UC San Diego

Applied math majors at UC San Diego

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20E Vector Calculus

Gradient

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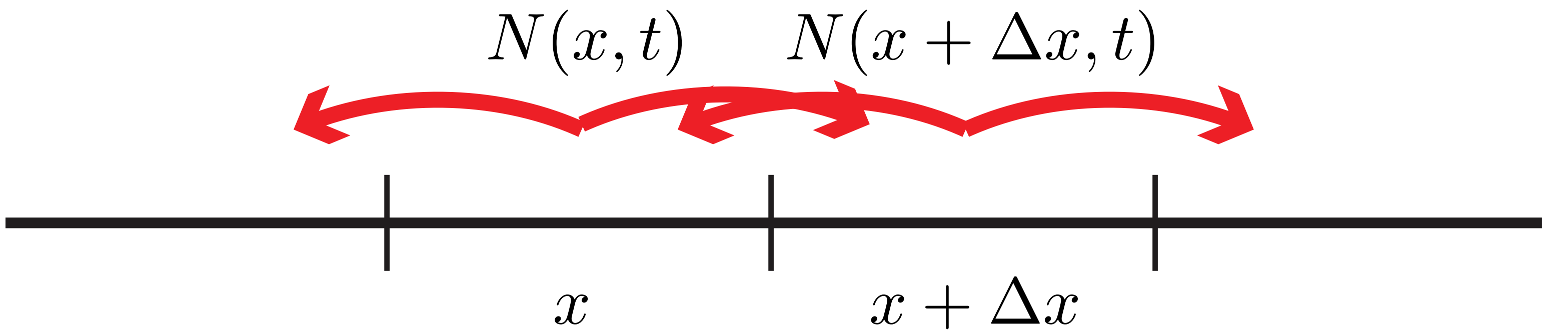
In this case $-\nabla f$ is proportional to the diffusion flux, J, *i.e.*, the stuff flows in the direction opposite the gradient, *e.g.*, from higher to lower concentration.

Diffusion flux

In 1 dimension, let $N(x,t) = f(x,t)\Delta x$ be the number of particles (amount of stuff) in an interval of length Δx at position x.



$$
\begin{aligned}
J &= \frac{1}{\Delta t}\Big(\frac{N(x,t)}{2} - \frac{N(x+\Delta x,t)}{2}\Big) \\
&= -\frac{(\Delta x)^2}{2\Delta t}\cdot\frac{1}{\Delta x}\Big(\frac{N(x+\Delta x,t)}{\Delta x} - \frac{N(x,t)}{\Delta x}\Big) \\
&= -\frac{(\Delta x)^2}{2\Delta t}\,\frac{f(x+\Delta x,t) - f(x,t)}{\Delta x} \to -\kappa\frac{\partial f}{\partial x}.
\end{aligned}
$$

Applying the Divergence Theorem

The change in the amount of stuff in a region V surrounded by a surface S is

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Since this must be true for every V, we have

$$\frac{\partial f}{\partial t} = -\nabla\cdot J = \kappa\nabla\cdot\nabla f = \kappa\nabla^2 f,$$

the diffusion equation. ∇^2 is the Laplacian.

Beyond 20E Vector Calculus

Partial differential equations

diffusion: $\frac{\partial f}{\partial t} = \kappa \nabla^2 f$

Laplace: $\nabla^2 \phi = 0$

wave: $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

Schrödinger: $i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \nabla^2 \psi + V(x,t)\psi$

incompressible Navier-Stokes: $\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \nabla^2 u = -\nabla w + g$

Numerical solution of PDEs

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so

$$f(x,t+\Delta t) \approx f(x,t) + \kappa \frac{\Delta t}{(\Delta x)^2}\big(f(x+\Delta x, t) - 2f(x,t) + f(x-\Delta x,t)\big).$$

Numerical linear algebra

Writing discretized $f(t)$ as a vector:

$$\begin{pmatrix} \vdots \\ f(x-\Delta x) \\ f(x) \\ f(x+\Delta x) \\ \vdots \end{pmatrix}(t+\Delta t) \approx \begin{pmatrix} \vdots \\ f(x-\Delta x) \\ f(x) \\ f(x+\Delta x) \\ \vdots \end{pmatrix}(t) + \kappa \frac{\Delta t}{(\Delta x)^2} \cdot$$

$$\cdot \begin{pmatrix} \ddots & & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ f(x-\Delta x) \\ f(x) \\ f(x+\Delta x) \\ \vdots \end{pmatrix}(t).$$

Better numerical methods involve more complicated linear algebra, *e.g.*, matrix inversion.

