Math 203, Problem Set 5. Due Monday, November 7.

1. (*Quadrics are rational.*) Show that any non-degenerate quadric \(Q \subset \mathbb{P}^n\) is birational to \(\mathbb{P}^{n-1}\).

*Hint:* Use the projection from a point \(p \in Q\) to a hyperplane.

2. (*Dimension of polynomial rings.*)

(i) Let \(A\) be a commutative ring and let \(p_1 \subset p_2 \subset p_3\) be three prime ideals in \(A[x]\). Show that the contractions \(p_1 \cap A, p_2 \cap A, p_3 \cap A\) cannot be equal.

*Hint:* Assume \(p = p_i \cap A, 1 \leq i \leq 3\). Consider \(S = A - p\), and let \(S^{-1}A[x] = A_p[x]\). The ideals \(S^{-1}p_1, S^{-1}p_2, S^{-1}p_3\) have the same contraction \(pA_p\) in \(A_p\). Consider the quotient \(A_p[x]/pA_p[x] = (A_p/pA_p)[x]\) and use that \(k[x]\) has dimension 1 for any field such as \(k = A_p/pA_p\).

(ii) Deduce that \(\dim A[x] \leq 2 \dim A + 1\).

3. (*Finite morphisms.*) Recall that in class we gave two definitions of finite morphisms: one for affine varieties and one for arbitrary varieties. This exercise shows that the two definitions agree for affine varieties.

To avoid confusion, in this exercise we use the first definition. Thus, we say a morphism of affine algebraic sets is finite if the induced morphism on coordinate rings is finite. Of course, once this exercise is solved, it won’t be necessary to make the distinction between the two definitions.

(i) Show that if

\[ f : Z \rightarrow W \]

is a finite morphism of affine algebraic sets, and \(W_g\) is a basic open set in \(W\), then

\[ f : f^{-1}(W_g) \rightarrow W_g \]

is finite as well.

(ii) Let \(f : X \rightarrow Y\) be a morphism of affine algebraic sets. Assume that there exists an affine open cover \(Y = \bigcup Y_i\) such that

\[ f : f^{-1}(Y_i) \rightarrow Y_i \]

is finite morphism of affine algebraic sets. Show that \(f\) is a finite morphism of affine algebraic sets.

In particular, the two definitions of finite morphisms given in class agree.

*Hint 1:* Using (i), show that you may assume \(Y_i\) to be principal open set in \(Y\).
Hint 2: We may assume $Y_i = Y_{g_i}$ are basic open sets, and that $f : f^{-1}(Y_{g_i}) \to Y_{g_i}$ is finite. Note $f^{-1}(Y_{g_i}) = X_{f^*g_i}$. Pick $\alpha_{i,k}$ a basis of the module $A(X)_{f^*g_i}$ over $A(Y)_{g_i}$, and clear denominators to assume $\alpha_{i,k} \in A(X)$. Show that $\alpha_{i,k}$ generate the module $A(X)$ over $A(Y)$.

(iii) Show that if $f : X \to Y$ is finite morphism of affine algebraic sets and $V \subset Y$ is an affine open then

$$f : f^{-1}(V) \to V$$

is a finite morphism of affine algebraic sets.

**Hint:** To prove $f^{-1}(V)$ is affine, you may wish to consider the intersection $(X \times V) \cap \Gamma_f$ in $X \times Y$.

(iv) In particular, let $X, Y$ be affine algebraic sets. Then, the following three statements are equivalent

(a) $f : X \to Y$ is a finite morphism of affine algebraic sets

(b) there exists an affine open cover $Y = \bigcup Y_i$ such that

$$f : f^{-1}(Y_i) \to Y_i$$

is finite morphism of affine algebraic sets

(c) for all affine covers $Y = \bigcup Y_i$ we have that

$$f : f^{-1}(Y_i) \to Y_i$$

is finite morphism of affine algebraic sets.

**Remark:** The same equivalence holds true without the assumption $X, Y$ are affine but arbitrary varieties, via an iteration of this argument.

4. **(Finite morphisms.)** Let $f_0, \ldots, f_m$ be homogeneous polynomials of degree $d > 0$ without common zeros on $X \subset \mathbb{P}^n$. Show that

$$f : X \to \mathbb{P}^m, \quad f(x) = [f_0(x) : \ldots : f_m(x)]$$

gives a finite morphism onto its image.

**Hint:** Use Veronese to reduce to the case of linear polynomials $d = 1$. 