Math 203, Problem Set 3. Due Wednesday October 24.

Hand in (at least) 3 *problems from the list below.*

For this problem set, you may assume that the ground field is algebraically closed.

1. (*Cubic curves are not rational.*) We have seen in the last problem set that irreducible conics in $\mathbb{A}^2$ are rational. In this problem, we show that most cubic curves are not.

Let $\lambda \in k \setminus \{0, 1\}$. Consider the cubic curve $E_\lambda \subset \mathbb{A}^2$ given by the equation

$$y^2 - x(x-1)(x-\lambda) = 0.$$

Show that E_λ is not birational to $\mathbb{A}^1$. In fact, show that there are no non-constant rational maps

$$F : \mathbb{A}^1 \dashrightarrow E_\lambda.$$

(i) Write

$$F(t) = \left(\frac{f(t)}{g(t)}, \frac{p(t)}{q(t)} \right)$$

where the pairs of polynomials (f, g) and (p, q) have no common factors. Conclude that

$$\frac{p^2}{q^2} = \frac{f(f-g)(f-\lambda g)}{g^3}$$

is an equality of fractions that cannot be further simplified. Conclude that $f, g, f-g, g-\lambda g$ must be perfect squares.

(ii) Conclude by proving the following:

Lemma: If f, g are polynomials in $k[t]$ without common factors and such that there is a constant $\lambda \neq 0, 1$ for which $f, g, f-g, f-\lambda g$ are perfect squares, then f and g must be constant.

Hint: Descent. Write $f = u^2, g = v^2$. Considering $f - g$ and $f - \lambda g$, prove that $u-v, u+v, u-\mu v, u+\mu v$ are also squares for some constant $\mu \neq 0, 1$. Show that suitable $\tilde{u}, \tilde{v}$ obtained as a linear combination of u and v verify the lemma, yet they have smaller degree than $\max(\deg f, \deg g)$.

Remark: We will see later that any cubic curve can be written in the form

$$y^2 - x(x-1)(x-\lambda) = 0, \text{ or } y^2 - x^3 = 0 \text{ or } y^2 - x^2(x-1) = 0, .$$

The latter curves are Z_2 and W_2 in the previous problem set, so they are birational to $\mathbb{A}^1$.

2. (*Isomorphisms of the affine and projective line*)

(i) Show that every isomorphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ is of the form $f(x) = ax + b$.

(ii) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$f(x) = \frac{ax + b}{cx + d}$$

for some $a, b, c, d \in k$, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

(iii) The isomorphisms of $\mathbb{P}^1$ act triply transitively. That is, given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

3. (*Conics though* 5 *points.*)

(i) Extend the result of the previous problem 2(iii) as follows. Four points in $\mathbb{P}^2$ are said to be in general position if no three are collinear (i.e. lie on a projective line in the projective plane). Show that if $p_1, \ldots, p_4$ are points in general position, there exists a linear change of coordinates

$$T : \mathbb{P}^2 \to \mathbb{P}^2 \text{ with}$$

$$T([1:0:0]) = p_1,\ T([0:1:0]) = p_2,\ T([0:0:1]) = p_3,\ T([1:1:1]) = p_4.$$

(ii) Given five distinct points in $\mathbb{P}^2$, no three of which are collinear, show that there is an unique irreducible projective conic passing though all five points. You may want to use part (i) to assume that four of the points are $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$.

(iii) Deduce that two distinct irreducible conics in $\mathbb{P}^2$ cannot intersect in 5 points. (We will see later that they intersect in exactly 4 points counted with multiplicity.)

Remark: For any degree d, fix $3d - 1$ points in $\mathbb{P}^2$ in "general position". You may ask how many rational curves of degree d in $\mathbb{P}^2$ pass through these $3d - 1$ points. Clearly, there is $N_1 = 1$ line through 2 points, and we have shown that $N_2 = 1$ conic through 5 points. The next few numbers are

$$N_3 = 12, N_4 = 620, N_5 = 87,304, N_6 = 26,312,976, N_7 = 14,616,808,192.$$

Thus, there are are 12 rational cubics through 8 points, 620 rational quartics through 11 points and so on. A general answer for arbitrary d was found in 1994 using ideas from physics/string theory. The area of algebraic geometry that computes these numbers is called enumerative geometry/Gromov-Witten theory.

4. (*Grassmannians.*) We will make the space of all lines in $\mathbb{P}^n$ into a projective variety. We define a set-theoretic map

$$\phi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with

$$N = \binom{n+1}{2} - 1$$

as follows. For every line $L \subset \mathbb{P}^n$, choose two distinct points

$$P = (a_0 \dots a_n) \text{ and } Q = (b_0 \dots b_n)$$

on L and define $\phi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the maximal minors of the matrix

$$\begin{pmatrix} a_0 : \dots : a_n \\ b_0 : \dots : b_n \end{pmatrix}$$

in any fixed order. Show that:

(i) The map ϕ is well-defined and injective. The map ϕ is called the Plucker embedding.

(ii) The image of ϕ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. You may want to recall the Gaussian algorithm which brings *almost* any matrix as above into the form

$$\begin{pmatrix} 1 \; 0 \; a'_2 \dots a'_n \\ 0 \; 1 \; b'_2 \dots b'_n \end{pmatrix}.$$

(iii) Show that $G(1,1)$ is a point, $G(1,2) = \mathbb{P}^2$, and $G(1,3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.

5. (*Introduction to moduli theory.*) Show that for any 3 lines L_1, L_2, L_3 in $\mathbb{P}^3$, there is a quadric $Q \subset \mathbb{P}^3$ containing all three of them.

(i) First, observe that any homogeneous degree 2 polynomial in 4 variables has 10 coefficients. These coefficients can be regarded as a point in the projective space $\mathbb{P}^9$. Show that this point only depends on the quadric Q and not on the polynomial defining it. Let us denote this point by p_Q. Show that any point $p \in \mathbb{P}^9$ determines a quadric in $\mathbb{P}^3$.

Remark: The projective space $\mathbb{P}^9$ is said to be the moduli space of quadrics in $\mathbb{P}^3$.

(ii) Consider a line $L \subset \mathbb{P}^3$. Show that there is a codimension 3 projective linear subpace

$$H_L \subset \mathbb{P}^9$$

such that

$$L \subset Q \text{ iff and only if } p_Q \in H_L.$$

(iii) Show that any three codimension 3 projective linear subspaces of $\mathbb{P}^9$ intersect. In particular, show that

$$H_{L_1} \cap H_{L_2} \cap H_{L_3} \neq \emptyset,$$

and conclude that L_1, L_2, L_3 are contained in a quadric Q.

(iv) Explain (briefly) that if L_1, L_2, L_3 are disjoint lines, then Q can be assumed to be irreducible.