Math 203, Problem Set 7. Due Wednesday, November 28.

Solve the following problems, and hand in solutions to three of them.

1. (*Distinguished open sets.*) Let $X = \mathrm{Spec}(A)$ be an affine scheme. For each $f \in A$, recall the basic open sets

$$X_f = \{\mathfrak{p} \in X : f \notin \mathfrak{p}\}.$$

Show that

(i) $X_f \subset X_g$ iff $\sqrt{(f)} \subset \sqrt{(g)}$. In particular,

$$X_f = \emptyset \iff f \text{ is nilpotent},$$

and

$$X_f = X \iff f \text{ is a unit}.$$

(ii) Show that X is quasi-compact i.e. any cover by open subsets admits a finite subcover. More generally, each X_f is quasi-compact.

Hint: You may want to recall that the radical of an ideal $\mathfrak{a} \subset A$ is the intersection of prime ideals containing $\mathfrak{a}$.

2. (*Reduced schemes.*) Show that for an affine scheme X the following are equivalent:

(i) X is reduced i.e. for all $U \subset X$ open, $\mathcal{O}_X(U)$ has no nilpotents;

(ii) for all $p \in X$, the local rings $\mathcal{O}_{X,p}$ have no nilpotents.

3. (*Specialization and generization. Generic points.*) Given two points x, y of a topological space X, we say that x is a specialization of y, and y is a generization of x, if $x \in \overline{\{y\}}$. A relation notion is that of generic points: if Y is a closed subset of X, we say η_Y is a generic point of Y if $\overline{\{\eta_Y\}} = Y$.

(i) If $X = \mathrm{Spec}\ (A)$, show that $\mathfrak{q}$ is a specialization of $\mathfrak{p}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Hence show that

$$\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p}) = \{\mathfrak{q} : \mathfrak{p} \subset \mathfrak{q}\}.$$

(ii) Assume that $X = \mathrm{Spec}(A)$ is an affine scheme. Show that any nonempty irreducible closed subset $Y \subset X$ admits a unique generic point η_Y.

4. (*Stalks over generic points.*) Let X be an affine scheme, and let Y be an irreducible closed subset of X. If η_Y is the generic point of Y, we write $\mathcal{O}_{X,Y}$ for the stalk $\mathcal{O}_{X,\eta_Y}$.

Show that $\mathcal{O}_{X,Y}$ is "the ring of rational functions on X that are regular at a general point of Y", i.e. it is isomorphic to the ring of equivalence classes of pairs (U, ϕ), where $U \subset X$ is open with $U \cap Y \neq \emptyset$ and $\phi \in \mathcal{O}_X(U)$. Two such pairs (U, ϕ) and (U', ϕ') are called equivalent if there is an open subset $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ such that $\phi|_V = \phi'_V$.

In particular, if X is a scheme that is a variety, then $\mathcal{O}_{X,\eta_X}$ is the function field of X.

5. (*Morphisms of affine schemes.*) Let $f : A \to B$ be a morphism of rings, and let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $\mathfrak{q}$ is a point of Y, then $f^{-1}(\mathfrak{q})$ is a prime ideal in A hence a point of X. Therefore, we have a well-defined morphism

$$f^{\star} : Y \to X.$$

(i) Show that $f^{\star}$ is continuous.

(ii) If f is surjective with kernel $\mathrm{Ker}(f) \subset A$, show that $f^{\star}$ is homeomorphism of Y onto the closed subset $Z(\mathrm{Ker}(f))$ of X. In particular, show that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{n})$ are homeomorphic, where $\mathfrak{n}$ is the nilradical of A.

(iii) If f is injective, show that $f^{\star}$ is dominant. More precisely, the image $f^{\star}(Y)$ is dense in X iff $\mathrm{Ker}(f) \subset \mathfrak{n}$.