Math 203, Problem Set 2. Due Friday, October 7.

For this problem set, you may assume that the ground field is \( k \) is algebraically closed.

1. (Identity principle.) The following question can be viewed as an analogue of the identity theorem in complex analysis.

   (i) Let \( F \to X \) be a sheaf on a topological space, and let \( s, t \in F(U) \) be two sections over an open set \( U \subset X \) whose germs \( s_x = t_x \in F_x \) for all \( x \in U \). Show that \( s = t \).

   (ii) Let \( X \) be an affine variety, \( U \subset X \) open and nonempty, and let \( s, t \in O_X(U) \) such that \( s_x = t_x \) for some point \( x \in U \). Show that \( s = t \).

2. (Isomorphisms of sheaves.) Let \( \alpha : F \to G \) be a morphism of sheaves on a topological space \( X \). Show that the following are equivalent:

   (i) \( \alpha \) is an isomorphism. That is, for all \( U \subset X \) open, \( \alpha_U : F(U) \to G(U) \) is an isomorphism.

   (ii) there exists an open cover \( X = \bigcup U_i \) such that for all \( i \), the restrictions \( \alpha|_{U_i} : F|_{U_i} \to G|_{U_i} \) are isomorphisms of sheaves.

   (iii) for all \( x \in X \) the maps on stalks \( \alpha_x : F_x \to G_x \) are isomorphisms.

3. (Stalks at subvarieties.) Let \( F \to X \) be a sheaf on a topological space, and let \( Y \subset X \) be an irreducible nonempty closed subset.

   Consider pairs \((s, U)\) with \( s \in F(U) \) and \( U \subset X \) open such that \( U \cap Y \neq \emptyset \). Define the equivalence relation \((s, U) \sim (t, V)\) iff there exists \( W \subset U \cap V \) such that

   \[ W \cap Y \neq \emptyset \text{ and } s|_W = t|_W. \]

   Define the stalk of \( F \) at \( Y \) to be the set of equivalence classes of such pairs.

   Show that if \( X \) is an affine variety, and \( Y \) is a nonempty irreducible subset, then the stalk \( O_{X,Y} \) is the localization of \( A(X) \) at the prime ideal \( I(Y) \).

4. (Vakil, Exercise 2.5B - Sheaves over the base of a topology.) Let \( \mathcal{B} \) be a base of a topology on \( X \), and let \( F \) denote a sheaf (of abelian groups) on the base \( \mathcal{B} \). That is, for each open set \( B \in \mathcal{B} \), we are given an abelian group \( F(B) \), and furthermore, for all \( B, B' \in \mathcal{B} \) with \( B \subset B' \) we are given morphisms

   \[ \rho_{B,B'} : F(B') \to F(B) \]

   satisfying the usual sheaf axioms (for the members of \( \mathcal{B} \)). Show that each sheaf \( F \) on the base \( \mathcal{B} \) extends to a sheaf \( F \) on \( X \) such that \( F(B) \simeq F(B) \) for \( B \in \mathcal{B} \).
Hint: First, define the stalks $F_p$ for the sheaf $F$ over the base. For any $U \subset X$ open, define $\mathcal{F}(U)$ to be the set of compatible germs. That is, set

$$\mathcal{F}(U) = \{(f_p) \in F_p : \text{ for all } p \in U, \text{ there exists } p \in B \subset U, \text{ with } B \in \mathfrak{B},$$

and a section $s \in F(B)$ such that $f_q = s_q$ for all $q \in B\}.$

Verify to your satisfaction that $\mathcal{F}$ is a sheaf and that $\mathcal{F}(B) \simeq F(B)$.

Remark: (Vakil, Exercise 2.5C.) Morphisms of sheaves are also determined by morphism of sheaves over the base. That is, given sheaves $\mathcal{F}, \mathcal{G}$ and homomorphisms

$$\alpha_B : \mathcal{F}(B) \to \mathcal{G}(B)$$

for all $B \in \mathfrak{B}$ compatible with restrictions of members of $\mathfrak{B}$, then there exists a morphism of sheaves

$$\alpha : \mathcal{F} \to \mathcal{G}$$

extending the $\alpha_B$’s. You do not need to write down a proof of this fact, but it is a good exercise to solve.

5. (Hartogs theorem and quasi-affine algebraic sets.)

(i) Show that all regular functions on $\mathbb{A}^2 \setminus \{(0,0)\}$ are given by polynomials.

(ii) Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not isomorphic to an affine algebraic set.

Hint: Argue by contradiction. Using your knowledge about the regular functions on $X$, what can you say about the inclusion $\iota : X \to \mathbb{A}^2$?

6. (Coordinate rings. Frobenius.)

(i) Show that the curve $X = \{x^2 - y^5 = 0\} \subset \mathbb{A}^2_C$ is not isomorphic to $\mathbb{A}^1_C$.

(ii) Show that if $k$ is an algebraically closed field of characteristic $p$, the Frobenius morphism

$$F : \mathbb{A}^1 \to \mathbb{A}^1, F(x) = x^p$$

is a homeomorphism but not an isomorphism.