Coloring, quantum mechanics, and Euclid

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Undergraduate Mathematics Colloquium
La Jolla, CA, 3 December 2003

UCSD

Coloring

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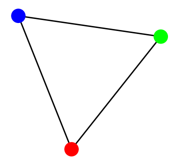

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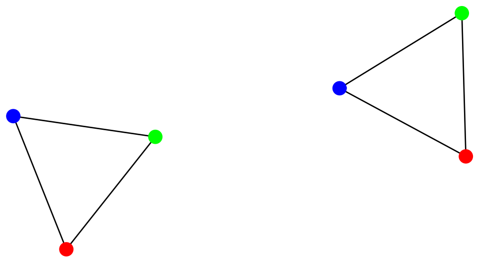

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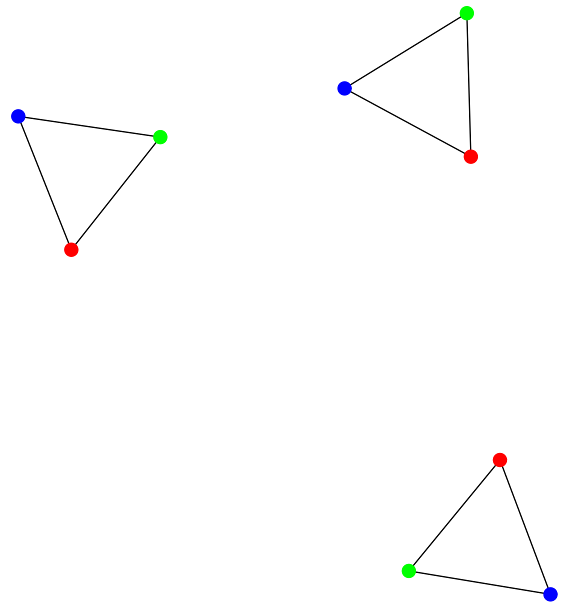

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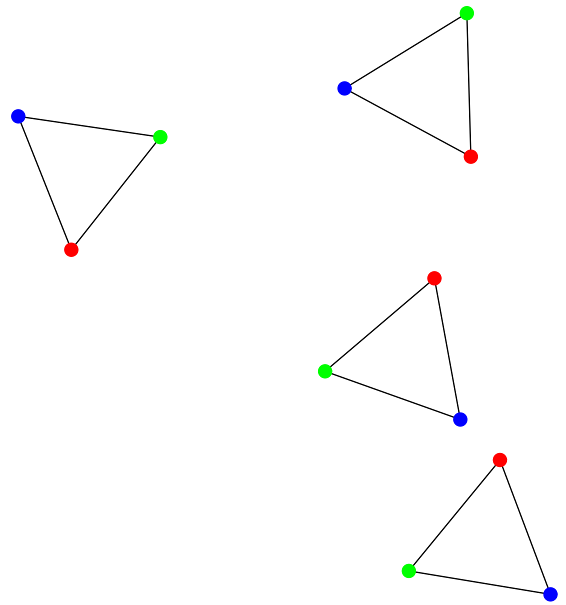

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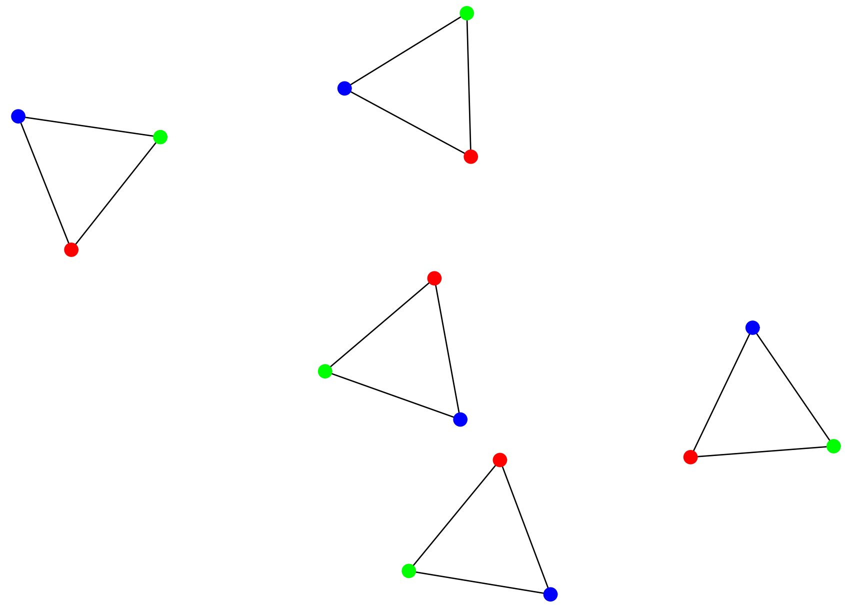

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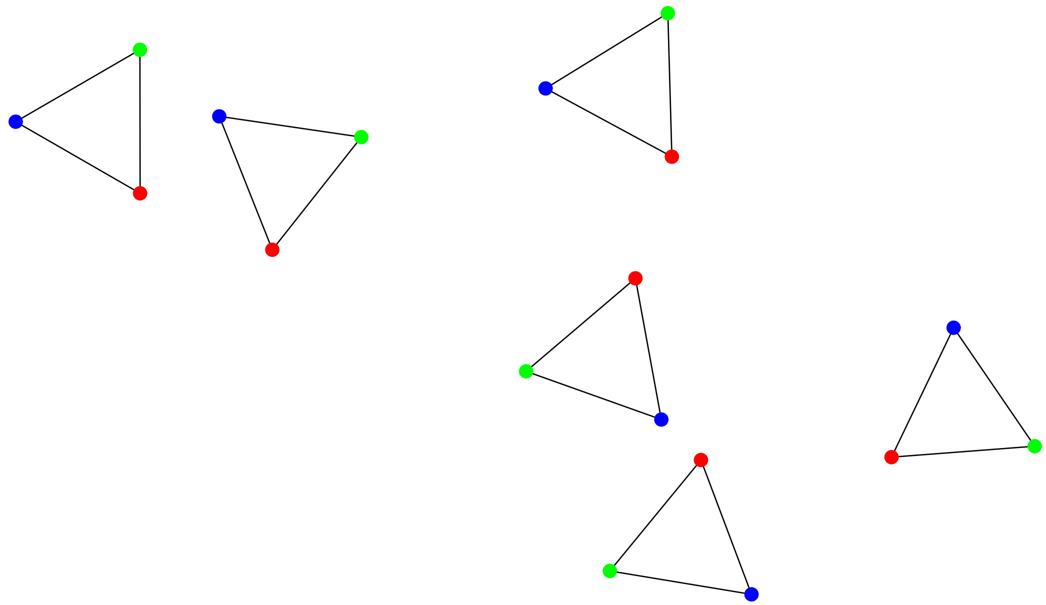

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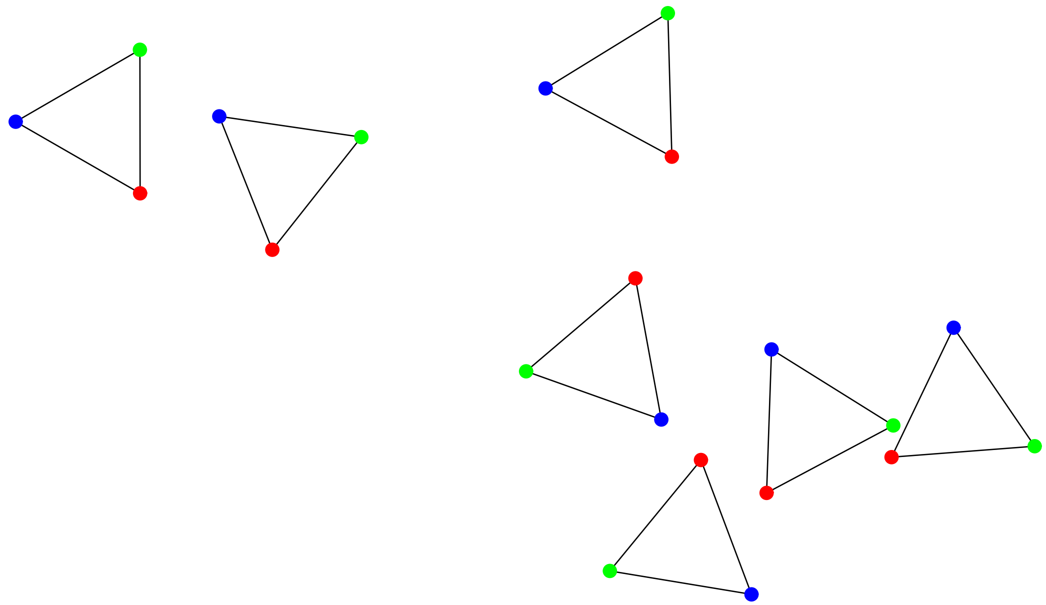

