom{n+d}{d} - 1 \), by means of the coefficients \( a_I \) of its defining equation
\[
f = \sum_{I} a_I X^I.
\]
We form the incidence correspondence
\[ J = \{(L, X) : L \subset X\} \subset \mathbb{G}(1, n) \times \mathbb{P}^N \]
and we let
\[ p : J \to \mathbb{G}(1, n), \ q : J \to \mathbb{P}^N \]
be the two projections.

We claim that the fibers of \( p \) have dimension \( N - (d + 1) \). Indeed, fix a line \( L \) and study \( p^{-1}(L) \). Without loss of generality, we may assume \( L \) is given by the equations
\[ x_0 = \ldots = x_{n-2} = 0. \]
If \( X \in p^{-1}(L) \) is given by the polynomial
\[ f = 0, \]
the requirement \( L \subset X \) means
\[ f(0 : \ldots : 0 : s : t) = 0 \]
for all \( s, t \). In particular, the \( d+1 \) coefficients of \( s^i t^{d-1} \) for \( 0 \leq i \leq d \) must vanish:
\[ a_{0,0,i,d-i} = 0, \]
while the other coefficients are arbitrary. Thus \( p^{-1}(L) \) has codimension \( d + 1 \) in \( \mathbb{P}^N \), as claimed. Also the fibers of \( p \) are irreducible so \( J \) is irreducible as well by Problem 3.

With this understood, we conclude by looking at the fibers of \( p \) that
\[ \dim J = \dim \mathbb{G}(1, n) + N - (d + 1) = (2n - 2) + N - (d + 1) < N. \]

Therefore, the morphism \( q \) is not surjective. In particular, the image \( q(J) \) is a proper subvariety of \( \mathbb{P}^N \). For hypersurfaces \( X \) belonging to the complement \( \mathbb{P}^n \setminus q(J) \), the preimage \( q^{-1}(X) \) is therefore empty, or in other words, for there are no lines lying on such hypersurfaces.

Aside: To show that \( J \) is closed in \( \mathbb{G}(1, n) \times \mathbb{P}^N \), we note that this is a local condition, so it suffices to work locally in an affine chart of \( \mathbb{G}(1, n) \). Thus we may assume we work over \( \mathbb{G}_{0,1} = \{p_{01} \neq 0\} \) where \( p_{ij} \) are the Plücker coordinates. By row operations we may assume that the line \( L \) is spanned by the vectors
\[ (1, 0, a_2, \ldots, a_n), \ (0, 1, b_2, \ldots, b_n). \]
Write \( f_c = \sum c_\alpha x^\alpha \) for the equation of a hypersurface \( X_c \) where \( c = (c_\alpha) \) is the corresponding moduli point. The condition \( L \subset X \) can be written as
\[ f_c(s, t, sa_2 + tb_2, \ldots, sa_n + tb_n) = 0. \]
We write
\[ f_c(s, t, sa_2 + tb_2, \ldots, sa_n + tb_n) = \sum F_{ij}(a, b, c)s^it^j \]
so
\[ J \cap (\mathbb{G}_{0,1} \times \mathbb{P}^N) \simeq J \cap (\mathbb{A}^{2n-2} \times \mathbb{P}^N) \]
is cut out by the equations
\[ F_{ij}(a, b, c) = 0, \]
so it is a closed subset of \( \mathbb{A}^{2n-2} \times \mathbb{P}^N \).

(ii) When \( d = n = 3 \), we have \( N = 19 \). For \( J = \{(L, X) : L \subset X\} \), we have \( \dim J = N = 19 \). We seek to show that
\[ q : J \to \mathbb{P}^{19}, \quad (L, X) \mapsto L \]
is surjective. Assume not, then \( q(J) \) is closed in \( \mathbb{P}^{19} \) hence its dimension is \( \leq 18 \).
Applying the theorem of fiber dimensions, we have that all nonempty fibers of \( q \) have dimension at least \( \dim J - \dim q(J) \geq 19 - 18 = 1 \). To obtain a contradiction, we exhibit a cubic surface with finitely many lines. For instance, we can let \( X \) to be the surface
\[ x^3 + y^3 + z^3 + w^3 = 0. \]
By symmetry, we may search for lines of the form \( x = az + bw \), \( y = cz + dw \) and substituting we find
\[ (az + bw)^3 + (cz + dw)^3 + z^3 + w^3 = 0. \]
This gives
\[ a^3 + c^3 + 1 = b^3 + d^3 + 1 = 0, \quad a^2b + c^2d = 0, \quad ab^2 + cd^2 = 0. \]
We claim that there are finitely many solutions for \( a, b, c, d \). If \( a = 0 \) then it is easy to conclude that \( d = 0 \) and \( b, c \) have to satisfy \( b^3 = c^3 = -1 \), and the solution set is finite. Assume now that neither \( a, b, c, d \) is zero. Then
\[ a^2b = -c^2d, \quad ab^2 = -cd^2 \implies a/b = c/d = t. \]
We find \( a = bt, c = dt \) so
\[ 0 = a^2b + c^2d = t(b^3 + d^3) = -t \]
or \( a = c = 0 \). Again, the set of solutions is finite.

