Problem 1. Let $V$ be a finite dimensional vector space and let $\omega \in \wedge^2 V$. Show that the following statements are equivalent

(i) $\omega = v \wedge w$ for two vectors $v, w \in V$

(ii) there exists a nonzero $v \in V$ such that $v \wedge \omega = 0$

(iii) $\omega \wedge \omega = 0$.

Use (iii) to show that the Grassmannian $G(1, 3)$ is a (nondegenerate) quadric in $\mathbb{P}^5$.

Answer: We show (i) $\iff$ (iii). It is clear that if $\omega = v \wedge w$ then $\omega \wedge \omega = 0$. Conversely, we will induct on $n = \dim V$, the base case $n = 2$ being clear. For the inductive step, if $\dim V$ is $n + 1$, let $e_0, \ldots, e_n$ be a basis for $V$. Let us write

$$\omega = e_0 \wedge \eta + \omega'$$

where $\omega', \eta$ do not contain the vector $e_0$. Thus

$$0 = \omega \wedge \omega = 2e_0 \wedge \eta \wedge \omega' + \omega' \wedge \omega'.$$

By examining tensors which do or do not contain $e_0$, this implies that

$$\omega' \wedge \omega' = 0, \quad \eta \wedge \omega' = 0.$$

Hence by induction

$$\omega' = v \wedge w,$$

with $v, w$ being in the subspace spanned by $e_1, \ldots, e_n$. Also, we know

$$\eta \wedge \omega' = \eta \wedge v \wedge w = 0.$$

This shows that $\eta$ cannot be independent of $v, w$ hence

$$\eta = av + bw.$$

Collecting terms we find

$$\omega = e_0 \wedge (av + bw) + v \wedge w = (v + be_0) \wedge (w + ae_0)$$

as claimed.

We show (i) $\iff$ (ii). In one direction, $\omega = v \wedge w$ implies $v \wedge \omega = v \wedge v \wedge w = 0$. Conversely, assume $v \wedge \omega = 0$ for $v \neq 0$. Without loss of generality, we write $v_1 = v$ and complete to a basis $v, v_2, \ldots, v_n$ of $V$. Write

$$\omega = \sum_{i<j} \omega_{ij} v_i \wedge v_j.$$

Then

$$v \wedge \omega = \sum_{i<j} \omega_{ij} v_1 \wedge v_i \wedge v_j$$
and in the sum we must have $i > 1$. Since $v \land \omega = 0$, it follows that $\omega_{ij} = 0$ for all $i, j > 1$. Thus

$$\omega = \sum_{k > 1} \omega_{1k}v_1 \land v_k = v \land w, \quad w = \sum_{k > 1} \omega_{1k}v_k.$$  

For the Grassmannian $G(1, 3)$, write

$$\omega = \sum_{1 \leq i < j \leq 4} \omega_{ij}e_i \land e_j.$$  

Then

$$\omega \land \omega = 2(\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23})e_1 \land e_2 \land e_3 \land e_4 = 0$$  

so $G(1, 3)$ is given by the following quadric in $\mathbb{P}^5$:

$$\omega_{12}\omega_{34} - \omega_{13}\omega_{24} + \omega_{14}\omega_{23} = 0.$$  


\[\text{Problem 2. Recall the Plücker embedding} \]

$$\Phi : G(k + 1, n + 1) \mapsto \mathbb{P}\left(\wedge^{k+1} \mathbb{C}^{n+1}\right), \quad W \mapsto \wedge^{k+1} W.$$  

Show that $\Phi$ is well-defined and injective. Show that the image of $\Phi$ is closed, hence $G(k, n) = G(k + 1, n + 1)$ is projective.

\begin{proof}
Let $v_1, \ldots, v_{k+1}$ be a basis for $W$, so that $\wedge^{k+1} W$ is spanned by $v_1 \land \ldots \land v_{k+1}$. This vector is not zero since $v_1, \ldots, v_{k+1}$ are independent. If another basis $v'_1, \ldots, v'_{k+1}$ is chosen, then

$$v_1 \land \ldots \land v_{k+1} = \det C \cdot v'_1 \land \ldots \land v'_{k+1}$$  

where $C$ is the change of basis. Thus the projective point $[v_1 \land \ldots \land v_{k+1}]$ is independent of choices, for each fixed $W$.