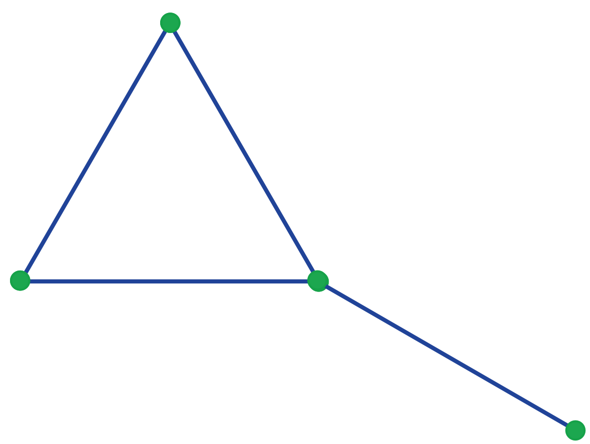
Graph theory

A graph is a set of vertices, V, with a set of edges, $E \subset V \times V$.

The degree of a vertex i is $|\{j \in V \mid (i,j) \in E\}|$.

The adjacency matrix of a graph is a $|V| \times |V|$ matrix A with

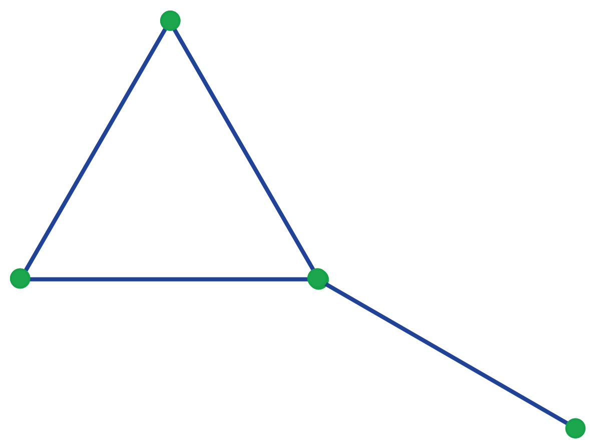
$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E; \\ 0 & \text{otherwise.} \end{cases}$$



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Applications of graph theory

By analogy with the discrete Laplacian, the graph Laplacian is A minus the diagonal matrix D with D_{ii} the degree of vertex i.



$$L = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$-LD^{-1} = I - AD^{-1}$ is a Markov matrix, *i.e.*, all nonnegative entries, summing to 1 in each column.

$$M = \begin{pmatrix} 0 & 1/2 & 1/2 & 1 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 0 \end{pmatrix}$$

Applications of graph theory

M_{ij} is the transition probability for a random walker to hop from j to i (which is why the entries are non-negative and sum to 1 for fixed j).

The equilibrium distribution is the eigenvector with eigenvalue 1, so that it is unchanged by a step of the random walk.

This eigenvector is very close to being Google's PageRank of a webpage in the web graph.

109 Proof

Malfatti's problem

Memoria sopra un problema stereotomico.
Memorie di Matematica e Fisica della Società Italiana, 10 p. 1ª (1803) pp. 235-244 - in 4°.

MEMORIA

SOPRA UN PROBLEMA STEREOTOMICO

DI GIANFRANCESCO MALFATTI.

Dato un Prisma retto triangolare di qualunque materia come di marmo, cavare da esso tre Cilindri dell' altezza del Prisma e della maggior grossezza possibile correspettivamente, e in conseguenza col minor avanzo possibile di materia avuto riguardo alla voluta grossezza.

Vi sono in Geometria alcuni problemi, la soluzione analitica de' quali non si può presentare senza tedio del lettore attesa la lunghezza e l' improbità de' calcoli, ai quali ha dovuto soggiacere il Geometra nella soluzione del suo problema; laddove dopo aver conosciuto il vero risultato, convertendo l' analisi in sintesi simbolica, ed il problema in teorema, succede parecchie volte che si possa per una via più agevole e piana dare di esso una comoda dimostrazione. Di questa specie è l' enunziato Problema che mi fu proposto non ha guari, e che mi parve sul principio di facile soluzione, osservando che esso riducevasi alla inscrizione di tre circoli nei due triangoli delle basi parallele del Prisma, cosicchè ciascun de' circoli toccasse gli altri due ed insieme due lati del triangolo. Intrapresa per tanto la soluzione di questo secondo Problema, mi vidi contro ogni mia aspettazione ingolfato in prolissi calcoli e scabrose formole, atte a stancar la pazienza d' un uomo meno di me ostinato. Superata però la difficoltà e avuti de' risultati assai semplici, tentai, cangiando il Problema in Teorema, di aprirmi una

Malfatti, “On a stereotomy problem” (1803)

“Given a triangular right prism of whatsoever material, say marble, take out from it three cylinders with the same heights of the prism but of maximum total volume, that is to say with the minimum scrap of material with respect to the volume.”

