Math 203, Problem Set 2. Due Monday October 15.

Hand in (at least) 3 *problems from the list below.*

For this problem set, you may assume that the ground field is algebraically closed.

1. Find the irreducible components of the affine algebraic set $xz - y^2 = z^3 - x^5 = 0$ in $\mathbb{A}^3$. What is the dimension of this affine algebraic set?

2. Which of the following algebraic sets are isomorphic and which ones are not:

(i) $\mathbb{A}^1$
(ii) $\mathcal{Z}(xy) \subset \mathbb{A}^2$
(iii) $\mathcal{Z}(x^2 + y^2) \subset \mathbb{A}^2$
(iv) $\mathcal{Z}(x^2 - y^5) \subset \mathbb{A}^2$
(v) $\mathcal{Z}(y - x^2, z - x^3) \subset \mathbb{A}^2$.

3. (*Conics.*) An algebraic set $\mathcal{Z} \subset \mathbb{A}^2$ defined by an irreducible polynomial f of degree 2 is called an irreducible conic.

(i) Show that after an affine change of coordinates in $\mathbb{A}^2$, any irreducible conic can be written in the form

$$Y - X^2 = 0 \text{ or } XY - 1 = 0.$$

(ii) Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two distinct irreducible conics in $\mathbb{A}^2$. Using (i), show that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ intersect in at most 4 points. Can you give examples of conics which intersect in 0, 1, 2, 3 or 4 points?

4. (*Hartogs theorem and quasi-affine algebraic sets.*) Show that the quasi-affine set $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not isomorphic to an affine algebraic set:

(i) Prove first the following algebro-geometric analogue of Hartog's theorem in complex analysis: if $f : X \to k$ is a regular function on X, then f must be a polynomial in $k[x, y]$, so it extends to $\mathbb{A}^2$.

(ii) If $\Phi : Y \to X$ is an isomorphism between an affine algebraic set Y and X, consider the composition $\Psi = \iota \circ \Phi : Y \to \mathbb{A}^2$, where $\iota : X \to \mathbb{A}^2$ is the inclusion. Show that the morphism Ψ induces an isomorphism on coordinate rings. Conclude that Ψ must be an isomorphism, hence ι must be an isomorphism, a contradiction.

5. Let $n \geq 2$, and let $S = \{a_1, \ldots, a_n\}$ be a finite set with n elements in $\mathbb{A}^1$.

(i) Show that the quasi-affine set $\mathbb{A}^1 \setminus S$ is isomorphic to an affine set. For instance, you may take X to be the affine algebraic set given by the equations

$$X_1(X_0 - a_1) = \ldots = X_n(X_0 - a_n) = 1.$$

Show that the projection onto the first coordinate

$$\pi : X \to \mathbb{A}^1 \setminus S,\ (X_0, \dots, X_n) \mapsto X_0$$

is an isomorphism.

(ii) Show that $\mathbb{A}^1 \setminus S$ is not isomorphic to $\mathbb{A}^1 \setminus \{0\}$ by proving that their rings of regular functions are not isomorphic.

Hint: Assume that

$$\Phi : A(X) \to k\left[t, t^{-1}\right]$$

is an isomorphism. Observe that X_i are invertible elements in $A(X)$ for all $1 \leq i \leq n$. Show that their images must be invertible in $k[t, t^{-1}]$. Prove that this implies that $\Phi(X_i) = t^{m_i}$ for some integers m_i. Derive a contradiction by comparing $\Phi(X_0 - a_i)$ for different values of i.

6. (*You may assume the result of problem 5.*) Let $n \geq 2$. Consider the affine algebraic sets in $\mathbb{A}^2$:

$$Z_n = \mathcal{Z}(y^n - x^{n+1})$$

and

$$W_n = \mathcal{Z}(y^n - x^n(x+1)).$$

Show that Z_n and W_n are birational but not isomorphic.

(i) Show that

$$f : \mathbb{A}^1 \to Z_n,\ \ f(t) = (t^n, t^{n+1})$$

is a morphism of affine algebraic sets which establishes an isomorphism between the open subsets

$$\mathbb{A}^1 \setminus \{0\} \to Z_n \setminus \{(0,0)\}.$$

Similarly, show that

$$g : \mathbb{A}^1 \to W_n,\quad g(t) = (t^n - 1, t^{n+1} - t).$$

is a morphism of affine algebraic sets. Find open subsets of $\mathbb{A}^1$ and W_n where g becomes an isomorphism.

(ii) Using (i), explain why Z_n and W_n are birational. Write down a birational isomorphism $Z_n \dashrightarrow W_n$.

(iii) Assume that there exists an isomorphism

$$h : Z_n \to W_n$$

such that $h((0,0)) = (0,0)$. Observe that this induces an isomorphism between the open sets

$$Z_n \setminus \{(0,0)\} \to W_n \setminus \{(0,0)\}.$$

Use part (i) and the previous problem to conclude this cannot be true if $n \geq 2$.

(iv) (*Optional.*) Repeat the argument above without the assumption that h sends the origin to itself. You may need to prove a stronger version of Problem 5.