To show $\Phi$ is injective, assume $W = \langle v_1, \ldots, v_{k+1} \rangle$ and $W' = \langle v'_1, \ldots, v'_{k+1} \rangle$. If $\Phi(W) = \Phi(W')$ then

$$v_1 \land \ldots \land v_{k+1} = cv'_1 \land \ldots \land v'_{k+1}$$  

for some constant $c \neq 0$. Thus, wedge product with $v'_1$ shows that

$$v'_1 \land v_1 \land \ldots \land v_{k+1} = cv'_1 \land v'_1 \land \ldots \land v'_{k+1} = 0.$$  

This shows that $v'_1, v_1, \ldots, v_{k+1}$ are linearly dependent. Since $v_1, \ldots, v_{k+1}$ are independent, it follows that

$$v'_1 \in \langle v_1, \ldots, v_k \rangle \implies v'_1 \in W.$$  

Similarly, $v'_j \in W$ and hence $W' \subset W$. Similarly $W \subset W'$. This shows $W = W'$

Next we show the Plücker image is closed.

\begin{lemma}
Let $\omega \in \wedge^{k+1} \mathbb{C}^{n+1}$, $\omega \neq 0$, and consider the linear map

$$f_\omega : \mathbb{C}^{n+1} \mapsto \wedge^{k+2} \mathbb{C}^{n+1}, v \mapsto v \land \omega.$$  

There exist $v_1, \ldots, v_{k+1}$ such that $\omega = v_1 \land \ldots \land v_{k+1}$ if and only if the rank of $f_\omega$ is at most $n - k$.

\end{lemma}
Assuming this result, it is easy to see that \( \text{Im } \Phi \) can be described by requiring that the minors of the matrix of \( f_\omega \) of size \( n - k + 1 \) vanish. Write

\[
\omega = \sum \omega_I e_I
\]

where \( I = \{i_1 < \ldots < i_{k+1}\} \) is a multindex and

\[
e_I = e_{i_1} \wedge \ldots \wedge e_{i_{k+1}}.
\]

The matrix of \( f_\omega \) in the standard bases can easily be expressed in terms of \( \omega_I \)'s via the rules

\[
e_t \wedge \omega = \sum_{t \not\in I} \omega_I e_t \wedge e_I.
\]

The matrix has entries depending on \( \omega_I \)'s. The vanishing of the minors gives polynomial equations between the \( \omega_I \)'s which define the Plücker image.

**Proof of the lemma.** By the rank-nullity theorem, the statement is equivalent to the fact that the null space of \( f_\omega \) has dimension at least \( k + 1 \). If \( \omega = v_1 \wedge \ldots \wedge v_{k+1} \) it is clear that

\[
f_\omega(v_j) = v_j \wedge \omega = v_j \wedge v_1 \wedge \ldots \wedge v_{k+1} = 0
\]

so the kernel contains the \((k + 1)\) independent vectors \( v_1, \ldots, v_{k+1} \).

Conversely, assuming that the kernel of \( f_\omega \) contains \((k + 1)\) independent vectors \( v_1, \ldots, v_{k+1} \), we must have

\[
v_j \wedge \omega = 0.
\]

Complete \( v_1, \ldots, v_{k+1} \) to a basis \( v_1, \ldots, v_{n+1} \) of \( \mathbb{C}^{n+1} \). Using multindex notation, write

\[
\omega = \sum \omega_I v_I
\]

where \( |I| = k + 1 \). Note that

\[
v_j \wedge \omega = \sum_{j \not\in I} \omega_I v_j \wedge v_I.
\]

Since \( v_j \wedge \omega = 0 \) for all \( 1 \leq j \leq k + 1 \), it follows that

\[
\omega_I = 0 \text{ if } j \not\in I.
\]

Thus

\[
\omega_I \neq 0 \implies j \in I
\]

for all \( j = 1, \ldots, k + 1 \). This shows that \( \{1, \ldots, k + 1\} \subset I \) hence \( I = \{1, \ldots, k + 1\} \). Thus

\[
\omega = cv_1 \wedge \ldots \wedge v_{k+1}.
\]

The statement follows by absorbing the constant \( c \) into one of the \( v_j \)'s.
Problem 3.  (i) Let $\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre morphism. Show that if $t, s \in \mathbb{P}^1$, then

\[ L_t = \Phi(\{t\} \times \mathbb{P}^1), \quad M_s = \Phi(\mathbb{P}^1 \times \{s\}) \]

are lines in $\mathbb{P}^3$ intersecting in the point $\Phi(t,s)$.