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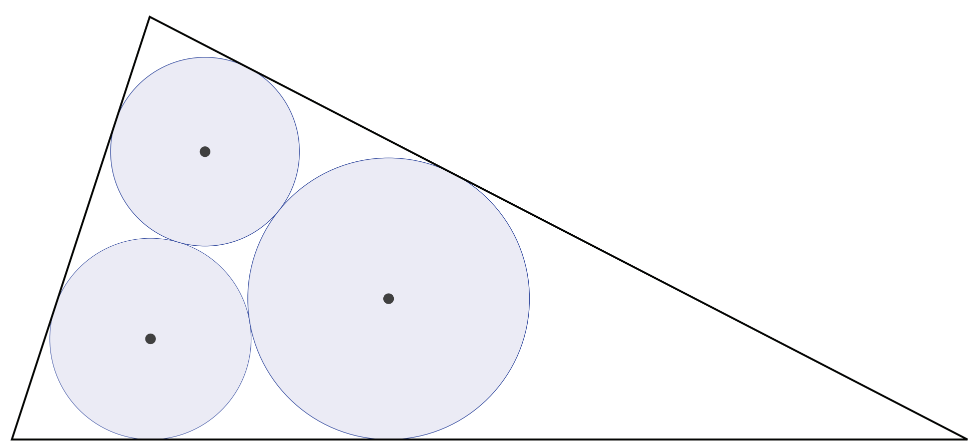
“Given a triangular right prism of whatsoever material, say marble, take out from it three cylinders with the same heights of the prism but of maximum total volume, that is to say with the minimum scrap of material with respect to the volume.”

“. . . the problem reduces to the inscription of three circles in a triangle in such a way that each circle touches the other two and at the same time two sides of the triangle . . .”



Malfatti, "On a stereotomy problem" (1803)

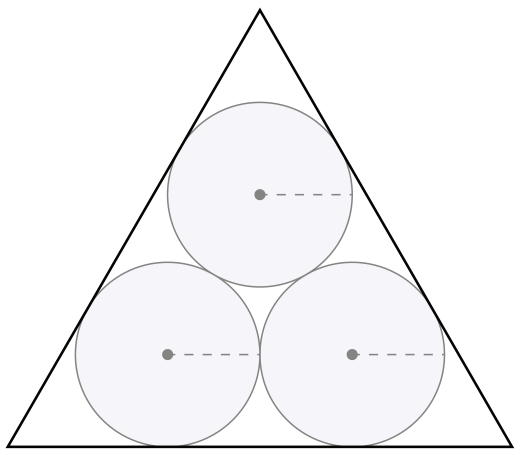
Let a, b, c be the side lengths of the triangle; $s = (a+b+c)/2$; r be the radius of the largest circle inscribable in the triangle; d, e, f be the distances from the center of this circle to the vertices opposite sides a, b, c, respectively.

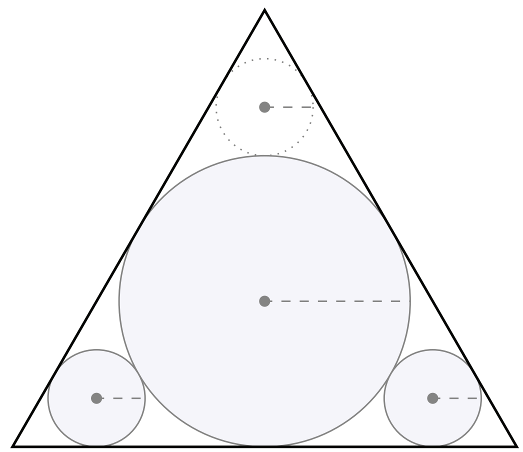
Then the radii of Malfatti's circles are

$$r_1 = \frac{r}{2(s-a)}(s+d-r-e-f)$$
$$r_2 = \frac{r}{2(s-b)}(s+e-r-d-f)$$
$$r_3 = \frac{r}{2(s-c)}(s+f-r-d-e)$$

Malfatti, “On a stereotomy problem” (1803)

But this is wrong. In 1930 (!) Lob and Richmond observed that in an equilateral triangle a different arrangement has larger area:





$$\frac{3\pi}{4(1+\sqrt{3})^2} \approx 0.316 \qquad \frac{11\pi}{108} \approx 0.320$$

Malfatti, "On a stereotomy problem" (1803)

In 1967 (!) Goldberg showed that Malfatti's solution is wrong for every triangle.

Not until 1994 (!) did Zalgaller and Los show that the greedy algorithm (draw the biggest possible circle at each step) gives the correct solution for every triangle.

CONJECTURE (Melissen 1997). The greedy algorithm solves the problem of finding the n circles in a triangle with maximum total area.

Beyond 109 Proof

Applicable math

analysis:	PDEs in physics, chemistry, biology, ...
algebra:	physics, codes, ...
probability:	gambling, finance, physics, chemistry, biology, ...
combinatorics:	networks, physics, chemistry, ...
geometry:	stone cutting, physics, chemistry, biology, ...
algebraic geometry:	codes, physics, chemistry, economics, ...
number theory:	codes, biology, ...
topology:	physics, economics, biology, ...

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Images of circles in triangles by Personline, via Wikimedia Commons.