Playing "20 questions" with a quantum computer

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Introduction

Computers which exploit quantum principles in their logic may be able to solve certain problems more efficiently than is possible classically.

R. P. Feynman, *International Journal of Theoretical Physics* **21** (1982) 467.
D. Deutsch, *Proceedings of the Royal Society of London A* **400** (1985) 97.
P. W. Shor, *SIAM Journal of Computing* **26** (1997) 1484.

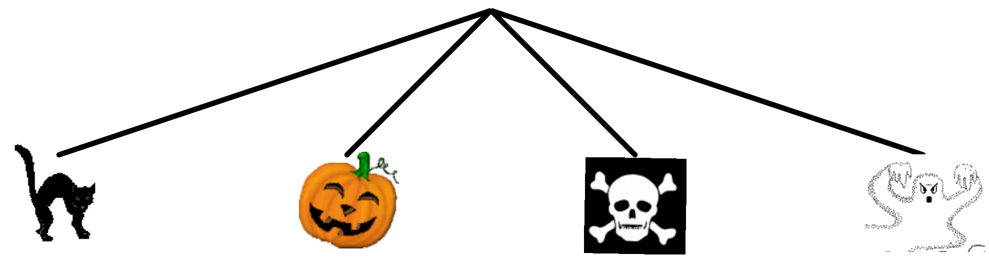
Outline

- example: the game of "20 questions"
- optimal classical strategies
- playing "20 questions" with a classical computer
- principles of quantum computing
- playing "20 questions" with a quantum computer

"20 questions"

Player 1 tries to determine what Player 2 is thinking by asking questions.

"Animal, vegetable or mineral?":



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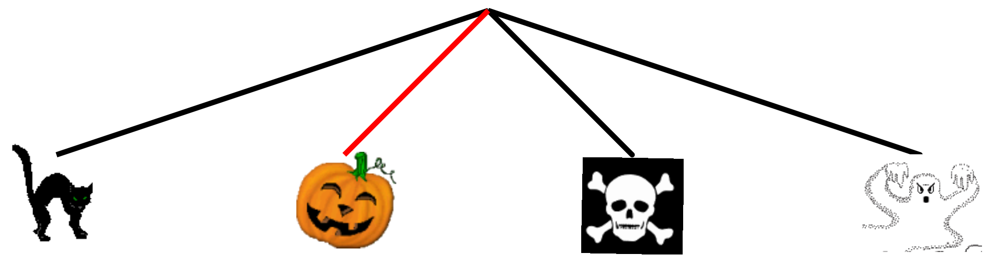

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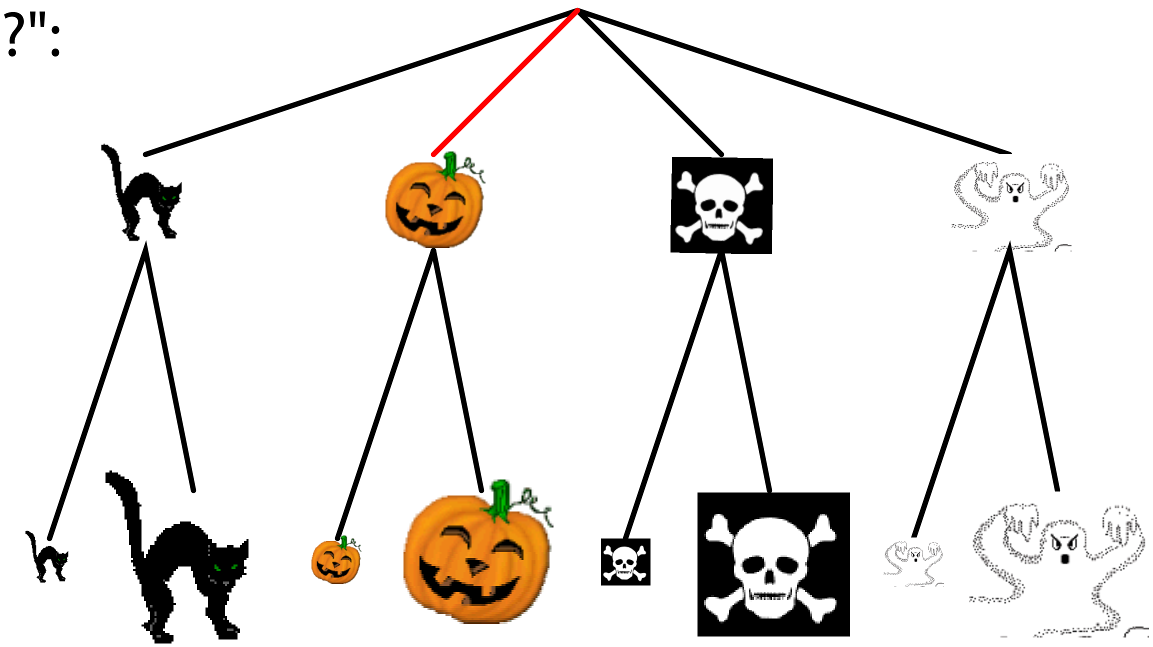
Player 2 responds:

