
For this problem set, you may assume that the ground field is $k = \mathbb{C}$.

1. (Singularities of cubics.)
   (i) Show that any singular irreducible cubic in $\mathbb{P}^2$ is isomorphic to either the nodal or the cuspidal cubics:
   \[ y^2z = x^2(x+z) \text{ or } y^2z = x^3. \]
   
   **Hint:** Assume the singularity is at $[0 : 0 : 1]$. Show that the cubic can be written as
   \[(\text{quadratic polynomial in } x, y) \cdot z = Q(x, y),\]
   where $Q$ is a cubic polynomial in $x, y$. Change coordinates suitably and write the cubic as
   \[ y^2z = \tilde{Q}(x, y) \text{ or } xyz = \tilde{Q}(x, y). \]
   Use the coordinate change $z \mapsto \lambda x + \mu y + \nu z$ to put the cubic into one of the forms
   \[ y^2z = (x + by)^3 \text{ or } xyz = (x + y)^3. \]
   Conclude by performing one more change of coordinates.

   (ii) Using (i), show that irreducible cubics in $\mathbb{P}^2$ can have at most 1 singular point.
   Exhibit a cubic in $\mathbb{P}^2$ with 3 singular points.
   
   **Remark:** It can be shown that an irreducible degree $d$ curve in $\mathbb{P}^2$ has at most $\binom{d}{2}$ singular points.

2. (Dual conics.) Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x, y, z]$. Consider the morphism
   \[ \Phi : C \to \mathbb{P}^2, p \mapsto \left[ \frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p) \right]. \]
   The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to $C$.

   (i) Why is $\Phi$ a well-defined morphism?
   (ii) If $C$ is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic.
   One way to prove this is to linearly change coordinates and assume the conic $C$ is $ax^2 + by^2 + cz^2 = 0$. What is $\Phi(C)$?
   (iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

3. (Resolving curve singularities.) Resolve the following $A_k$ plane curve singularity by subsequent blow-ups
   \[ y^2 - x^{k+1} = 0. \]
Remark: We have the following terminology on isolated “simple” singularities of hypersurfaces in \( \mathbb{A}^{n+2} \):

- type \( A_k \): \( x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0 \);
- type \( D_k \): \( x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0 \);
- type \( E_6 \): \( x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0 \);
- type \( E_7 \): \( x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0 \);
- type \( E_8 \): \( x^5 + y^3 + z_1^2 + \ldots + z_n^2 = 0 \).

(The names suggest a connection with the Weyl groups of type \( A, D, E \).)

4. (Tangent cones.) Let \( X \subset \mathbb{A}^n \) be an affine variety and let \( p \in X \). Let \( \mathfrak{m} \) be the maximal ideal of \( \mathcal{O}_{X,p} \). Show that there exists an isomorphism

\[
\frac{k[x_1, \ldots, x_n]}{i^{\infty}} \rightarrow \bigoplus_{k \geq 0} \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}}.
\]

Remark: If \( i^{\infty} \) is radical, then the coordinate ring \( A(C_{X,p}) \) of the tangent cone of \( X \) at \( p \) is isomorphic to the graded algebra \( \bigoplus_{k \geq 0} \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}} \), so the tangent cone is intrinsic.

However, the tangent cone is better defined as a scheme

\[
\text{Spec} \left( \frac{k[x_1, \ldots, x_n]}{i^{\infty}} \right) = \text{Spec} \left( \bigoplus_{k \geq 0} \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}} \right)
\]

whose underlying variety is the tangent cone defined in class. The definition of schemes will be pursued in Math 203B, but the idea is that we do not need to restrict only to radical ideals.

5. (Exceptional hypersurface.) Consider the blowup of the affine variety \( X \subset \mathbb{A}^n \) at \( p \in X \). Show that the exceptional hypersurface is the projectivization of the tangent cone

\[ E \cong \mathbb{P}(C_{X,p}). \]

You may want to generalize the argument we had in class for plane curves.