Conclude that if $Q \subset \mathbb{P}^3$ is a nondegenerate quadric, there are two families of lines (called rulings) $\{L_t\}$ and $\{M_s\}$ in $\mathbb{P}^3$ lying on the quadric $Q$. Furthermore, any point of $Q$ is the intersection of two lines, one from each family.

(ii) Show that if $Q \subset \mathbb{P}^5$ is a nondegenerate quadric, there exist two families of planes in $\mathbb{P}^5$ lying on the quadric $Q$.

Answer:  (i) The second paragraph of part (i) follows from the first since all nondegenerate quadrics $Q \subset \mathbb{P}^3$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via a linear change of coordinates. It suffices to show $L_t$ are lines in $\mathbb{P}^3$. Write $t = [a : b]$, let $[z : w]$ be the coordinates of the second $\mathbb{P}^1$, and let $[x_0 : x_1 : x_2 : x_3]$ be the coordinates in $\mathbb{P}^3$. Then points in $L_t$ are of the form
\[ \Phi([a : b], [z : w]) = [az : aw : bz : bw]. \]

We can describe $L_t$ as the line with equation
\[ bx_0 - ax_2 = bx_1 - ax_3 = 0. \]

(ii) All nondegenerate quadrics in $\mathbb{P}^5$ are isomorphic, and we have shown in $G(1,3)$ is a non-degenerate quadric in $\mathbb{P}^5$ in Question 1. Thus, we may assume that $Q = G(1,3)$ under its Plücker embedding.

We show that for each $p \in \mathbb{P}^3$, the locus
\[ X_p = \{[\ell] \in G(1,3) : p \in \ell\} \]

is a plane in $\mathbb{P}^5$ lying in $Q$. To see that $X_p$ is a plane, we use Plücker coordinates. The argument is the same as in Problem 4(i).

Without loss of generality, we may assume $p = [1 : 0 : 0 : 0]$, otherwise we change coordinates appropriately. We will think of lines $L$ in terms of their Plucker coordinates
\[ z_{ij} = a_ib_j - a_jb_i \]
where $a, b$ are two points on $L$ with
\[ a = [a_0 : \ldots : a_3], b = [b_0 : \ldots : b_3]. \]

In Problem 4(i), we show that the condition $p \in \ell$ is given by the equations
\[ p_i z_{jk} - p_j z_{ik} + p_k z_{ij} = 0 \]
for all $i < j < k$. In our case, these equations are
\[ z_{12} = z_{13} = z_{23} = 0. \]
These equations describe a 2-plane in $\mathbb{P}^5$.

Similarly, if $H \subset \mathbb{P}^3$ is a hyperplane, then the locus

$$Y_H = \{[\ell] \in G(1, 3) : \ell \subset H\}$$

is a plane in $\mathbb{P}^5$ lying in $Q$. Indeed, without loss of generality, we may assume $H = \{x_0 = 0\}$. The line $\ell$ has to be spanned by vectors

$$a = \sum_{i=1}^{3} a_i e_i, \quad b = \sum_{i=1}^{3} b_i e_i$$

with $a_0 = b_0 = 0$, so that the Plücker coordinates are

$$z_{01} = z_{02} = z_{03} = 0.$$

These equations describe a plane in $\mathbb{P}^5$.

Conversely, any point in $\mathbb{P}^5$ with

$$z_{01} = z_{02} = z_{03} = 0$$

comes from a line in $H$. Indeed, for a point with coordinates $(z_{ij})$ as above, let

$$\omega = \sum_{1 \leq i,j \leq 3} z_{ij} e_i \wedge e_j.$$

Viewing $\omega$ as an element in $\wedge^2 \mathbb{C}^3$, by dimension reasons $\omega \wedge \omega = 0$ so $\omega = a \wedge b$ for $a, b \in \mathbb{C}^3$.