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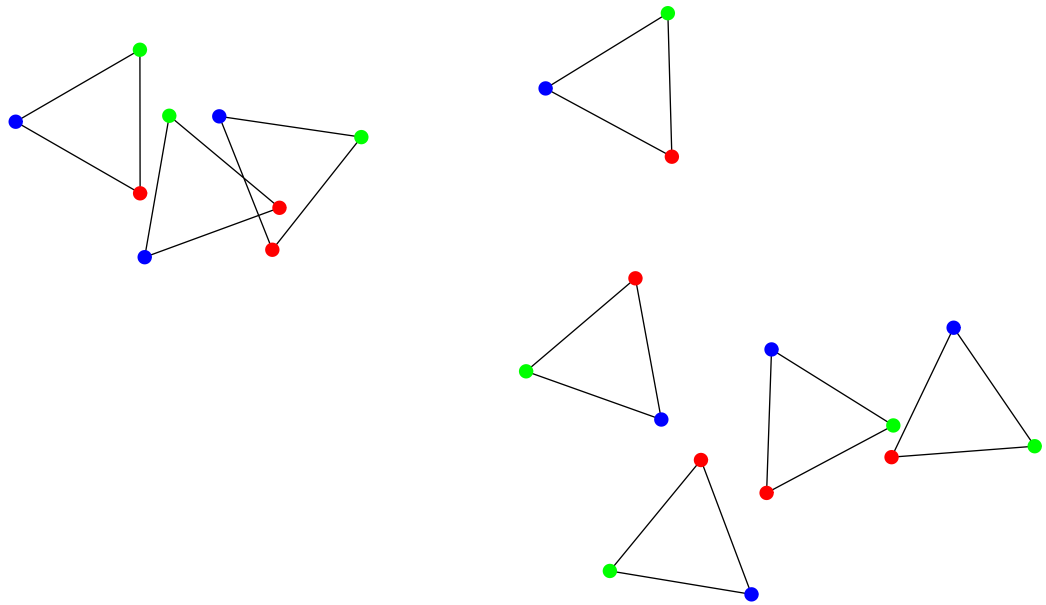

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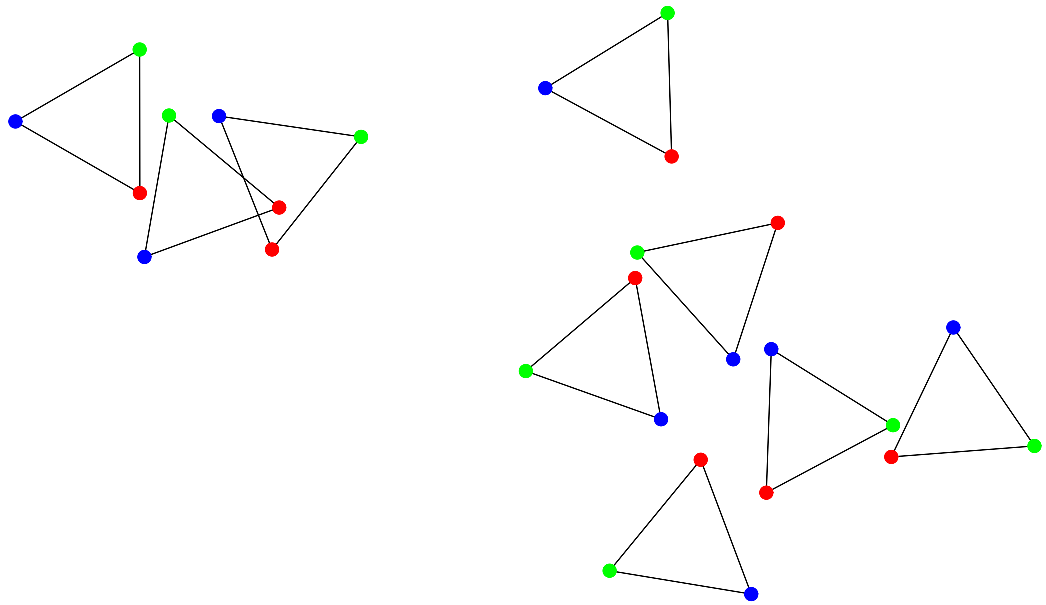

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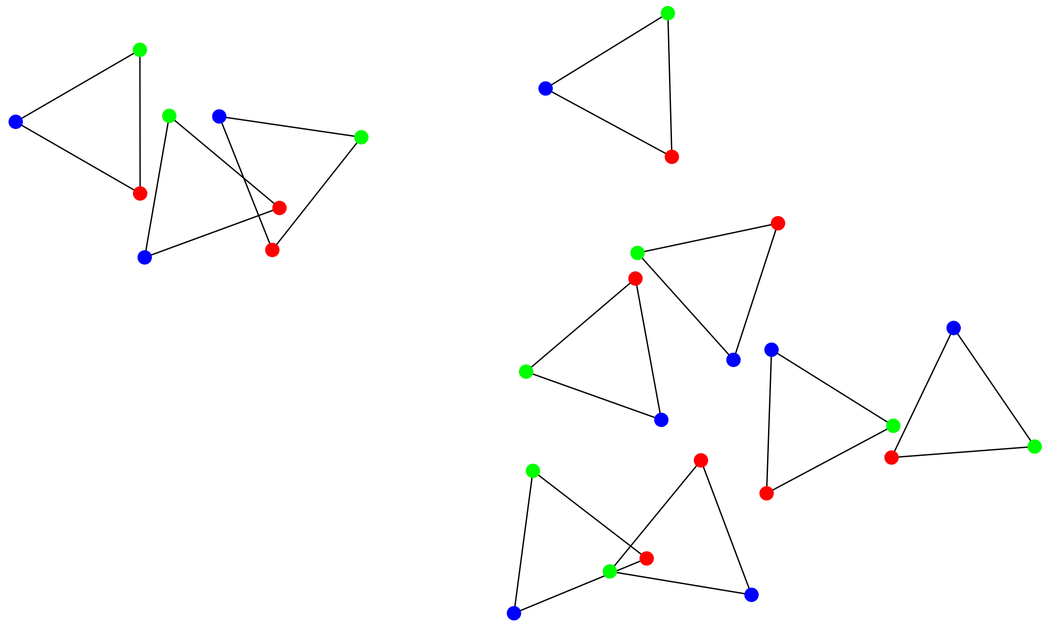

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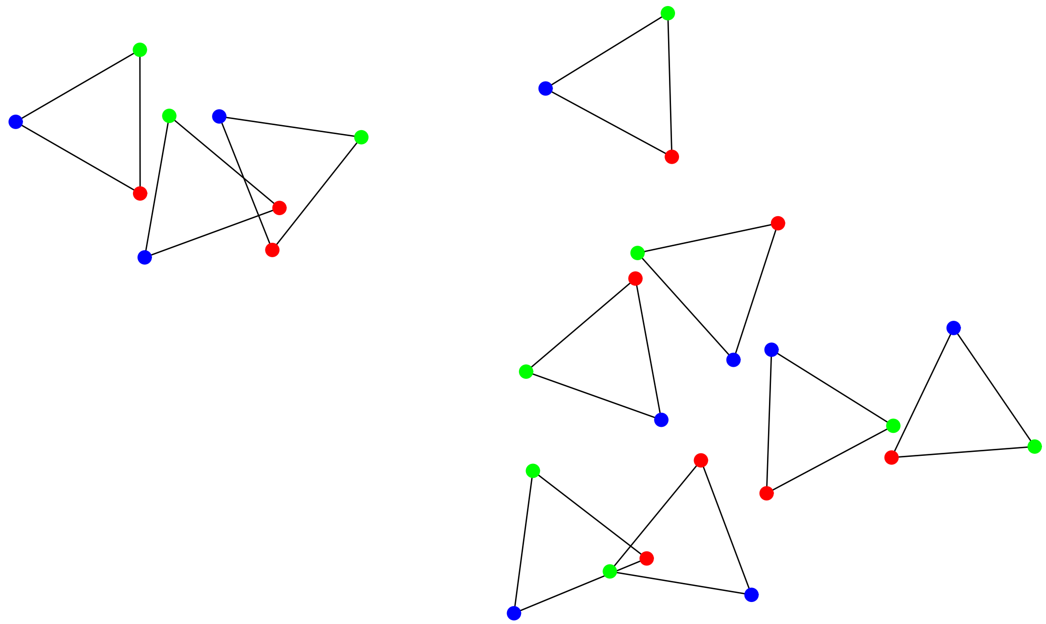

So far, so good . . . but we need to color all the points in the plane.

Seurat's attempt (1884–1886)

