Math 203, Problem Set 3. Due Monday, October 17.

For this problem set, you may assume that the ground field is \( k = \mathbb{C} \).

1. An algebraic set \( Z \subset \mathbb{A}^2 \) defined by an irreducible polynomial \( f \) of degree 2 is called an irreducible conic.

   (i) Show that any irreducible conic is isomorphic to
   \[
   Y - X^2 = 0 \quad \text{or} \quad XY - 1 = 0
   \]
   after an affine change of coordinates in \( \mathbb{A}^2 \). In particular, any conic is rational.
   
   \text{Hint: Write } f(x, y) = a_1 x^2 + a_2 y^2 + a_3 xy + a_4 x + a_5 y + a_6, \text{ and complete the square.}

   (ii) Let \( Z_1 \) and \( Z_2 \) be two distinct irreducible conics in \( \mathbb{A}^2 \). Using (i), show that \( Z_1 \) and \( Z_2 \) intersect in at most 4 points. (This is a particular case of Bézout’s theorem.) Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

2. (Cubic curves are not rational.) We claimed in a previous lecture, but did not prove, that (most) cubic plane curves are not rational.

Let \( \lambda \in k \setminus \{0, 1\} \). Consider the cubic curve \( E_\lambda \subset \mathbb{A}^2 \) given by the equation
\[
y^2 - x(x - 1)(x - \lambda) = 0.
\]

Remark: All cubics can be written as
\[
y^2 = x^3 \quad \text{or} \quad y^2 = x(x - 1)^2 \quad \text{or} \quad y^2 = x(x - 1)(x - \lambda), \quad \lambda \neq 0, 1
\]
after a linear change of coordinates. You can check this for yourself by completing the cube in the cubic equation; this is similar to what we did for conics. The first two curves are called the cuspidal and nodal cubics.

Show that \( E_\lambda \) is not birational to \( \mathbb{A}^1 \). In fact, show that there are no non-constant rational maps
\[
F : \mathbb{A}^1 \rightarrow E_\lambda.
\]

(i) Write
\[
F(t) = \left( \frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right)
\]

where the pairs of polynomials \( (f, g) \) and \( (p, q) \) have no common factors. Conclude that
\[
\frac{p^2}{q^2} = \frac{f(f - g)(f - \lambda g)}{g^3}
\]
is an equality of fractions that cannot be further simplified. Conclude that
\( f, g, f - g, g - \lambda g \) must be perfect squares.

(ii) Conclude by proving the following:

**Lemma:** If \( f, g \) are polynomials in \( k[t] \) without common factors and such that
there is a constant \( \lambda \neq 0,1 \) for which \( f, g, f - g, f - \lambda g \) are perfect squares, then
\( f \) and \( g \) must be constant.

**Hint:** Descent. Write \( f = u^2, g = v^2 \). Considering \( f - g \) and \( f - \lambda g \), prove
that \( u - v, u + v, u - \mu v, u + \mu v \) are also squares for some constant \( \mu \neq 0,1 \). Show
that suitable \( \tilde{u}, \tilde{v} \) obtained as a linear combination of \( u \) and \( v \) verify the lemma,
yet they have smaller degree than \( \max(\deg f, \deg g) \).

**Remark:** You may know from complex analysis or number theory that after a change
of coordinates most cubics can be written in Weierstraß normal form
\[
y^2 = 4x^3 - g_2x - g_3, \quad g_2^3 - 27g_3^2 \neq 0.
\]
Such cubics can be parametrized \( (x, y) = (\wp(z), \wp'(z)) \) via the Weierstraß \( \wp \)-function
\[
\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right),
\]
for some lattice \( \Lambda \). Of course, this does not contradict what we have shown above,
namely that there is no parametrization of the cubic by rational fractions.

3. Let \( n \geq 2 \), and let \( S = \{a_1, \ldots, a_n\} \) be a finite set with \( n \) elements in \( \mathbb{A}^1 \).
   (i) Show that the quasi-affine set \( \mathbb{A}^1 \setminus S \) is isomorphic to an affine set. For instance,
you may take \( X \) to be the affine algebraic set given by the equations
   \[
   X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.
   \]
   (ii) Show that \( \mathbb{A}^1 \setminus S \) is not isomorphic to \( \mathbb{A}^1 \setminus \{0\} \) by proving that their rings of
   regular functions are not isomorphic.

