Math 203, Problem Set 9. Due Friday, December 2.

1. (Cremona transformations.) Consider the Cremona birational automorphism of $\mathbb{P}^2$ given by

$$C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1].$$

Let $\mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$ where $C$ is undefined. Show that

(i) Show that $C$ extends to an isomorphism $\widetilde{C} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$.

(ii) Let $E_1, E_2, E_3$ be the exceptional lines for the blowup, and let $L_{ij}$ be the strict transform of the line through $P_i$ and $P_j$. Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under $\widetilde{C}$?

*Hint:* Show that the equations of the blowup $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are given by

$$x_0y_0 = x_1y_1 = x_2y_2.$$ 

It may help to find the equations of the exceptional lines $E_1, E_2, E_3$ and those of the strict transforms $L_{23}, L_{12}, L_{13}$.

2. (Degree of the Segre embedding.) Show that the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ has degree $\binom{n+m}{n}$.

3. (Arithmetic genus.) Let $X \subset \mathbb{P}^n$ be a projective variety with Hilbert polynomial $\chi_X$. Define the arithmetic genus of $X$ to be

$$p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1).$$

(i) Show that the genus of $\mathbb{P}^n$ is zero.

(ii) If $X$ is a hypersurface of degree $d$ in $\mathbb{P}^n$, show that $p_a(X) = \binom{d-1}{n}$. In particular, a cubic in $\mathbb{P}^2$ has genus 1.

(iii) If $X$ is a complete intersection of two surfaces of degree $a$ and $b$ in $\mathbb{P}^3$ then

$$p_a(X) = \frac{1}{2}ab(a + b - 4) + 1.$$ 

In particular, intersection of two quadrics in $\mathbb{P}^3$ has genus 1.

*Hint:* For (iii), you will need to write down a suitable exact sequence.

4. (Enumerative geometry of lines.) Given four general lines in $\mathbb{P}^3$, show that there are exactly 2 lines which intersect all four of them.

*Remark:* This means you need to show that there exists a nonempty open (hence dense) subset $U \subset G(1,3)^4$ such that for all $(L_1, L_2, L_3, L_4) \in U$, the number of lines intersecting $L_1, L_2, L_3, L_4$ is exactly 2.
Hint: Recall that the Grassmannian $G(1, 3)$ is a quadric in $\mathbb{P}^5$ via the Plücker embedding. You may wish to show that a quadric intersects 4 general hyperplane in 2 points.

Remark: The number of lines in $\mathbb{P}^n$ which intersect $2(n - 1)$ fixed general codimension 2 linear hyperplanes equals the Catalan number

$$C_n = \frac{1}{n} \binom{2n - 2}{n - 1}.$$

5. (Varieties of minimal degree.) Let $X$ be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree $d$ and codimension $c$ in $\mathbb{P}^n$.

(i) (Intersecting $X$ with hyperplanes to cut down the dimension), show inductively that $d \geq c + 1$.

(ii) The del Pezzo-Bertini theorem classifies all varieties for which equality holds in (i). Here, verify that equality holds for rational normal curves in $\mathbb{P}^n$, and for the image $v(\mathbb{P}^2)$ of the Veronese embedding $v: \mathbb{P}^2 \to \mathbb{P}^5$.

(iii) Can you classify the varieties of degree 2?