**Math 203, Problem Set 1. Due Friday October 5.**

*Hand in (at least)* 3 *problems from the list below.*

*For this problem set, you may assume that the ground field is algebraically closed. .*

**1.** Show that the Zariski topology on $\mathbb{A}^2$ is not the product of the Zariski topologies on $\mathbb{A}^1 \times \mathbb{A}^1$.

**2.** Let $\mathbb{A}^3$ be the 3-dimensional affine space with coordinates $x, y, z$. Find, with proof, the ideals of the following algebraic sets:

(i) The union of the three coordinate axes.
(ii) The union of the $z$-axis and the $(x, y)$-plane.

**3.** Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map *i.e.* $f(p) = (f_1(p), \dots, f_m(p))$ for $p \in \mathbb{A}^n$, where $f_1, \dots, f_m$ are polynomials in $n$ variables. Are the following true or false:

(i) The image $f(X) \subset \mathbb{A}^m$ of an affine algebraic set $X \subset \mathbb{A}^n$ is an affine algebraic set.
(ii) The inverse image $f^{-1}(X) \subset \mathbb{A}^n$ of an affine algebraic set $X \subset \mathbb{A}^m$ is an affine algebraic set.
(iii) If $X \subset \mathbb{A}^n$ is an affine algebraic set, then the graph $\Gamma = \{(x, f(x)) : x \in X\} \subset \mathbb{A}^{n+m}$ is an affine algebraic set.

**4.** Let $X_1, X_2$ be affine algebraic sets in $\mathbb{A}^n$. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, *i.e.* find affine algebraic sets $X_1, X_2$ such that

$$I(X_1 \cap X_2) \neq I(X_1) + I(X_2).$$

**5.** Let $X$ be the image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \to (t^3, t^4, t^5)$. Show that $I(X)$ cannot be generated by fewer than 3 elements.

*Remark:* Note that $X$ has dimension 1 even though it is cut out by at least 3 equations. We say that $X$ is not a *complete intersection*.