Un dimanche après-midi à l'Ile de la Grande Jatte

Seurat's attempt, three-colorized

Un dimanche après-midi à l'Ile de la Grande Jatte

Seurat's attempt, three-colorized

Un dimanche après-midi à l'Ile de la Grande Jatte

OK

Seurat's attempt, three-colorized

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OK; OK

Seurat's attempt, three-colorized

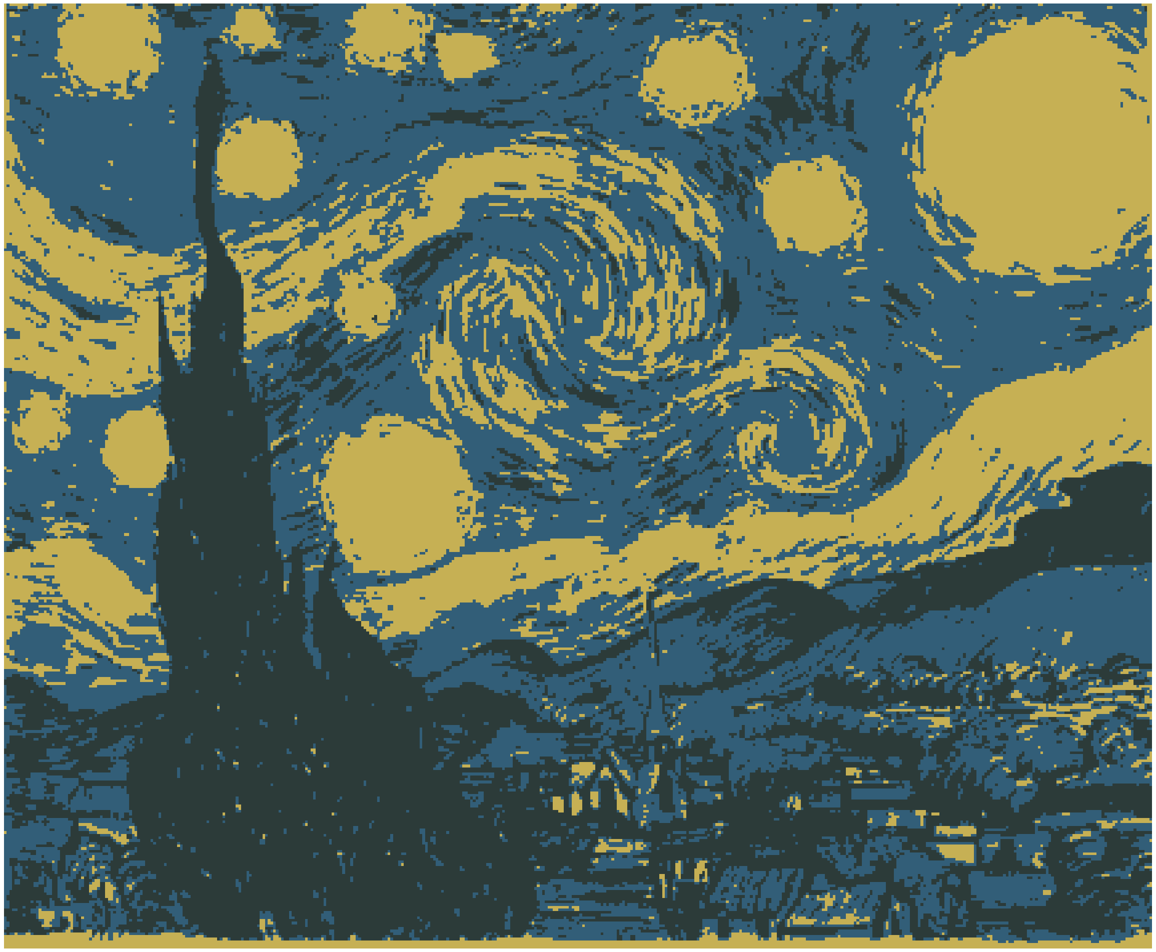
Un dimanche après-midi à l'Ile de la Grande Jatte

OK; OK; not OK

Van Gogh's attempt (1889)

Starry night

Van Gogh's attempt, three-colorized



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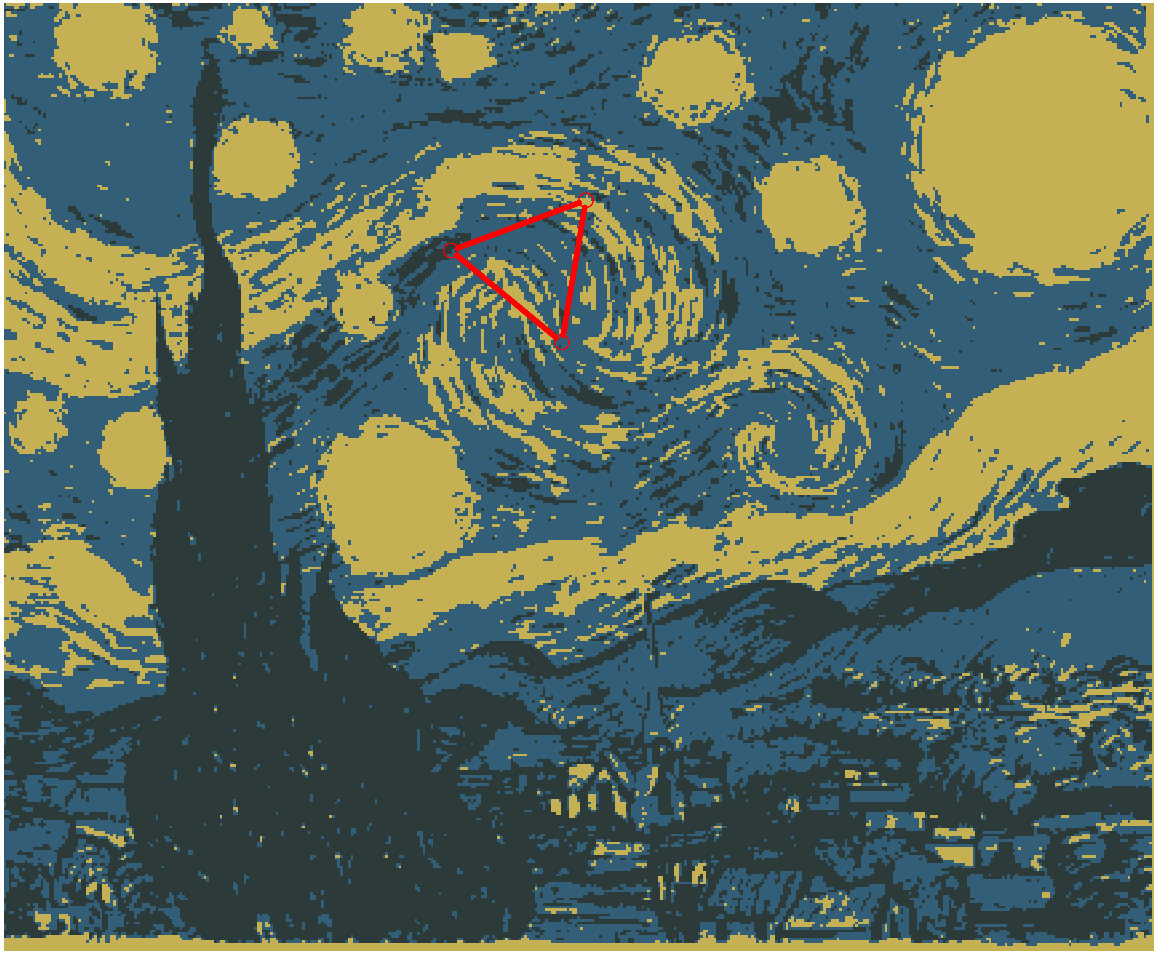

Starry night

OK

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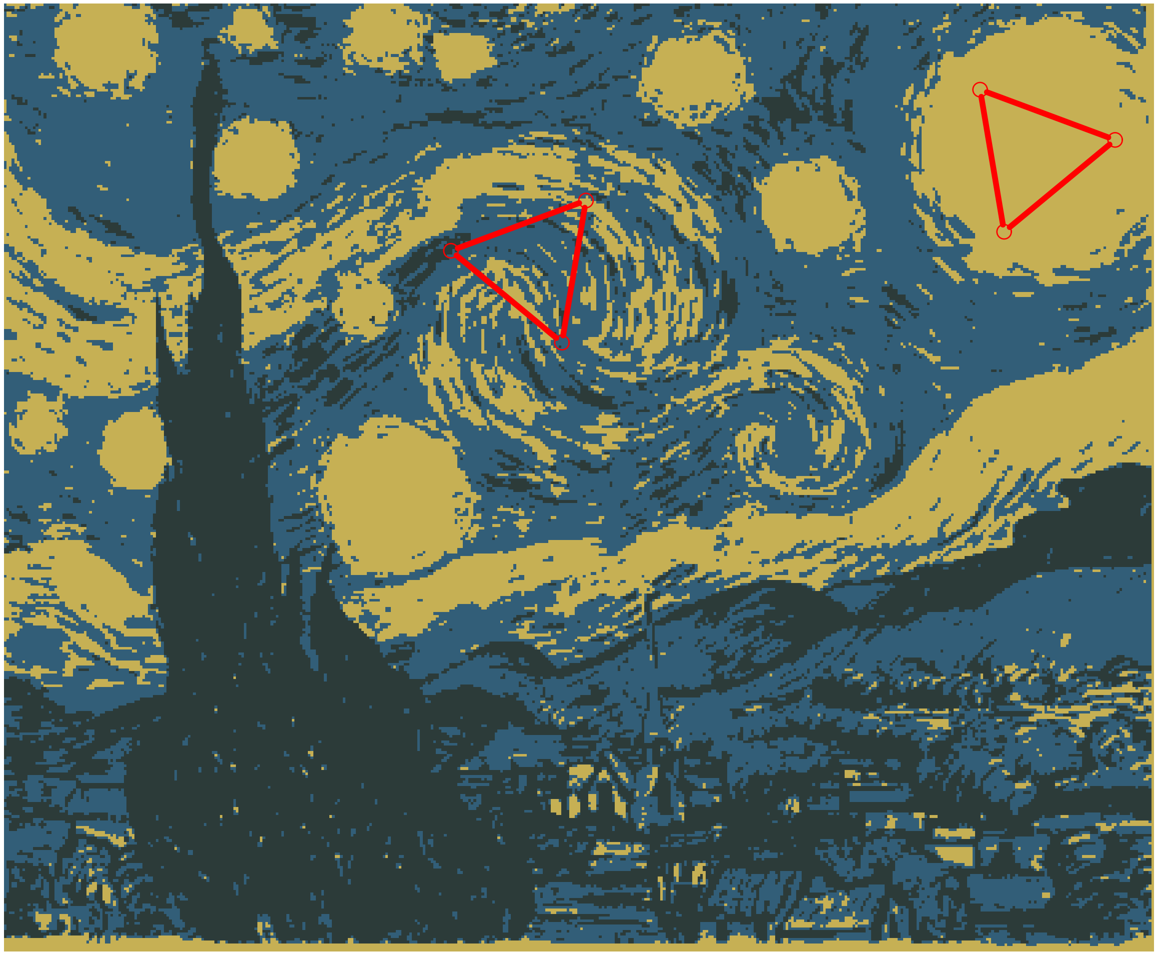