Now view $a, b$ as vectors in $\mathbb{C}^4$ by making the first coordinate 0, let $\ell \subset H$ be spanned by $a, b$. The Plücker point of $\ell$ is the point we started with.

\[\square\]

**Problem 4.** Let $G(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in the previous homework. Show that:

(i) The set $\{(L, P) : P \in L\} \subset G(1, n) \times \mathbb{P}^n$ is closed.

(ii) If $Z \subset G(1, n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.

(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting $X$ and $Y$ is a closed subset of $\mathbb{P}^n$. It is called the join $J(X, Y)$ of $X$ and $Y$.

**Answer:**

(i) We let

$$J = \{(P, L) : P \in L\} \subset \mathbb{P}^n \times G(1, n).$$

We will think of lines $L$ in terms of their Plücker coordinates

$$z_{ij} = a_i b_j - a_j b_i$$

where $a, b$ are two points on $L$ with

$$a = [a_0 : \ldots : a_n], \quad b = [b_0 : \ldots : b_n].$$
In fact, it will be useful to form the vectors 
\[ a = \sum a_i e_i, \quad b = \sum b_i e_i. \]

Similarly, a point \( P \in \mathbb{P}^n \) will have an associated vector 
\[ p = \sum p_i e_i. \]

Now, if \( P \in L \), then \( p = sa + tb \) hence
\[ p \wedge a \wedge b = 0. \]

Then
\[
\left( \sum p_i e_i \right) \wedge \left( \sum a_i e_i \right) \wedge \left( \sum b_i e_i \right) = \left( \sum p_i e_i \right) \wedge \left( \sum_{j<k} z_{jk} e_j \wedge e_k \right)
\]
\[ = \sum_{i<j<k} (p_i z_{jk} - p_j z_{ik} + p_k z_{ij}) e_i \wedge e_j \wedge e_k. \]

The conclusion is that \( J \) is defined by the equations
\[ p_i z_{jk} - p_j z_{ik} + p_k z_{ij} = 0 \]
which are bihomogeneous in the variables. Thus, \( J \) is projective.

(ii) Let
\[ p : J \to G(1,n), \quad q : J \to \mathbb{P}^n \]
be the natural projections. Then, for any \( Z \) closed in \( G(1,n) \), the preimage \( p^{-1}(Z) \) is also closed. Thus \( q(p^{-1}(Z)) \) is closed by the main theorem of projective varieties. This set consists in points \( P \) lying on lines \( L \) such that \( L \in Z \), hence it can be identified with the union of all lines in \( Z \).

(iii) We let \( A \) be the set of lines intersecting \( X \) and \( B \) be the set of lines intersecting \( Y \). We show \( A \) and \( B \) are closed in \( G(1,n) \), hence so is \( Z = A \cap B \). The join \( J(X,Y) \) is simply the union of lines contained in \( Z \) hence it must be closed in \( \mathbb{P}^n \) by item (ii).

It suffices to prove \( A \) is closed in \( G(1,n) \). Indeed, we can think of \( A \) as the projection under \( p \) of the set
\[
\{(P,L) : P \in L \} \cap X \times G(1,n) = J \cap q^{-1}(X).
\]

Hence
\[ A = p(J \cap q^{-1}(X)) \]
which is closed because \( p \) is closed and \( q \) is continuous.

Problem 5. Show that \( \mathbb{P}^1 \times \mathbb{A}^1 \) and \( \mathbb{P}^2 \setminus \{x\} \) are neither affine nor projective.
Answer: Write \( t \) for the coordinate on \( \mathbb{A}^1 \). We claim that the regular functions on \( \mathbb{P}^1 \times \mathbb{A}^1 \) are polynomials \( f(t) \). This will show that \( \mathbb{P}^1 \times \mathbb{A}^1 \) is not projective, because projective varieties only have constants as regular functions. It also shows \( X = \mathbb{P}^1 \times \mathbb{A}^1 \) is not affine since it were affine, its coordinate ring would be

\[
A(X) = k[t] = A(\mathbb{A}^1).
\]

Then \( X \cong \mathbb{A}^1 \), but this is clearly impossible for dimension reasons.

