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MATH 180A. INTRODUCTION TO PROBABILITY LECTURE 26

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§5.3 Solutions

3. W, X, Y, Z are i.i.d. $N(0,1)$ random variables.

   a. $P(W+X > Y+Z+1) = P(W+X-Y-Z > 1) = P(W > 1/2)$, since $W+X-Y-Z$ is an $N(0,4)$ random variable by the theorem on page 363, so if we rescale it by a factor of 2, it is an $N(0,1)$ random variable. $P(W > 1/2) \approx 0.3085$, using the table of Φ values in Appendix 5.

   b. $P(4X+3Y < Z+W) = P(4X+3Y-Z-W < 0) = 1/2$ since $4X+3Y-Z-W$ is a mean 0 random variable, again by the theorem on page 363.

   c. $\mathsf{E}[4X+3Y-2Z^2-W^2+8] = 4\mathsf{E}[X]+3\mathsf{E}[Y]-2\mathsf{E}[Z^2]-\mathsf{E}[W^2]+8 = 0+0-2-1+8 = 5.$

   d. $\mathsf{SD}[3Z-2X+Y+15] = \mathsf{SD}[3Z-2X+Y] = \sqrt{3^2+2^2+1^2} = \sqrt{14}.$

7. Measure time in minutes after 8am. The bus arrival time, B, is an $N\big(10,(2/3)^2\big)$ random variable. The author's arrival time, A, is an independent $N\big(9,(1/2)^2\big)$ random variable.

   a. $P(A < 10) = \Phi(2) \approx 0.9772$, since 1 minute is 2 standard deviations.

   b. $P(A < B) = P(A-B < 0) = P\big((A-9)-(B-10) < 1\big)$. The two terms in parentheses are normal random variables with mean 0 and standard deviations 1/2 and 2/3, respectively, so their difference is a normal random variable with mean 0 and standard deviation $\sqrt{(1/2)^2 + (2/3)^2} = 5/6$. So $P\big((A-9)-(B-10) < 1\big) = \Phi(6/5) \approx 0.8849$.

c. $P(B < 9 \mid B < 9 \text{ or } B > 12) = P(B < 9)/\big(P(B < 9) + P(B > 12)\big) = \Phi(-3/2)/\big(\Phi(-3/2) + 1 - \Phi(3)\big) \approx 0.0668/(0.0668 + 0.0013) \approx 0.9809$.

11. Brownian motion.

a. Start by considering $t = 1/n$, for $n \in \mathbb{N}$. Then X_1, the displacement after time 1, is the sum of n independent displacements $X_{1/n}$, so $n\sigma_{1/n}^2 = \sigma^2$, which means $\sigma_{1/n} = \sqrt{1/n}\sigma$.

Now consider $t = k/n$ for $k \in \mathbb{N}$. Now $X_{k/n}$, the displacement after time k/n, is the sum of k independent displacements $X_{1/n}$, so $\sigma_{k/n}^2 = k\sigma_{1/n}^2 = (k/n)\sigma^2$, and thus $\sigma_{k/n} = \sqrt{k/n}\sigma$.

We conclude that for any positive rational t, $0 \leq t \in \mathbb{Q}$, the standard deviation of X_t is $\sigma_t = \sqrt{t}\sigma$. If we define X_t for real numbers as the limit of a sequence of random variables X_s for $s \in \mathbb{Q}$ and $s \to t$, then this formula for the standard deviation holds for irrational t as well.

b. $R_t/(\sqrt{t}\sigma) = \sqrt{(X_t/\sqrt{t}\sigma)^2 + (Y_t/\sqrt{t}\sigma)^2}$ is the square root of the sum of the squares of two i.i.d. $N(0,1)$ random variables, the distribution of which is calculated on pages 358–359. From this we can calculate the mean and the standard deviation by doing the appropriate integrals.

c. Now $\sigma = 1$ (millimeter) with t measured in seconds. Integrate the pdf from (b) from 2 to ∞.