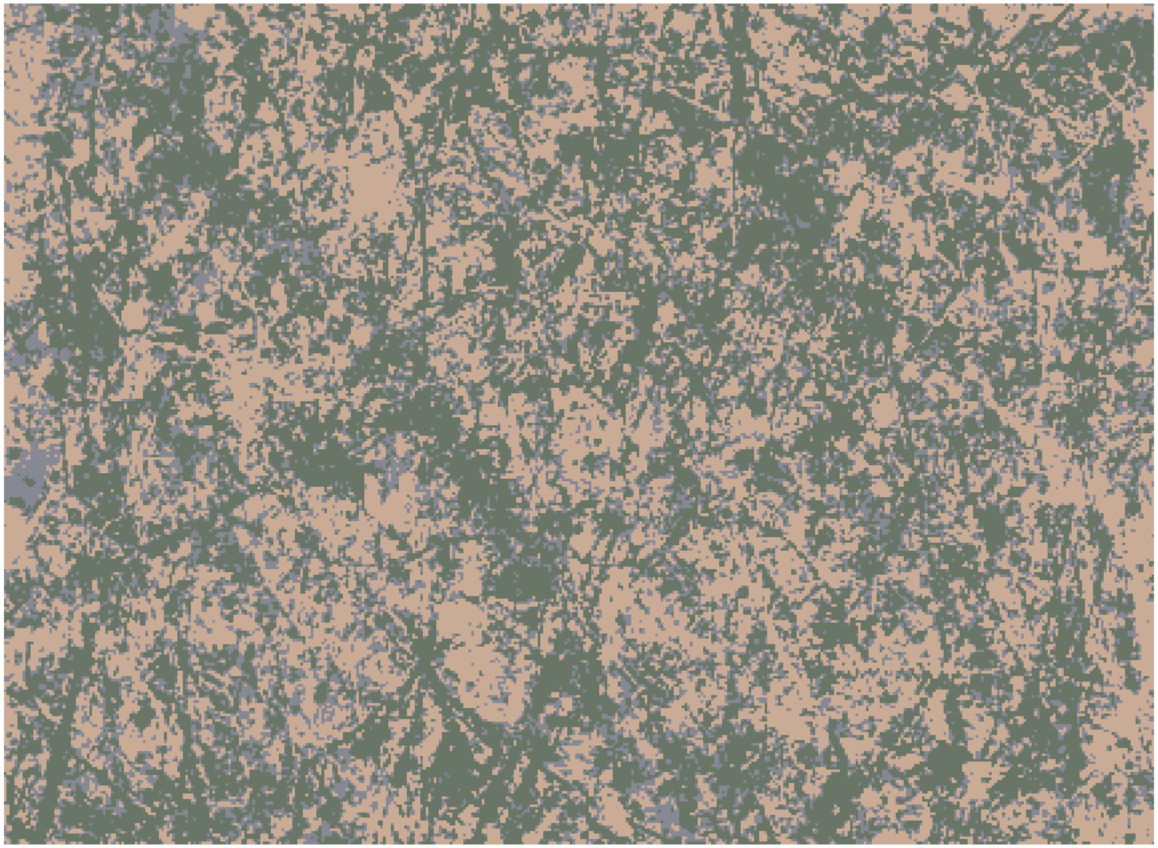
Starry night

OK; not OK

Pollock's attempt (1950)

Lavender mist

Pollock's attempt, three-colorized



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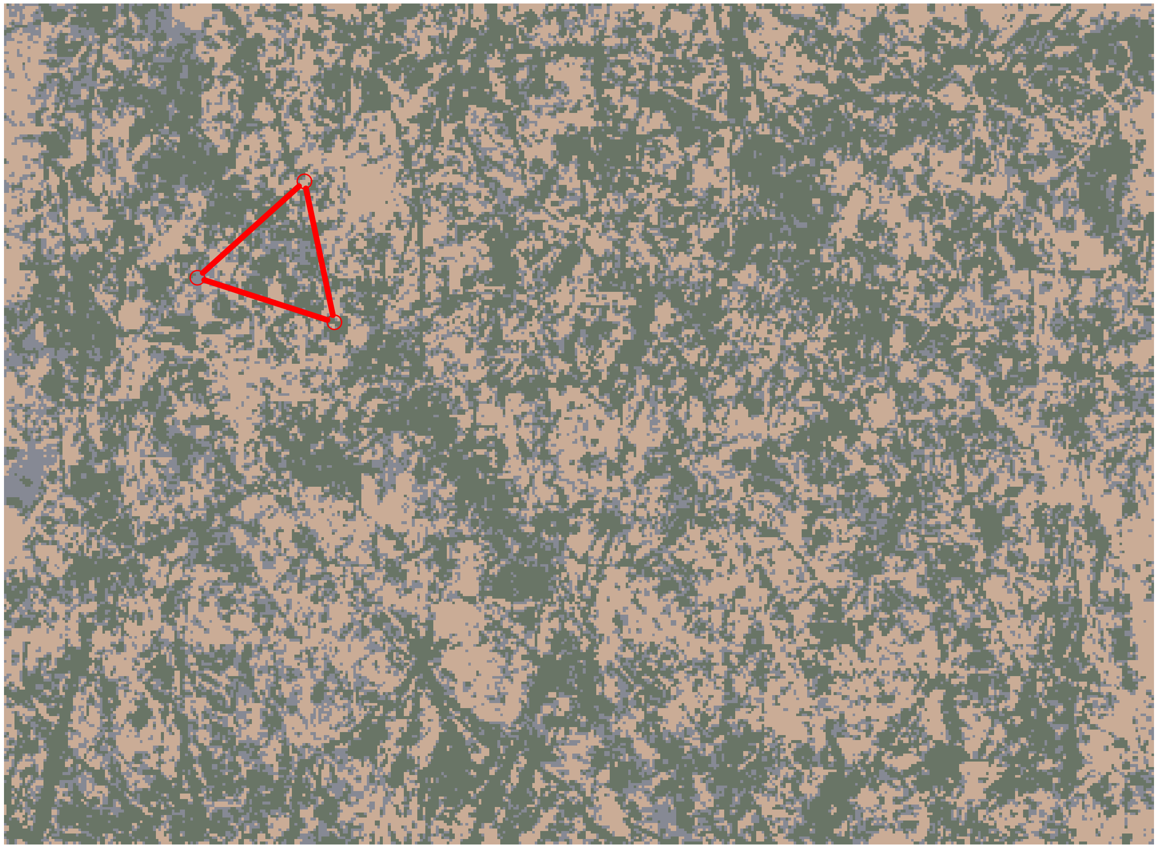
Pollock's attempt, three-colorized



Lavender mist

OK

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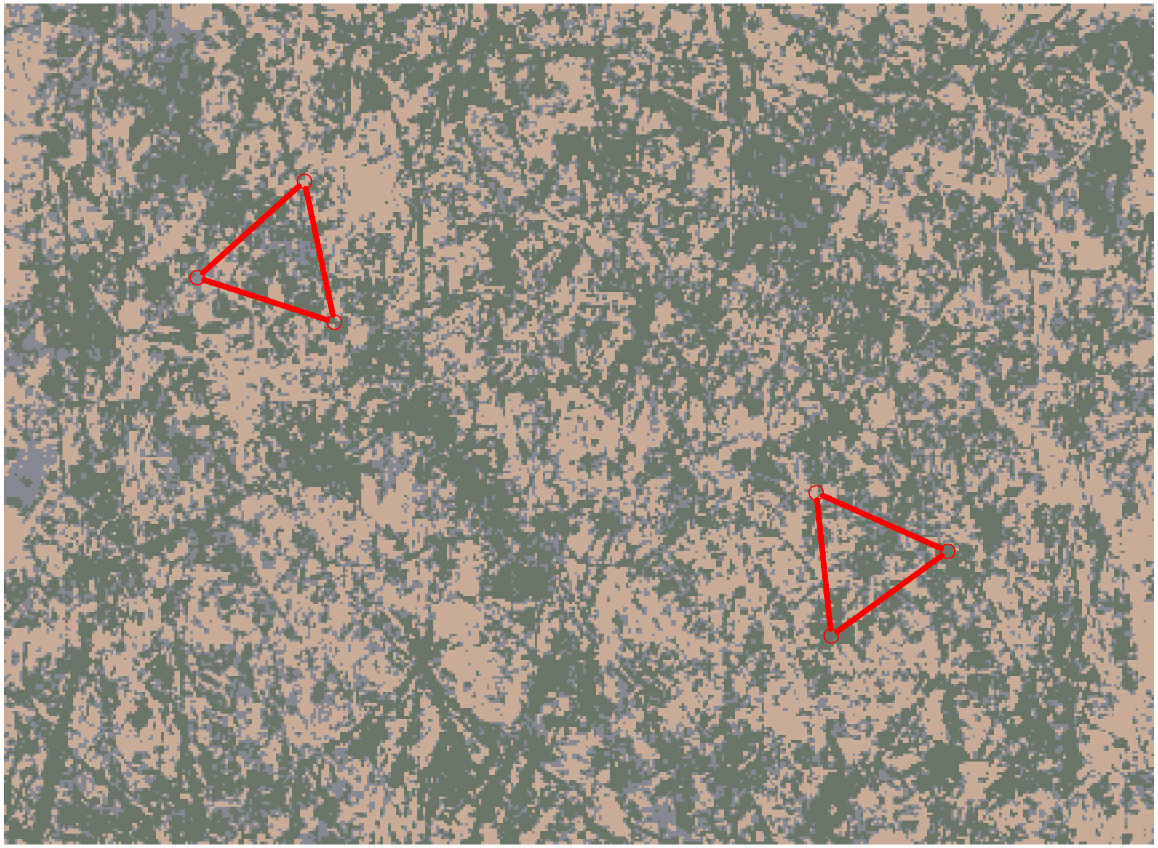