Indeed, let \( U, V \) be two affine opens covering \( \mathbb{P}^1 \). We have

\[
U \cong \mathbb{A}^1, \quad V \cong \mathbb{A}^1
\]

with coordinates \( z, w \) and \( w = \frac{1}{z} \) over overlaps. Let \( \phi \) be a regular function on \( \mathbb{P}^1 \times \mathbb{A}^1 \). Then \( \phi \) is regular on \( U \times \mathbb{A}^1 \cong \mathbb{A}^2 \) so

\[
\phi = p(z, t)
\]

for some polynomial \( p \). Similarly, \( \phi \) is regular on \( V \times \mathbb{A}^1 \cong \mathbb{A}^2 \) so

\[
\phi = q(w, t)
\]

for some polynomial \( q \). Over \( (U \cap V) \times \mathbb{A}^1 \) we must have

\[
p(z, t) = q \left( \frac{1}{z}, t \right).
\]

The powers of \( z \) on the left have nonnegative exponents, while the powers of \( z \) on the right have nonpositive exponents, so the exponents must be 0. Thus,

\[
p(z, t) = q \left( \frac{1}{z}, t \right) = f(t) \quad \implies \quad \phi = f(t)
\]

for some polynomial \( f \).

For \( Y = \mathbb{P}^2 \setminus \{ x \} \), we claim that all regular functions are constant. This will show that \( Y \) cannot be affine because

\[
A(Y) = k = A(\text{point}) \implies Y \cong \text{point}
\]

which is clearly impossible for dimension reasons. Indeed, if \( \phi \) is regular on \( \mathbb{P}^2 \setminus \{ x \} \), then consider the restriction of \( \phi \) to \( U = \mathbb{A}^2 \setminus \{ 0 \} \). This extends to a regular function on \( \mathbb{A}^2 \) by the removable singularity theorem. Thus \( \phi \) extends to a regular function on \( \mathbb{P}^2 \), showing then that \( \phi \) must be constant.

To see \( Y \) is not projective, assume otherwise. Let \( L = \{ \ell = 0 \} \) be a line in \( \mathbb{P}^2 \) through \( x \). Then, \( Z = L \setminus \{ x \} \) is closed in \( Y \) so it must be projective. This is not true since \( Z = L \setminus \{ x \} \cong \mathbb{A}^1 \). \( Z \) admits nonconstant regular functions, so it cannot be projective. This contradiction shows that \( Y \) is not projective.

\[\square\]

**Problem 6.** Let \( n \geq 2 \). The set \( X \) of degree \( d \) homogeneous polynomials in \( n + 1 \) variables can be identified with a projective space \( \mathbb{P}^N \), by recording the coefficients in some order. What is \( N \)?

Using the fundamental theorem of elimination theory, show that the set of irreducible polynomials form an open dense subset of \( X \).
The space $V_d$ of degree $d$ polynomials in $n + 1$ variables has dimension $\binom{n + d}{d}$. Consider the morphism

$$\phi_k : \mathbb{P}(V_k) \times \mathbb{P}(V_{d-k}) \rightarrow \mathbb{P}(V_d), \quad (f, g) \mapsto f \cdot g.$$ 

Clearly, $\phi_k$ is a morphism. This can be seen by writing

$$f = \sum a_I z^I, \quad g = \sum b_J z^J \implies f \cdot g = \sum_{I+J=K} \left( \sum_{I} a_I b_J \right) z^K$$

which shows

$$\phi_k(a_I, b_J) = (c_K), \quad c_K = \sum_{I+J=K} a_I b_J.$$ 

This is a morphism. By the main theorem of projective geometry,

$$Y_k = \text{Image } \phi_k$$

is closed. The reducible polynomials are given as

$$Y = \bigcup_{k=1}^{d-1} Y_k.$$ 

This set is therefore also closed in $X$.

