Math 203, Problem Set 6. Due Monday, November 19.

For this problem set, you may assume that the ground field is $k = \mathbb{C}$. Solve the following problems, and hand in solutions to three of them.

1. (*Resolving curve singularities.*) Resolve the following A_k plane curve singularity by subsequent blow-ups

$$y^2 - x^{k+1} = 0.$$

Remark: We have the following terminology on isolated "simple" singularities of hypersurfaces in $\mathbb{A}^{n+2}$:

- type A_k: $x^{k+1} + y^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type D_k: $x^{k-1} + xy^2 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_6: $x^4 + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_7: $x^3y + y^3 + z_1^2 + \ldots + z_n^2 = 0$;
- type E_8: $x^5 + y^3 + z_1^2 + \ldots + z_n^2 = 0$.

(The names suggest a connection with the Weyl groups of type A, D, E.)

2. (*On del Pezzo surfaces.*) Show that the blowup of $\mathbb{P}^2$ at two points is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. .

Hint: Think of $\mathbb{P}^1 \times \mathbb{P}^1$ as the quadric $Q = \{x_0x_3 = x_1x_2\}$ in $\mathbb{P}^3$ and blowup the ideal (x_0, x_1, x_2) corresponding to the point $[0 : 0 : 0 : 1]$ to obtain

$$\widetilde{Q} \subset Q \times \mathbb{P}^2 \subset \mathbb{P}^3 \times \mathbb{P}^2.$$

Similarly, blowup $\mathbb{P}^2$ at the ideal $(y_0^2, y_0y_1, y_0y_2, y_1y_2)$ corresponding to the two points $[0 : 1 : 0]$ and $[0 : 0 : 1]$, to obtain

$$\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

Show that the isomorphism

$$\mathbb{P}^3 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^3$$

exchanging the factors restricts to an isomorphism $\widetilde{Q} \to \widetilde{\mathbb{P}^2}$.

3. (*Tangent cones.*) Let $X \subset \mathbb{A}^n$ be an affine variety and let $p \in X$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p}$. Show that the coordinate ring $A(C_{X,p})$ of the tangent cone of X at p is isomorphic to the graded algebra $\bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$.

4. (*Exceptional hypersurface.*) Consider the blowup of the affine variety $X \subset \mathbb{A}^n$ at $p \in X$. Show that the exceptional hypersurface is the projectivization of the tangent cone

$$E \cong \mathbb{P}(C_{X,p}).$$

You may want to generalize the argument we had in class for plane curves.

5. (*Cremona transformations.*) Consider the Cremona birational automorphism of $\mathbb{P}^2$ given by

$$C([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1].$$

Let $\widetilde{\mathbb{P}^2}$ be the blowup of $\mathbb{P}^2$ at the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$ where C is undefined. Show that

(i) Show that C extends to an isomorphism

$$\widetilde{C} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}.$$

(ii) Let E_1, E_2, E_3 be the exceptional lines for the blowup, and let L_{ij} be the strict transform of the line through P_i and P_j. Draw the incidence graph of the configuration of lines. What happens to each of the 6 lines under $\widetilde{C}$?

Hint: Show that the equations of the blowup $\widetilde{\mathbb{P}^2} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are given by

$$x_0y_0 = x_1y_1 = x_2y_2 = 0.$$

It may help to find the equations of the exceptional lines E_1, E_2, E_3 and those of the strict transforms L_{23}, L_{12}, L_{13}.

(iii) The group of automorphisms of the field of fractions in two variables

$$K(\mathbb{P}^2) \cong K(\mathbb{A}^2) \cong k(t_1, t_2)$$

is called the Cremona group. Therefore, the elements of the Cremona group correspond to birational self-isomorphisms of $\mathbb{P}^2$.

Explain that the Cremona transformation C corresponds to the involution of $k(t_1, t_2)$ sending

$$(t_1, t_2) \to (t_1^{-1}, t_2^{-1}).$$

Furthermore, show that $GL_2(\mathbb{Z})$ is a subgroup of the Cremona group, in such a fashion that $-I_2$ corresponds to the Cremona transformation C.

Remark: The Cremona group is not yet fully understood (especially when the number of indeterminantes is bigger than 2).