Math 203, Problem Set 4. Due Friday, November 2.

For this problem set, you may assume that the ground field is algebraically closed.

Solve all problems below. Hand in at least 3 *problems from the list below subject to the rules:*

(i) *you need to hand in either Problem* 2 *or Problem* 3*;*
(ii) *you need to hand in either Problem* 4 *or Problem* 6*;*
(iii) *I recommend that if you hand in Problem* 4*, then you solve Problem* 5 *as well.*

1. (*Quadrics are rational.*) We have seen in class that an irreducible projective quadric Q corresponds to a symmetric square matrix A such that

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \text{ for } \mathbf{x} \in \mathbb{P}^n.$$

We say that the quadric Q is nondegenerate if the matrix A is non-degenerate.

(i) Show that a non-degenerate irreducible quadric Q in $\mathbb{P}^3$ can be written in the form

$$xy = zw$$

after a suitable change of homogeneous coordinates. Conclude that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Conclude that Q is rational.

(ii) In general, show that any non-degenerate irreducible quadric $Q \subset \mathbb{P}^{n+1}$ is birational to $\mathbb{P}^n$.

Hint: Consider the linear projection from a point $p \in Q$ to a hyperplane H.

2. *(Elliptic curves and intersection of quadrics.)*

In general, the intersection of two quadrics in $\mathbb{P}^3$ is an elliptic curve. Check this for the following two quadrics in $\mathbb{P}^3$:

$$Q_1 = \mathcal{Z}(xw - yz), Q_2 = (yw - (x - z)(x - \lambda z)).$$

More precisely, show that the intersection of the two quadrics is isomorphic to the elliptic curve $\overline{E}_\lambda \subset \mathbb{P}^2$ given by

$$y^2 z - x(x - z)(x - \lambda z) = 0.$$

3. *(Twisted curves and complete intersections.)*

A variety Y of dimension r in $\mathbb{P}^n$ is a *strict complete intersection* if the ideal $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

(i) Show that a strict complete intersection is a set theoretic complete intersection.

(ii) Show that the twisted cubic T in $\mathbb{P}^3$ can be written as the set-theoretic intersection of the quadric and the cubic

$$Q = \mathcal{Z}(y^2 - xz), C = \mathcal{Z}(z^3 + xw^2 - 2yzw).$$

In particular, T is a set theoretic complete intersection.

(iii) Show that T is the intersections of the three quadrics

$$Q_1 = \mathcal{Z}(xz - y^2), Q_2 = \mathcal{Z}(xw - yz), Q_3 = \mathcal{Z}(yw - z^2).$$

Show that any two of these quadrics will not intersect in the twisted cubic. In particular $I(T)$ contains three elements of degree 2.

(iv) Possibly using (iii), explain that the ideal of the twisted cubic $I(T)$ cannot be generated by two elements, hence T is not a strict complete intersection.

Remark: It is an unsolved problem to show that every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic complete intersection. This is called the *Hartshorne conjecture.*

4. (*Joins.*) Let $\mathbb{G}(1, n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in the previous homework. Show that:

(i) The set $\{(L, P) : P \in L\} \subset \mathbb{G}(1, n) \times \mathbb{P}^n$ is closed.

(ii) If $Z \subset G(1, n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.

(iii) Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting X and Y is a closed subset of $\mathbb{P}^n$. It is called the join $J(X, Y)$ of X and Y.

5. (*Intersections in projective space.*) Let X and Y be two subvarieties of $\mathbb{P}^n$. Show that if $\dim X + \dim Y \geq n$, then $X \cap Y$ is not empty.

Hint: Let H_1, H_2 be two disjoint linear subspaces of dimension n in $\mathbb{P}^{2n+1}$, and consider $X \subset H_1 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ and $Y \subset H_2 \cong \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ as subvarieties of $\mathbb{P}^{2n+1}$. Show that the join $J(X, Y)$ in $\mathbb{P}^{2n+1}$ has dimension $\dim X + \dim Y + 1$. Then construct $X \cap Y$ as a suitable intersection of $J(X, Y)$ with $n + 1$ hyperplanes.

6. (*Lines on hypersurfaces.*) Let $d > 2n - 3$. Show that a general degree d hypersurface in $\mathbb{P}^n$ contains no lines. (That is, show that at least one such hypersurface contains no lines. This implies that in fact a random such hypersurface contains no lines.)