The set of irreducible polynomials is therefore open being the complement. It is also nonempty, hence dense. To see nonemptiness, the polynomial $x_0^d + \ldots + x_n^d$ is irreducible for $n \geq 2$ (in characteristic zero). This can be proved in several ways. For instance, when $n = 2$, it suffices to view

$$x_0^d + x_1^d + x_2^d$$

as a polynomial in $k[x_1, x_2][x_0]$ and apply the Eisenstein criterion to the prime ideal

$$p = (x_1 + \zeta x_2),$$

where $\zeta$ is a root of $-1$. Clearly,

$$x_1^d + x_2^d \in p \setminus p^2,$$

so the criterion applies. For $n > 2$, use induction to conclude $p = (x_1^d + \ldots + x_n^d)$ is prime, and apply Eisenstein to the polynomial $x_0^d + (x_1^d + \ldots + x_n^d)$ in $k[x_1, \ldots, x_n][x_0]$ for the ideal $p$.

**Problem 7.** *(Rational varieties.)* The definition of birational isomorphisms given in class extends to the projective category. Two projective varieties $X$ and $Y$ are birational if there are rational maps

$$f : X \dashrightarrow Y, \quad g : Y \dashrightarrow X,$$

which are rational inverses to each other. Just as in the affine case, a birational isomorphism $f : X \dashrightarrow Y$ induces an isomorphism of the fields of rational functions $f^* : K(Y) \rightarrow K(X).$
(i) Explain that if $X$ is rational, then

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Show that $\mathbb{P}^n \times \mathbb{P}^m$ is rational, by constructing an explicit birational isomorphism with $\mathbb{P}^{n+m}$. Show that if $X$ and $Y$ are rational, then $X \times Y$ is rational.

(iii) Show that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(iv) The group of automorphisms of the field of fractions in two variables $k(x, y)$ is called the Cremona group. Explain that the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$. Explain that the Cremona involution

$$(x, y) \rightarrow (x^{-1}, y^{-1})$$

extends to an automorphism of $k(x, y)$. What is the corresponding birational involution of $\mathbb{P}^2$? Where is this birational automorphism regular?

(v) More generally, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group.

**Answer:**

(i) Clearly, $K(\mathbb{A}^n) \cong k(t_1, \ldots, t_n)$. Thus $X$ is rational iff

$$K(X) \cong k(t_1, \ldots, t_n).$$

(ii) Let $U \subset \mathbb{P}^n$ be the open set where the coordinate $x_0 \neq 0$. Then $U \cong \mathbb{A}^n$, showing that $\mathbb{A}^n$ and $\mathbb{P}^n$ are birational since they have an isomorphic open subset.

Similarly, let $V \subset \mathbb{P}^m$ be the open set where the coordinate $y_0 \neq 0$. We have $U \cong \mathbb{A}^n$, $V \cong \mathbb{A}^m$. Thus $U \times V \cong \mathbb{A}^{n+m}$ is an open subset of $\mathbb{P}^n \times \mathbb{P}^m$, hence $\mathbb{P}^n \times \mathbb{P}^m$ and $\mathbb{A}^{n+m}$ are birational.

(iii) Two closed subsets $\{a\} \times \mathbb{P}^1$ and $\{b\} \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ have nonempty intersection. This is false in $\mathbb{P}^2$: any two curves intersect by the weak Bezout theorem. Hence $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ cannot be isomorphic.

(iv) We explained in (ii) that

$$K(\mathbb{P}^2) = k(x, y)$$

hence any automorphism of $k(x, y)$ corresponds to an automorphism of $K(\mathbb{P}^2)$ which in turn gives a birational isomorphism of $\mathbb{P}^2$. The involution

$$(x, y) \rightarrow (x^{-1}, y^{-1})$$

corresponds to the birational map

$$f[x : y : z] = [x^{-1} : y^{-1} : z^{-1}] = [yz : xz : xy].$$

This map is regular on $\mathbb{P}^2 \setminus \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Indeed, to show that the map is regular at the points where $(x, y) \neq (0, 0)$, we rewrite it in the form

$$f[x : y : z] = \left[ \begin{array}{c} z \\ x \\ y \end{array} : \begin{array}{c} z \\ x \\ y \end{array} : 1 \right].$$
(v) The automorphism

$$ (x, y) \rightarrow (x^a y^b, x^c y^d) $$

where

$$ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) $$

belongs to the Cremona group. Its inverse is

$$ (x, y) \rightarrow (x^{a'} y^{b'}, x^{c'} y^{d'}) $$

where

$$ A^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} $$

is the inverse of the matrix above.