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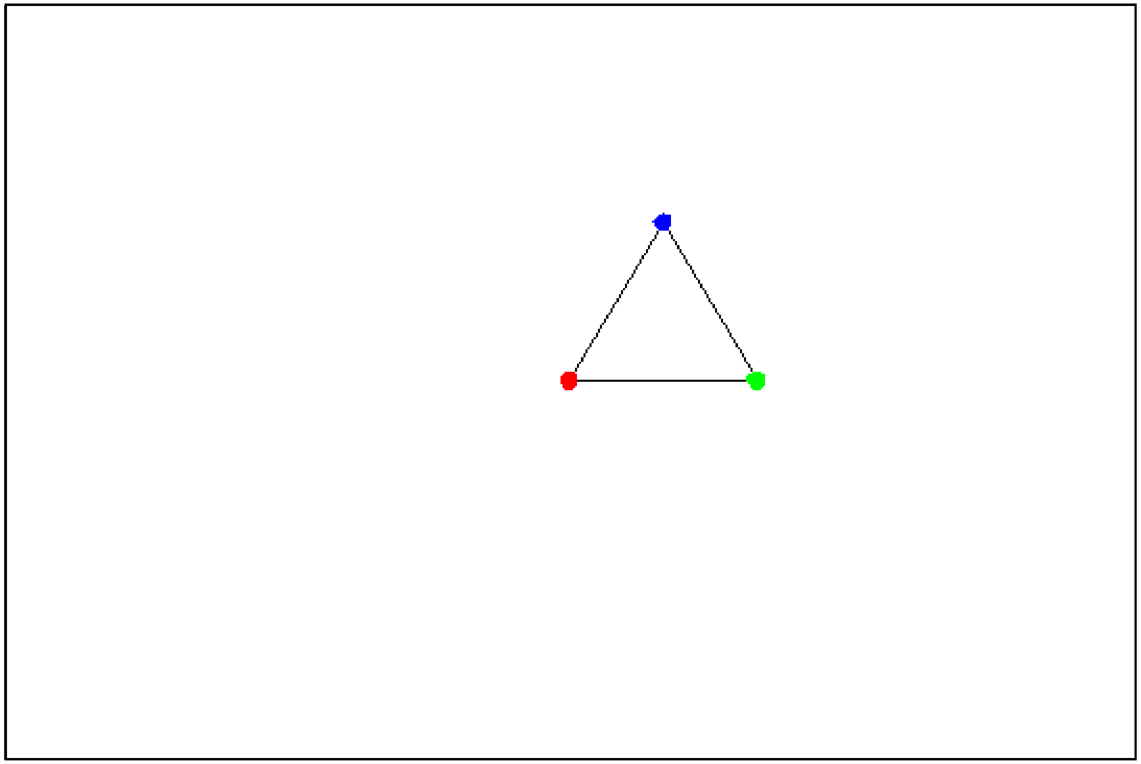
OK; not OK

Proof of impossibility

Suppose the points of the plane can be three-colored so that every equilateral triangle with sides of length one has one vertex of each color:



Then every point a distance 1 from a red point must be green or blue.

Proof of impossibility (cont.)

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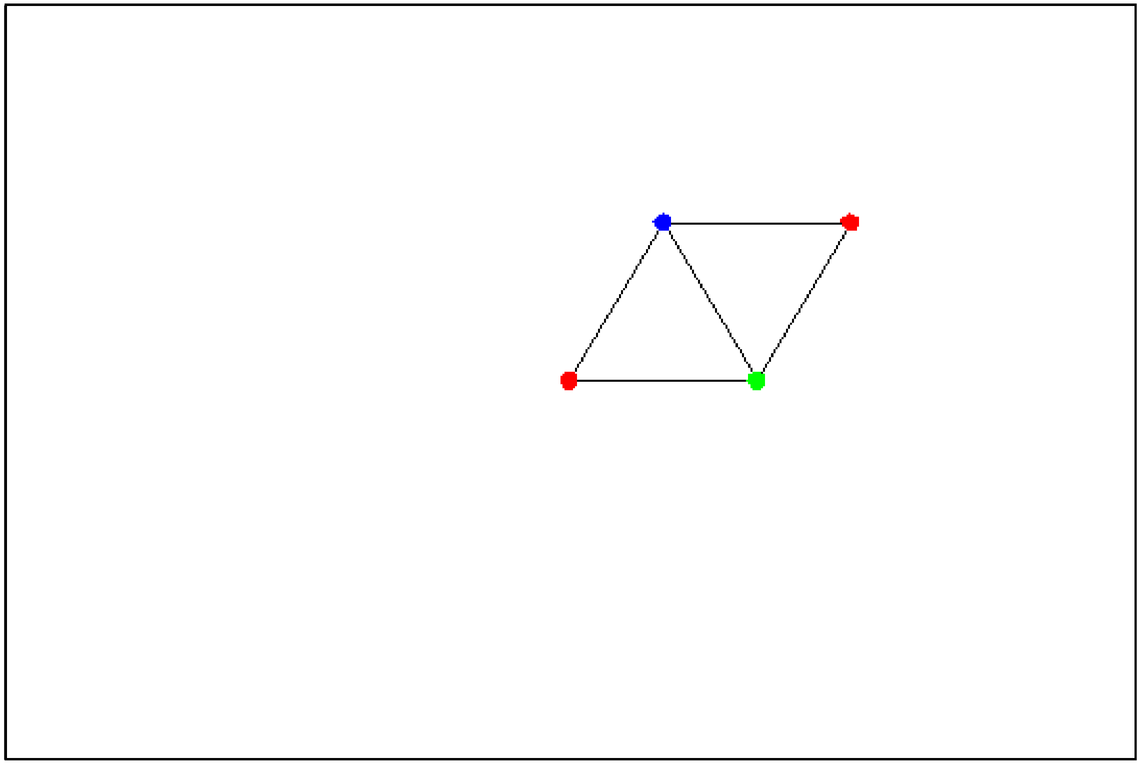
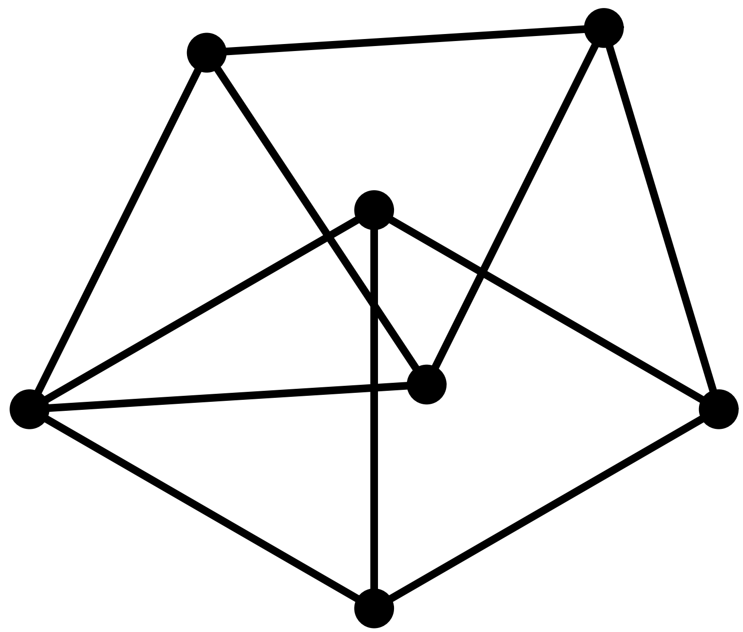


And every point a distance $\sqrt{3}$ from a red point must be red . . . which implies that there are two red points a distance 1 apart, which is a contradiction.

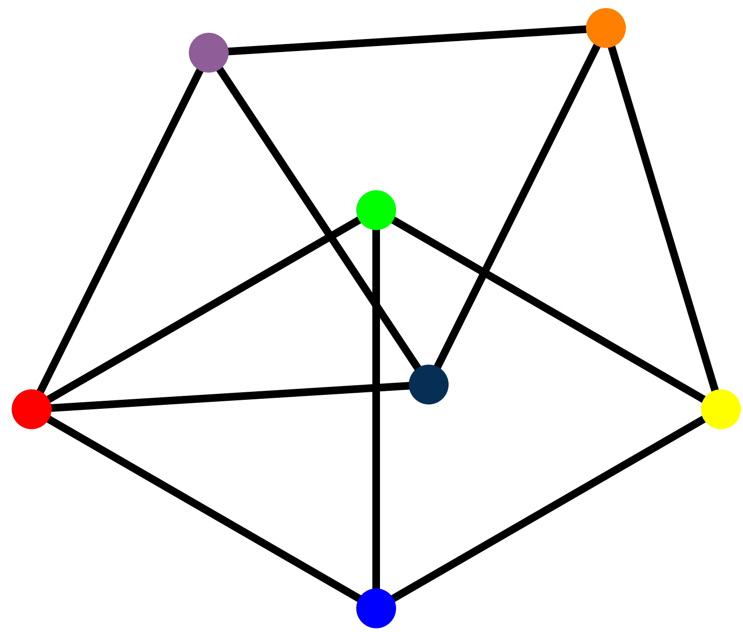
Combinatorial graphs

A graph G consists of a set V of vertices and set E of edges, each of which is a pair $\{u, v\}$, $u, v \in V$.



Digression: graph colorings

A k-coloring of a graph G is a map $f : V \to C$, where C is a set with $k \in \mathbb{N}$ elements (the colors), such that if $\{u, v\} \in E$ then $f(u) \neq f(v)$.