   **Hint:** Assume that
   \[
   \Phi : A(X) \to k[t, t^{-1}]
   \]
is an isomorphism. Observe that \( X_i \) are invertible elements in \( A(X) \) for all
\( 1 \leq i \leq n \). Show that their images must be invertible in \( k[t, t^{-1}] \). Prove that
this implies that \( \Phi(X_i) = t^{m_i} \) for some integers \( m_i \). Derive a contradiction by
comparing \( \Phi(X_0 - a_i) \) for different values of \( i \).

4. Let \( n \geq 2 \). Consider the affine algebraic sets in \( \mathbb{A}^2 \):
   \[
   Z_n = \mathcal{Z}(y^n - x^{n+1})
   \]
and
\[ W_n = Z(y^n - x^n(x + 1)). \]
Show that \( Z_n \) and \( W_n \) are birational but not isomorphic.

(i) Show that
\[
 f : \mathbb{A}^1 \to Z_n, \quad f(t) = (t^n, t^{n+1})
\]
is a morphism of affine algebraic sets which establishes an isomorphism between
the open subsets
\[
 \mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.
\]
Similarly, show that
\[
 g : \mathbb{A}^1 \to W_n, \quad g(t) = (t^n - 1, t^{n+1} - t).
\]
is a morphism of affine algebraic sets. Find open subsets of \( \mathbb{A}^1 \) and \( W_n \) where \( g \)
becomes an isomorphism.

(ii) Using (i), explain why \( Z_n \) and \( W_n \) are birational.

(iii) Assume that there exists an isomorphism
\[
 h : Z_n \to W_n
\]
such that \( h((0,0)) = (0,0) \). Observe that this induces an isomorphism between
the open sets
\[
 Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.
\]
Use part (i) and the previous problem to conclude this cannot be true if \( n \geq 2 \).

(iv) (Optional - do not hand in.) Repeat the argument above without the assumption
that \( h \) sends the origin to itself. You may need to prove a stronger version of
Problem 3.

5. (Quotients.) Taking quotients in algebraic geometry is subtle. We will explain how
to take quotients by finite groups.

Let \( X \) be an affine variety, and let \( G \) be a finite group. Assume that \( G \) acts on \( X \)
algebraically, i.e. that for every \( g \in G \), we are given a morphism \( g : X \to X \) (denoted
by the same letter for simplicity of notation), such that
\[
 (gh)(p) = g(h(p))
\]
for all \( g, h \in G \) and \( p \in X \).

(i) Let \( g \in G \) act on the coordinate rings \( A(X) \) via
\[
 f \mapsto f^g \text{ with } f^g(p) = f(g(p)).
\]
Let \( A(X)^G \) be the subalgebra of \( A(X) \) consisting of all \( G \)-invariant functions on
\( X \). Show that \( A(X)^G \) is a finitely generated \( k \)-algebra.
Hint 1: Let $x_1, \ldots, x_n$ be the coordinate functions on $X$, and let $|G| = m$. Using the polynomial

$$P(t) = \prod_{g \in G} (t - x_i^g)$$

show that there exists an identity

$$x_i^m + a_{i1}x_i^{m-1} + \ldots + a_{im} = 0$$

where $a_{ij} \in A(X)^G$. Use this to conclude that each $f \in A(X)$ can be written as

$$f = \sum f_I x^I$$

where $f_I$ are polynomials in the $a_{ij}$ and the indices $I = (i_1, \ldots, i_n)$ have $0 \leq i_j \leq m - 1$.

Hint 2: Use the averaging operator

$$\pi(f) = \frac{1}{|G|} \sum_{g \in G} f^g.$$

If $f \in A(X)^G$ is written in the form in Hint 1, observe

$$f = \pi(f) = \sum_I f_I \pi(x^I),$$

and conclude.

(ii) By (i), there is an affine variety $Y$ with coordinate ring $A(X)^G$, together with a morphism

$$\pi : X \to Y$$

determined by the inclusion

$$A(X)^G \hookrightarrow A(X).$$

Show that $Y$ can be considered as the quotient of $X$ by $G$, denoted $X/G$, in the following sense: if $p, q \in X$ then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(iii) Let

$$\mu_n = \left\{ \exp \left( \frac{2\pi ik}{n} \right), k \in \mathbb{Z} \right\}$$

be the group of $n$-th roots of unity. Let $\mu_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Describe $\mathbb{C}/\mu_n$ and $\mathbb{C}^2/\mu_n$ as affine algebraic sets.

6. (Intersections of affine opens.) If $X$ is a variety, and $U, V$ are two affine open sets, then $U \cap V$ is an affine open set.

Hint: Construct $U \cap V$ as $(U \times V) \cap \Delta$ where $\Delta \subset X \times X$ is the diagonal.