

Math 203, Problem Set 8. Due Friday, December 7.

Solve the following problems, and hand in solutions to three of them.

1. (*Degree of the Segre embedding.*) Show that the Segre embedding

$$\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)-1}$$

has degree $\binom{n+m}{n}$.

2. (*Arithmetic genus.*) Let $X \subset \mathbb{P}^n$ be a projective scheme with Hilbert polynomial χ_X . Define the arithmetic genus of X to be

$$p_a(X) = (-1)^{\dim X} (\chi_X(0) - 1).$$

- (i) Show that the genus of \mathbb{P}^n is zero.
(ii) If X is a hypersurface of degree d in \mathbb{P}^n , show that $p_a(X) = \binom{d-1}{n}$. In particular, a cubic in \mathbb{P}^2 has genus 1.
(iii) If X is a complete intersection of two surfaces of degree a and b in \mathbb{P}^3 then

$$p_a(X) = \frac{1}{2}ab(a+b-4) + 1.$$

In particular, intersection of two quadrics in \mathbb{P}^3 has genus 1.

Remark: To compare (ii) and (iii), recall that a cubic in \mathbb{P}^2 is isomorphic to an intersection of two quadrics in \mathbb{P}^3 as shown in a previous homework.

3. (*Enumerative geometry of lines.*) Given four general lines in \mathbb{P}^3 , show that there are exactly 2 lines which intersect all four of them.

Hint: Recall that the Grassmannian $G(1, 3)$ is a quadric in \mathbb{P}^5 via the Plücker embedding.

Challenge: You may attempt to prove the following generalization:

Show that the number of lines in \mathbb{P}^n which intersect $2(n-1)$ fixed general codimension 2 linear hyperplanes equals the Catalan number

$$C_n = \frac{1}{n} \binom{2n-2}{n-1}.$$

4. (*Varieties of minimal degree.*) Let X be a non-degenerate (i.e., not contained in any hyperplanes) projective variety of degree d and codimension c in \mathbb{P}^n .

- (i) (Intersecting X with hyperplanes to cut down the dimension), show inductively that

$$d \geq c + 1.$$

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(iii) Show that equality holds for rational normal curves in \mathbb{P}^n , and for the image $v(\mathbb{P}^2)$ of the Veronese embedding

$$v : \mathbb{P}^2 \rightarrow \mathbb{P}^5.$$

(iii) Can you classify the varieties of degree 2?

Remark: The del Pezzo-Bertini theorem classifies all varieties for which equality holds in (i).