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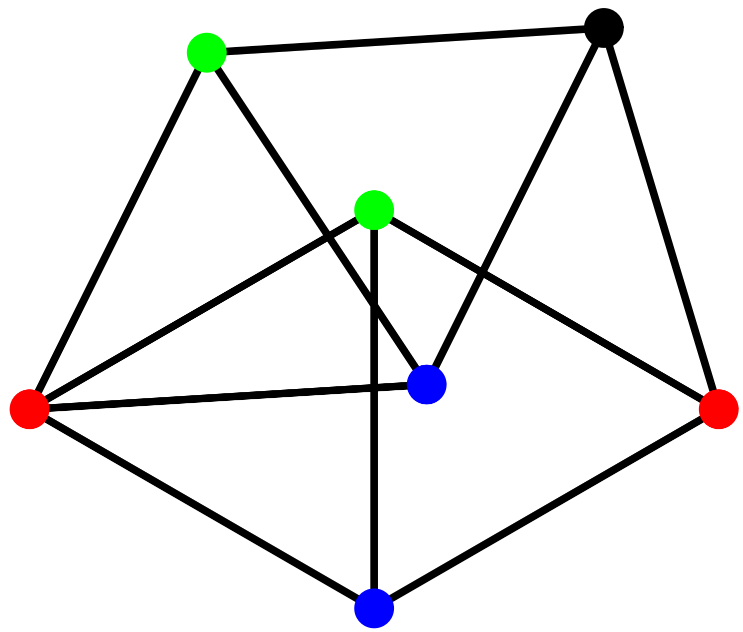
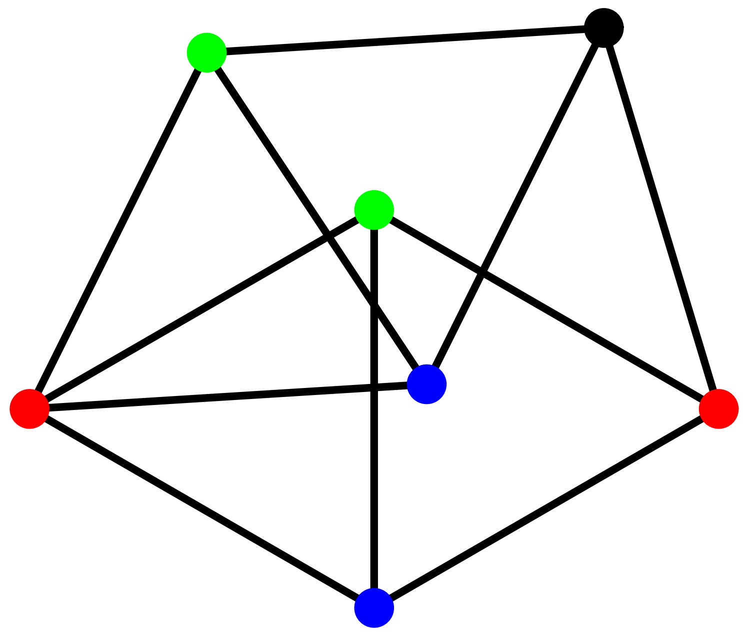


A 4-coloring.

Digression: chromatic number

The chromatic number, $\chi(G)$, of a graph G is the smallest $k \in \mathbb{N}$ such that G has a k-coloring.



This graph has chromatic number 4, because it has no 3-coloring.

Digression: the Hadwiger-Nelson problem (1961)

Let G be the infinite graph with all the points of the plane as vertices and edges $\{u, v\}$ for all pairs of points distance 1 apart. What is $\chi(G)$?

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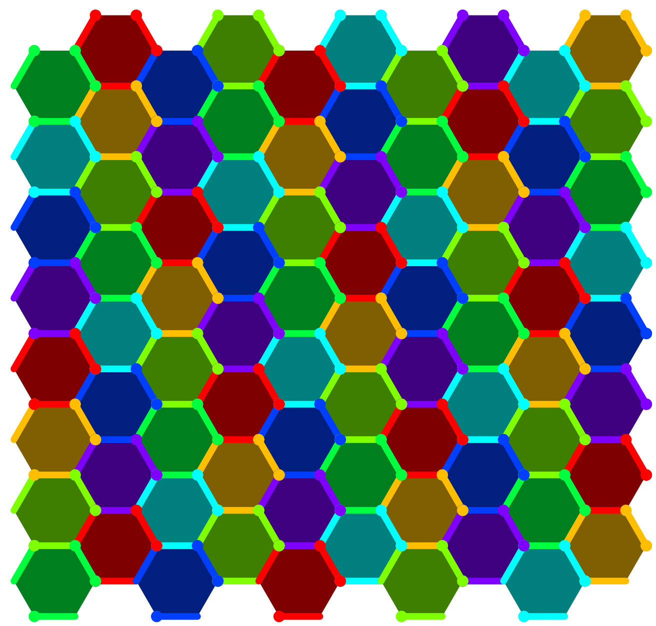
Isbell (1950): $\chi(G) \leq 7$.

> Proof by construction:
> (diameter 1 hexagons)



No progress since 1950!

Compactness

Rado (1949), Gottschalk (1951): Let C be a finite set and let M be an infinite set. Let $\mathcal{V}$ be the class of all finite subsets of M, and for each $V \in \mathcal{V}$, let $f_V : V \rightarrow C$. Then there exists $f : M \rightarrow C$ such that for all $V \in \mathcal{V}$ there is a $V \subset W \in \mathcal{V}$ such that $f(v) = f_W(v)$ for all $v \in V$.

For $V \in \mathcal{V}$, let F_V be the set of all $f \in X = \times_{v \in M} C$ such that there exists $V \subset W \in \mathcal{V}$ satisfying $f(v) = f_W(v)$, for all $v \in V$. For the discrete topology on C, Tychonoff's theorem implies X is compact. Since $\{F_V \mid V \in \mathcal{V}\}$ is a class of nonempty closed subsets of X, with the property that any finite number of them have nonempty intersection, there exists some $f \in \cap_{V \in \mathcal{V}} F_V$. ■

Erdös (1950), de Bruijn & Erdös (1951): In particular, if every finite subgraph of G is k-colorable, G is also k-colorable.

Application to Nelson's problem

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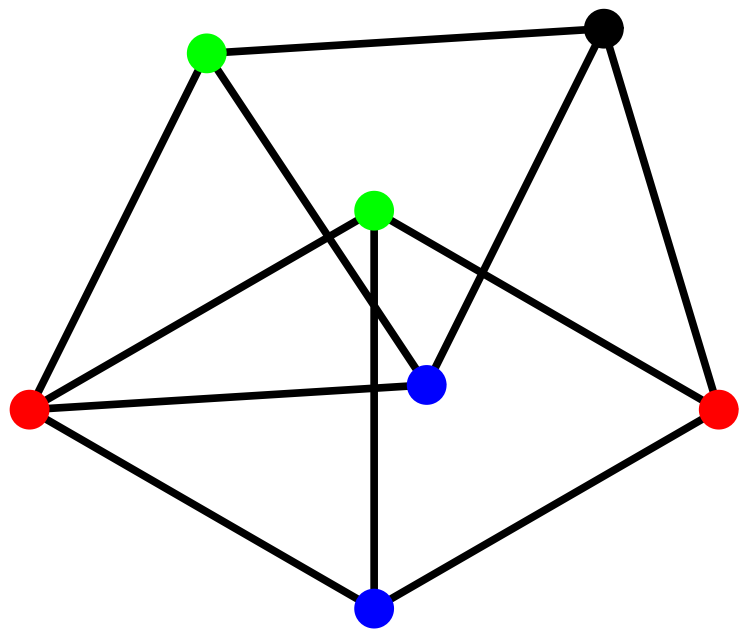

Moser & Moser (1961)

Quantum mechanics

Mathematical foundations of quantum mechanics
[Hilbert, von Neumann, Nordheim (1928), von Neumann (1932)]

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For a quantum system in state ψ, the outcome of a measurement corresponding to an orthonormal basis $\{e_i\}$ is probabilistic; it is the unit vector e_i with probability $|\langle \psi | e_i \rangle|^2$. (Since ψ is a unit vector, the probabilities sum to 1.)

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This can be (and is, in classical physics) understood as the outcome of the measurement on a specific system being deterministic, but the system being one in some ensemble of systems with outcome frequencies corresponding to the probability density.

That is, the outcome of a measurement is determined by some hidden variables that specify which system is being measured, and hence the outcome of any measurement.

Hidden variable models for quantum mechanics

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1. The hidden variable values describing the state of a system would specify the result of any measurement, *i.e.*, determine which element of any orthonormal basis is the outcome of the corresponding measurement.

2. The hidden variable values describing the state of a system would have a probability density that reproduces the quantum mechanical probabilities for the outcomes of each measurement.

Noncontextual hidden variable models

Gleason (1957), Bell (1966) and Kochen & Specker (1967) showed that hidden variable models with certain properties cannot exist.

The values of the hidden variables are required to determine whether each unit vector will or will not be the outcome of a measurement, if that unit vector is an element of the corresponding orthonormal basis. If this determination for a given unit vector is independent of the measurement of which it is a possible outcome, the hidden variables are called noncontextual.

