Math 203, Problem Set 5. Due Friday, November 9.

For this problem set, you may assume that the ground field is algebraically closed.

Solve all problems below. Hand in at least 3 *problems from the list. Please hand in at least one of the last three problems: Problem* 4*,* 5 *or* 6*.*

1. (*Hyperelliptic curves.*) Let $a_1, \ldots, a_{2g+1}$ be pairwise distinct constants. Find the singularities of the projective *hyperelliptic curve of genus* g*:*

$$y^2 z^{2g-1} = (x - a_1 z) \ldots (x - a_{2g+1} z).$$

Remark: When $g = 1$, we get the elliptic curve $\overline{E}_\lambda$ of the previous problem set for $a_1 = 0, a_2 = 1, a_3 = \lambda$.

2. (*Pencils of conics and singularities.*) Let Q_1 and Q_2 be two distinct *nonsingular* conics in $\mathbb{P}^2$. The family of conics

$$Q_{\lambda,\mu} = \lambda Q_1 + \mu Q_2$$

where $[\lambda : \mu] \in \mathbb{P}^1$ is called a *pencil* of conics.

(i) Recall that any conic $Q \subset \mathbb{P}^2$ determines and is determined by the symmetric matrix A of coefficients with

$$Q(\mathbf{x}) = \mathbf{x}^T \cdot A \cdot \mathbf{x}$$

Possibly by diagonalizing A (and therefore Q), show that

$$Q \text{ is singular if and only if } \det A = 0.$$

(ii) Letting $A_{\lambda,\mu}$ be the matrix associated to the conic $Q_{\lambda,\mu}$, show that $\det A_{\lambda,\mu}$ is a cubic polynomial in λ, μ. Prove that any pencil of conics contains (at most) 3 singular conics.

(iii) Let p_1, p_2, p_3, p_4 be points in $\mathbb{P}^2$ such that no three of them lie on a line. Show that the set of conics through p_1, p_2, p_3, p_4 is a pencil. (Feel free to change coordinates to prove this fact). What are the singular conics in this pencil? Can you draw them?

3. (*Singularities of hypersurfaces.*) Show that a *general* hypersurface of degree d in $\mathbb{P}^n$ is non-singular:

(i) For any hypersurface $\mathcal{Z}(f) \subset \mathbb{P}^n$ of degree d, view the coefficients of f as a point p_f in a large dimensional projective space $\mathbb{P}^N$ (This projective space is called *the moduli space* of degree d hypersurfaces). Let

$$X = \{(f, p) \in \mathbb{P}^N \times \mathbb{P}^n : p \text{ is a singular point of } f\}.$$

Show that X is a projective algebraic set in $\mathbb{P}^N \times \mathbb{P}^n$.

(ii) Conclude that the image $\pi(X)$ of X under the projection onto $\mathbb{P}^N$ is a projective algebraic set. What is $\pi(X)$? Conclude that the subset of $\mathbb{P}^N$ corresponding to smooth hypersurfaces is open and *nonempty.*

Remark: This will prove that the hypersurface is singular provided that the coefficients of f satisfy certain polynomial relations. Therefore, if you pick f randomly, these polynomial relations will most likely not be satisfied and your hypersurface is non-singular. This is the explanation of the word *general.*

4. (*Singularities of cubics.*)

(i) Show that any singular irreducible cubic in $\mathbb{P}^2$ is isomorphic to either the nodal or the cuspidal cubics:

$$y^2z = x^2(x+z) \text{ or } y^2z = x^3.$$

Hint: Assume the singularity is at $[0:0:1]$. Show that the cubic can be written as

$$(\text{quadratic polynomial in } x, y) \cdot z = Q(x, y),$$

where Q is a cubic polynomial in x, y. Change coordinates suitably and write the cubic as

$$y^2z = \widetilde{Q}(x, y) \text{ or } xyz = \widetilde{Q}(x, y).$$

Use the coordinate change $z \mapsto \lambda x + \mu y + \nu z$ to put the cubic into one of the forms

$$y^2z = (x + by)^3 \text{ or } xyz = (x + y)^3.$$

Conclude by performing one more change of coordinates.

(ii) Using (i), show that *irreducible cubics* in $\mathbb{P}^2$ can have at most 1 singular point. Exhibit a cubic in $\mathbb{P}^2$ with 3 singular points.

Remark: We will show later that an *irreducible* degree d curve in $\mathbb{P}^2$ has at most $\binom{d-1}{2}$ singular points.

5. (*Dual conics.*) Let $C \subset \mathbb{P}^2$ be a non-singular curve, given as the zero locus of a homogeneous polynomial $f \in k[x, y, z]$. Consider the morphism

$$\Phi : C \to \mathbb{P}^2, p \mapsto \left[\frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p)\right].$$

The image $\Phi(C) \subset \mathbb{P}^2$ is called the dual curve to C.

(i) Why is Φ a well-defined morphism? Find a geometric description of Φ, independent of coordinates.

(ii) If C is an irreducible conic, prove that its dual $\Phi(C)$ is also an irreducible conic. One way to prove this is to linearly change coordinates and assume the conic C is $ax^2 + by^2 + cz^2 = 0$. What is $\Phi(C)$?

(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines.

6. (*Analytic singularities.*) Consider the singular plane curves Z and W given by the equations

$$y^2 - x^2(x+1) = 0 \text{ and } xy = 0$$

respectively.

(i) Explain briefly why Z and W are not isomorphic. Explain that $(0,0)$ is an ordinary double point for both of these curves. What are the tangent directions at $(0,0)$ for Z and W? Sketch (the real points of) Z and W. Do Z and W look *alike* near the origin?

(ii) Show that there are *formal power series*

$$\tilde{x} = f_1 + f_2 + f_3 + \ldots \text{ and}$$

$$\tilde{y} = g_1 + g_2 + g_3 \ldots$$

in the variables x and y such that the equation of Z becomes

$$\tilde{x}\tilde{y} = 0.$$

Hint: Construct the degree i homogeneous parts f_i and g_i inductively. Show you can pick

$$f_1 = y - x, g_1 = x + y.$$

Next, you would need

$$f_2(x+y) + g_2(y-x) = -x^3.$$

Why can you construct f_2 and g_2? Continue in this fashion.

Remark: If we work over an arbitrary field k it doesnt make sense to ask if the power series $\tilde{x}$ and $\tilde{y}$ converge, hence the terminology *formal power series.* Convergence may be arranged if you work over the complex numbers, but you don't have to prove it.

Remark: It turns out the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is invertible *e.g.* you can solve for x, y in terms of formal power series in $\tilde{x}, \tilde{y}$. In fact, this statement is generally true about any power series

$$\tilde{x} = ax + by + \ldots, \tilde{y} = cx + dy + \ldots$$

provided that $ad - bc \neq 0$. Therefore, the assignment

$$(x, y) \to (\tilde{x}, \tilde{y})$$

is a *formal* change of coordinates, establishing a *formal isomorphism* between Z and W. We say that Z and W are *analytically equivalent.*

Remark: Over the complex numbers, convergence may be arranged near the origin, if x, y are small, and thus the word *formal* may be replaced by *local analytic isomorphism* near the origin.

(iii) Explain briefly why any ordinary double point singularity in $\mathbb{A}^2$ is analytically equivalent to the node $\tilde{x}\tilde{y} = 0$.

Remark: It can be shown that any double point is analytically equivalent to the singularity $\tilde{y}^2 = \tilde{x}^r$, for some r. The case $r = 2$ corresponds to the case which concerned us above.