(iii) In this case, we have \( d = 4, n = 3, N = 34 \). Let \( J = \{(L, X) : L \subset X\} \).

In this case, the above computation show that \( \dim J = N - 1 \). We claim that the image \( q(J) \) is a codimension 1 subvariety of \( \mathbb{P}^N \). We complete the proof letting \( \Phi \) be a polynomial cutting out \( q(J) \).
To prove \( q(J) \) is of dimension 33, assume otherwise, namely that the dimension is 32 or lower. By the theorem of dimension of fibers, for all \([X] \in q(J)\), the fiber \( q^{-1}([X]) \) has dimension at least \( 33 - 32 = 1 \). In other words all quartics that contain at least one line in fact contain infinitely many lines. One counterexample however is the quartic

\[
x^4 + y^4 + z^4 + w^4 = 0.
\]

By symmetry, we may search for lines of the form \( x = az + bw, y = cz + dw \) and substituting we find

\[
(az + bw)^4 + (cz + dw)^4 + z^4 + w^4 = 0.
\]

This gives

\[
a^4 + c^4 + 1 = b^4 + d^4 + 1 = 0, a^2b^2 + c^2d^2 = 0, a^3b + c^3d = 0, ab^3 + cd^3 = 0.
\]

We claim that there are finitely many solutions for \( a, b, c, d \). If \( a = 0 \) then it is easy to conclude that \( d = 0 \) and \( b, c \) have to satisfy \( b^4 = c^4 = -1 \), and the solution set is finite (and nonempty). Assume now that neither \( a, b, c, d \) is zero. Then

\[
a^3b = -c^3d, ab^3 = -cd^3 \implies (a/b)^2 = (c/d)^2
\]

and in addition \((ab)^2 = -(cd)^2\) so multiplying we find \( a^4 = -c^4 \) which contradicts \( a^4 + c^4 = -1 \).

\[\square\]

**Problem 3.** Assume that \( f : X \to Y \) is a surjective morphism of projective algebraic sets such that \( Y \) is irreducible and all fibers of \( f \) are irreducible of the same dimension. Show that \( X \) is irreducible as well.

**Answer:** Write \( n \) for the common dimension of the fibers. Write \( X_1, \ldots, X_r \) for the irreducible components of \( X \). Then

\[
\bigcup f(X_i) = Y
\]

and \( f(X_i) \) are closed in \( Y \) since \( f \) is a morphism of projective sets, hence a closed map. Since \( Y \) is irreducible, there exists \( i \) such that \( f(X_i) = Y \). Assume that \( X_1, \ldots, X_s \) are chosen so that

\[
f(X_1) = \ldots = f(X_s) = Y
\]

but \( f(X_j) \neq Y \) for \( j > s \). Construct \( U_1, \ldots, U_s \) nonempty open sets in \( Y \) such that

\[
y \in U_i, 1 \leq i \leq s \implies \dim(f|_{X_i})^{-1}(y) = n_i = \dim X_i - \dim Y.
\]

In fact, even for \( j > s \) we can define \( U_j = Y \setminus f(X_j) \) and for \( y \in U_j \) we have

\[
(f|_{X_j})^{-1}(y) = \emptyset.
\]
Write
\[ U = \bigcap_{i=1}^{r} U_i, \]
which is open and nonempty. For \( y \in U \), \( f^{-1}(y) \) is irreducible and nonempty, and is covered by \( X_1, \ldots, X_r \) so it must exist \( i_0 \) such that
\[ f^{-1}(y) \subset X_{i_0}. \]
It is clear from the choice of \( i_0 \) that the entire fiber over \( y \) can be computed in \( X_{i_0} \) so that
\[ (f|_{X_{i_0}})^{-1}(y) = f^{-1}(y) \neq \emptyset \]
so
\[ f|_{X_{i_0}} : X_{i_0} \to Y \]
must be surjective by the definition of \( U_{i_0} \), and \( i_0 \leq s \). Furthermore \( n = n_{i_0} \) is the common dimension of the fibers since the fiber dimension can be calculated at \( y \) and \( (f|_{X_{i_0}})^{-1}(y) = f^{-1}(y) \). If \( z \in Y \), then
\[ (f|_{X_{i_0}})^{-1}(z) \subset f^{-1}(z) \]
and the left hand side is at least of dimension \( n_{i_0} = \dim X_{i_0} - \dim Y \) by the theorem on dimension of fibers. But \( f^{-1}(z) \) is irreducible and \( n = n_{i_0} \) dimensional, so must have equality. Thus
\[ f^{-1}(z) \subset X_{i_0} \]
for all \( z \in Y \). This shows that there are no components in \( X \) other than \( X_{i_0} \), so \( X \) is irreducible.

\[ \square \]

**Problem 4.** Consider the singular plane curves \( Z \) and \( W \) given by the equations
\[ y^2 - x^2(x + 1) = 0 \]
and \( xy = 0 \) respectively.