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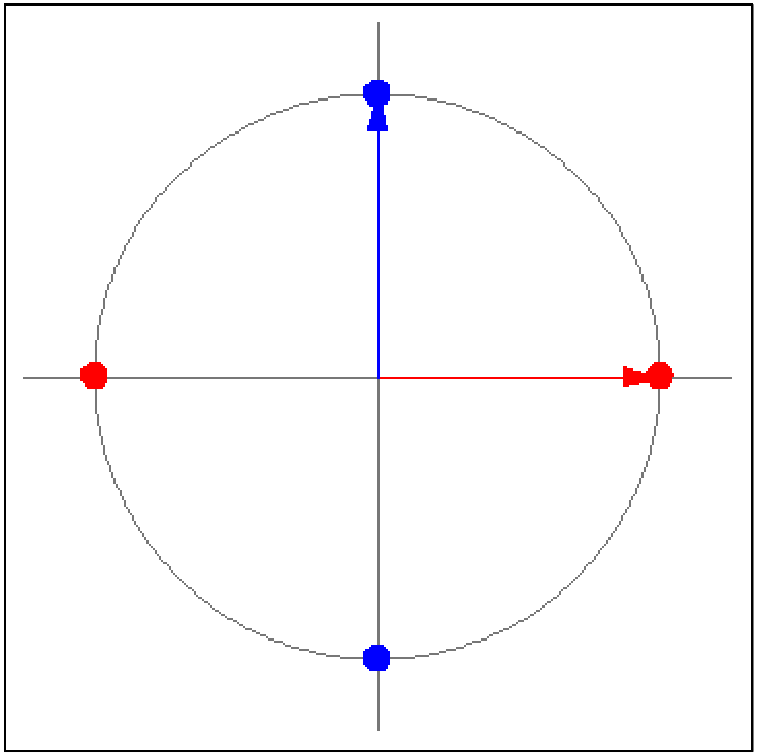
Thus a specific set of values for noncontextual hidden variables must determine whether each unit vector is or is not the outcome of every measurement containing it. That is, the unit vectors can be colored blue or red in such a way that exactly one element of each orthonormal basis is blue.

Coloring in two dimensions

In two dimensions, it is easy to color the unit vectors so that exactly one element of each orthonormal basis is blue and the other is red:



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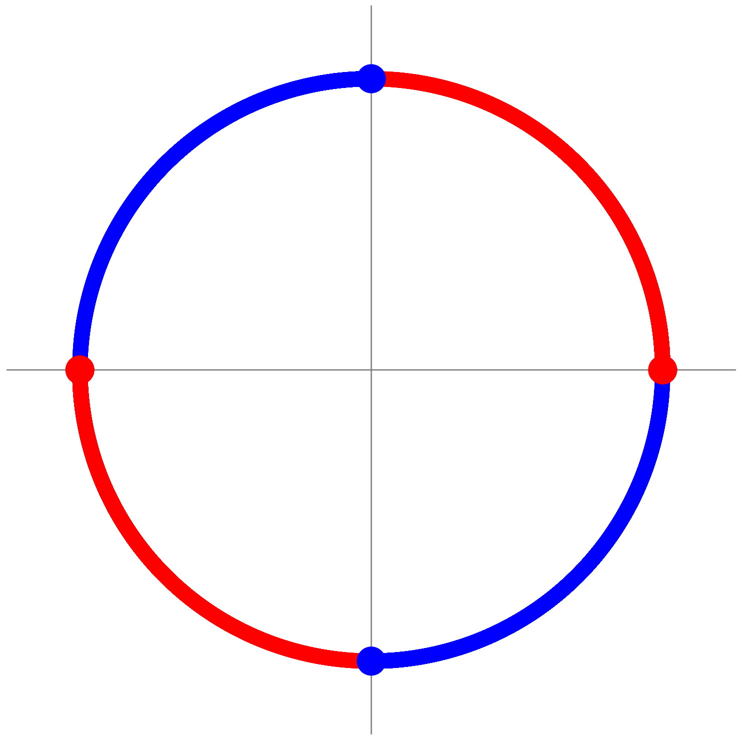
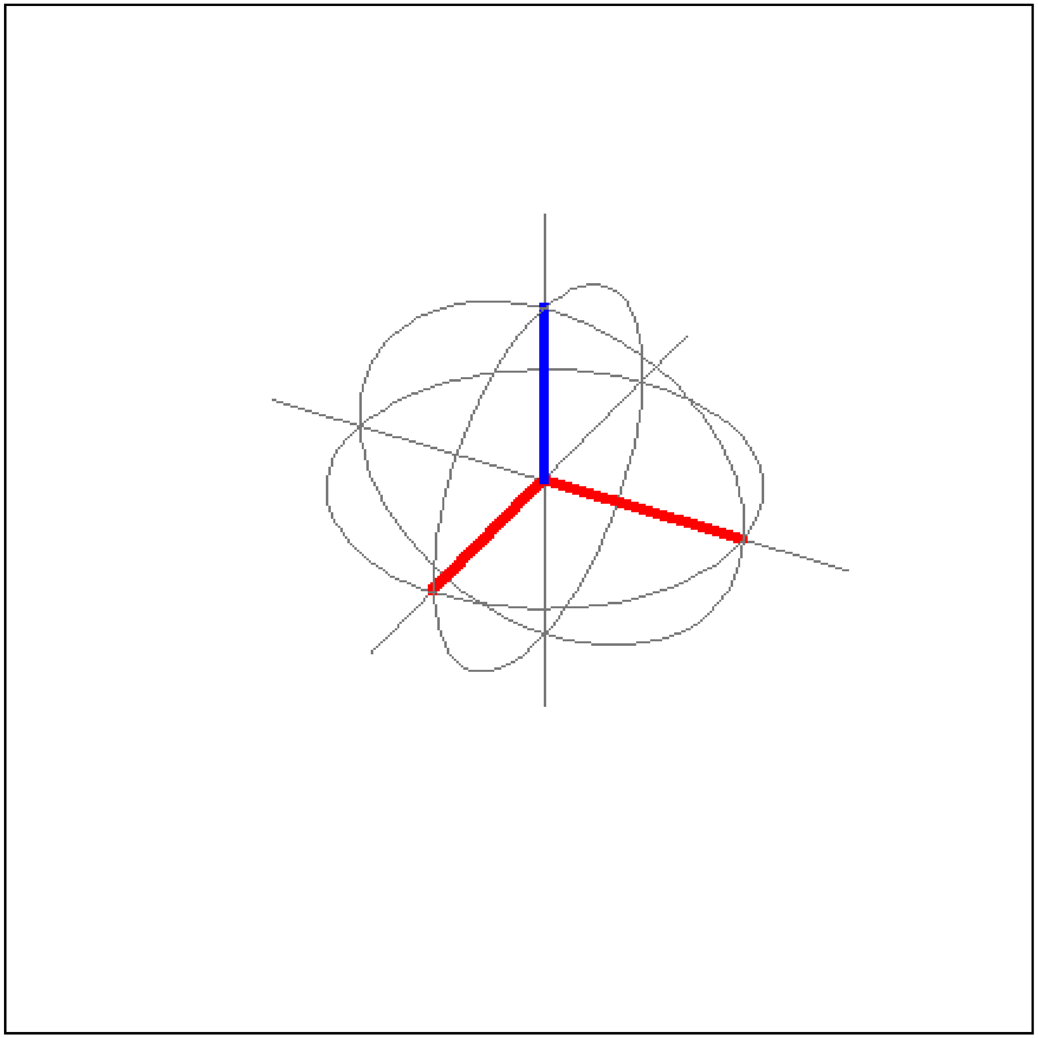


This is only one of many possible colorings; a noncontextual hidden variable model would also require an appropriate probability density on such colorings, but this can be constructed [Bell (1966)].

Coloring in three dimensions

But in three dimensions, it is impossible [Gleason (1957), Bell (1966), Kochen & Specker (1967)]:



If the green angle between the blue $\hat{z}$ and the red unit vector in the y-z plane is less than $\arctan(\frac{1}{2})$, the two orthogonal vectors $(\hat{x}-\hat{z})/\sqrt{2}$ and $-(\hat{x}+\hat{z})/\sqrt{2}$ (shown in magenta) must be red, and hence $\hat{y}$ must be blue, a contradiction.

Coloring in three dimensions (cont.)

But any such coloring of the unit vectors in three dimensions must have a red vector within an arbitrarily small angle of a blue vector. So there can be no such coloring, and hence no noncontextual hidden variable model for quantum systems described by vector spaces of dimension at least three.

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Kochen & Specker (1967) found a set of 117 unit vectors in three dimensions that cannot be colored this way. (This set is related to the not 3-colorable graph we found for the plane.)

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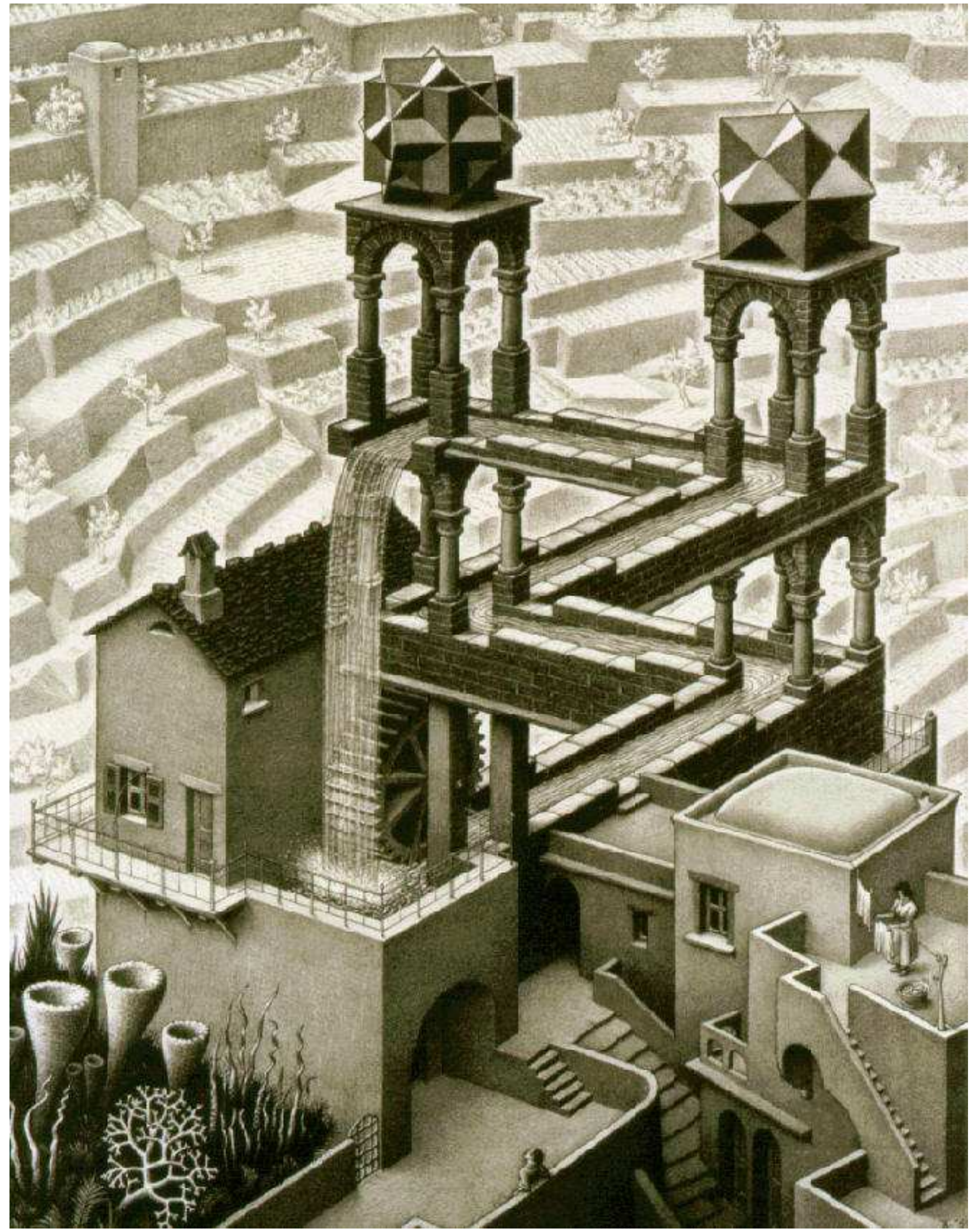
Peres (1991) found a 'nicer' set of 33 unit vectors in three dimensions that cannot be colored this way.

Penrose (1992?) noticed that this set of vectors was discovered much earlier.

Escher's discovery (1961)

Consider the 3 interpenetrating cubes on the top of the left pillar. Each cube has 4 lines from the mutual center to its vertices, 6 lines to the centers of its edges, and 3 lines to the centers of its faces. Three of the lines are shared by all three cubes, giving $3 \times (4+6+3) - 6 = 33$ lines. These are Peres' vectors.



Waterfall

Irrationality

Notice that some of Peres' unit vectors have irrational coordinates: For example, if we choose as coordinate axes the three lines that are shared by all three cubes, the unit vectors in the direction of the vertices of the cube resting on a face are $(\pm 1, \pm 1, \pm 1)/\sqrt{3}$.

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The non-repeating decimal expansion of irrational numbers raises the issue of finite precision *versus* infinite precision; results from computational complexity suggest one should be wary of models that incorporate the latter.

Finite precision measurement

Furthermore, making a measurement corresponding exactly to a specific orthonormal basis would require aligning an experimental apparatus with infinite precision.

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Since this seems likely to be difficult, perhaps a putative noncontextual hidden variable model need not assign outcomes to every possible unit vector, but only to unit vectors in a dense subset.

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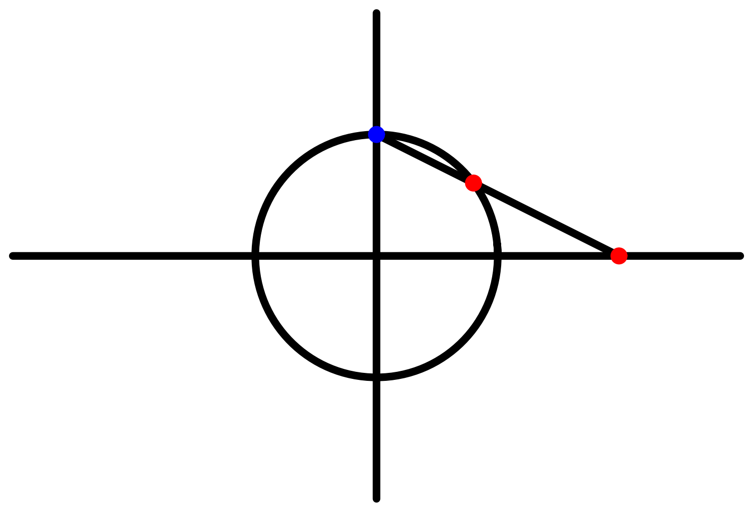
Completing the rationals to include the irrationals requires that

> *we transcend the proximably observable facts and . . . introduce ideal elements into the description of physical systems* [Jauch (1968)]

So let's not!

Density

The rational unit vectors are dense in S^2 since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and rational vectors in S^2 map bijectively to rational points in affine $\mathbb{R}^2$ (*i.e.*, $(x/z, y/z)$ for $x, y, z \in \mathbb{Z}$ and gcd$(x, y, z) = 1$)—*via* stereographic projection:



Coloring the rational unit vectors

[Hales & Straus (1982), Godsil & Zaks (1988)]

Rational directions in three dimensions are defined by vectors (x, y, z), where $x, y, z \in \mathbb{Z}$ and gcd$(x, y, z) = 1$. A rational unit vector points in this direction if and only if $x^2 + y^2 + z^2 = r^2$ for some $r \in \mathbb{Z}$.

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If z is odd, color the unit vector blue, otherwise color it red. (Notice that we could 3-color the unit vectors if we wanted to.)

Coloring the rational unit vectors (cont.)

[Hales & Straus (1982), Godsil & Zaks (1988)]

To check that exactly one element of each orthonormal basis is colored blue, consider two elements, and notice that the angle between their directions (x, y, z) and (x', y', z') is $\pi/2$, *i.e.*,

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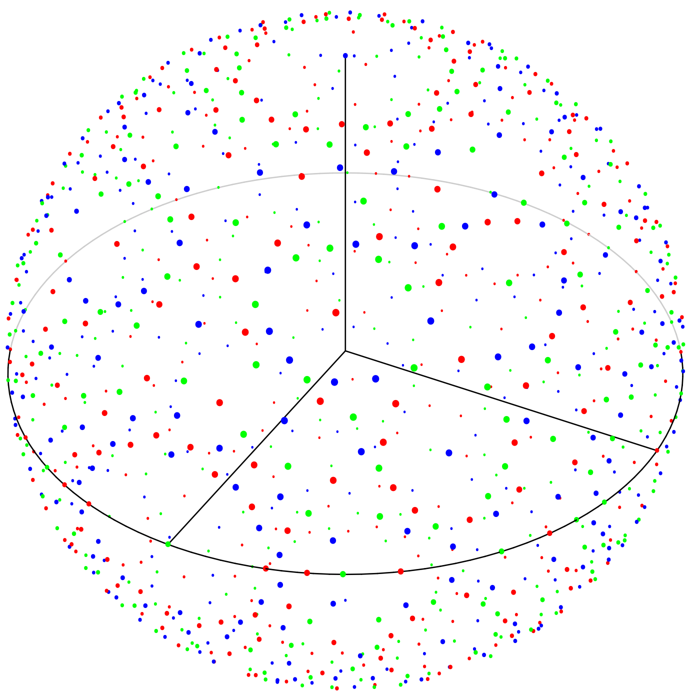
Thus only one of the three elements in each orthonormal basis has an odd z-component and is therefore colored blue.

Coloring the rational unit vectors (cont.)



and Euclid

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But Kent (1999) and Clifton & Kent (2000) were inspired by this argument to construct a complete noncontextual hidden variable model.

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What these models show is that there is a previously unstated additional condition on the noncontextual hidden variable models ruled out by Bell and Kochen-Specker, namely that probabilities are continuous as a function of measurement basis [Mermin (1999)].

Peres' comment

Meyer's claim that "finite precision measurement nullifies the Kochen-Specker theorem" (that is, makes it irrelevant to physics) and some of its generalizations have caused considerable controversy that lasts until today. Meyer's proposal was to replace the set of all directions in space by the dense subset of rational directions, arguing that a finite precision measurement cannot decide whether or not a number is rational.

Let us apply the same argument to ordinary geometry and consider only points with rational coordinates. Then the line $x = y$ *and the unit circle* $x^2 + y^2 = 1$ *are both dense but they do not intersect, in contradiction to Euclid's postulates.* [Peres (2003)]

What about Euclid?

Euclid actually omits the postulate that is necessary to ensure that the line and and the circle intersect. In fact, the proof of Proposition 1 in Book I is (relatively) well-known to assume the existence of the intersection of two circles without ever having stated explicitly the necessary postulate!

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Actually, Euclid wasn't even so comfortable with the idea that irrationals might be numbers.

The most Euclid seems likely to want to allow is constructible numbers. Amusingly, the components of the unit vectors in the non-colorable sets of Kochen-Specker and Peres seem to be constructible.

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Recent attempts to construct quantum theories of gravity hint that spacetime may be discrete, and that the set of possible directions may not be continuous [Major (1999)].

Further reading

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