(i) Explain briefly why \( Z \) and \( W \) are not isomorphic. Explain that \((0,0)\) is an ordinary double point for both of these curves. What are the tangent directions at \((0,0)\) for \( Z \) and \( W \)? Sketch (the real points of) \( Z \) and \( W \). Do \( Z \) and \( W \) look alike near the origin?

(ii) Show that there are formal power series
\[ \tilde{x} = f_1 + f_2 + f_3 + \ldots \quad \text{and} \]
\[ \tilde{y} = g_1 + g_2 + g_3 \]
in the variables \( x \) and \( y \) such that the equation of \( Z \) becomes
\[ \tilde{x}\tilde{y} = 0. \]
(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

**Answer:**

(i) First, $Z$ and $W$ are not isomorphic because $Z$ is irreducible and $W$ is reducible.

Now, $Z$ is defined by $y^2 - x^2(x + 1) = 0$. The tangent lines can be found by considering the lowest degree terms $y^2 - x^2$. This factors as $(y - x)(y + x)$. So the 2 tangent lines are $y - x = 0$ and $y + x = 0$.

Similarly, $W = (xy = 0)$ is the union of two lines. The tangent lines are just these two lines $x = 0$ and $y = 0$.

(ii) We wish to find $f_i$ and $g_i$ such that

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).$$

First, compare the degree 2 terms, then

$$f_1g_1 = y^2 - x^2.$$

Hence, we can take

$$f_1 = y - x \text{ and } g_1 = y + x.$$  

Comparing the degree 3 terms we have

$$-x^3 = (y - x)g_2 + (y + x)f_2.$$  

The polynomials $g_2 = x^2/2$ and $f_2 = -x^2/2$ will work.

Suppose we have found $f_i$ and $g_i$ for $1 \leq i \leq d - 1$ and

$$(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1)$$

up to degree $d$. Comparing degree $d + 1$ terms, we have:

$$f_1g_d + f_2g_{d-1} + \cdots + f_dg_1 = 0.$$  

Now only $f_d$ and $g_d$ are unknown, others are fixed, we can rearrange the equation:

$$(y - x)g_d + (y + x)f_d = -(f_{2d-1} + \cdots + f_{d-1}g_2).$$

Notice that

$$f_{2d-1} + \cdots + f_{d-1}g_2$$

is a homogeneous polynomial of degree $d + 1$. Let

$$-(f_{2d-1} + \cdots + f_{d-1}g_2) = ax^d + yR(x, y).$$

Isolating $x^{d+1}$ and dividing the remaining term by $y$ to obtain $R$, then we need

$$x(f_d - g_d) + y(f_d + g_d) = ax^{d+1} + yR(x, y).$$

This is possible by letting

$$f_d = \frac{1}{2} \left( \frac{a}{2} x^d + R(x, y) \right) \quad \text{and} \quad g_d = \frac{1}{2} \left( -\frac{a}{2} x^d + R(x, y) \right).$$
Therefore we can find \( f_d \) and \( g_d \) and
\[
(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1)
\]
up to degree \( d + 1 \). Inductively, there exist
\[
\tilde{x} = f_1 + f_2 + f_3 + \ldots
\]
\[
\tilde{y} = g_1 + g_2 + g_3 + \ldots
\]
such that
\[
(f_1 + f_2 + f_3 + \ldots)(g_1 + g_2 + g_3 + \ldots) = y^2 - x^2(x + 1).
\]

(iii) Suppose \( C = (H = 0) \) is a curve which has an ordinary double point, we can change coordinates to assume that the singularity is at the origin. Because \( C \) has a double point at the origin,
\[
H(x, y) = H_2(x, y) + H_3(x, y) + \cdots \quad \text{deg}H_i = k
\]
where \( H_2 \) is a homogeneous polynomial of degree 2, with distinct factors
\[
H_2 = (ax + by)(cx + dy).
\]
Change coordinates so that
\[
x' = ax + by, y' = cx + dy.
\]
In the new coordinates,
\[
H = x'y' + H_3' + H_4' + \ldots.
\]
By the same method as in (ii), we inductively find
\[
\tilde{x} = x' + f_2 + f_3 + \ldots
\]
\[
\tilde{y} = y' + g_2 + g_3 + \ldots
\]
such that
\[
H = \tilde{x}\tilde{y}.
\]
\[\square\]