Then **YES/NO** questions like,
"Bigger than a breadbox?":



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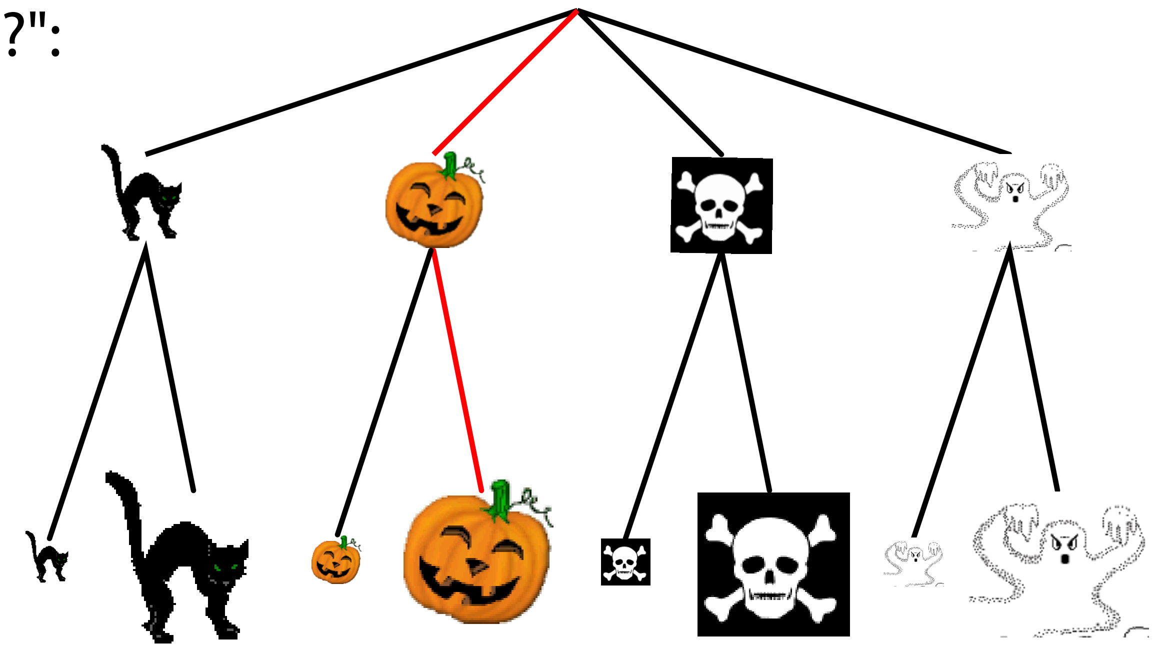

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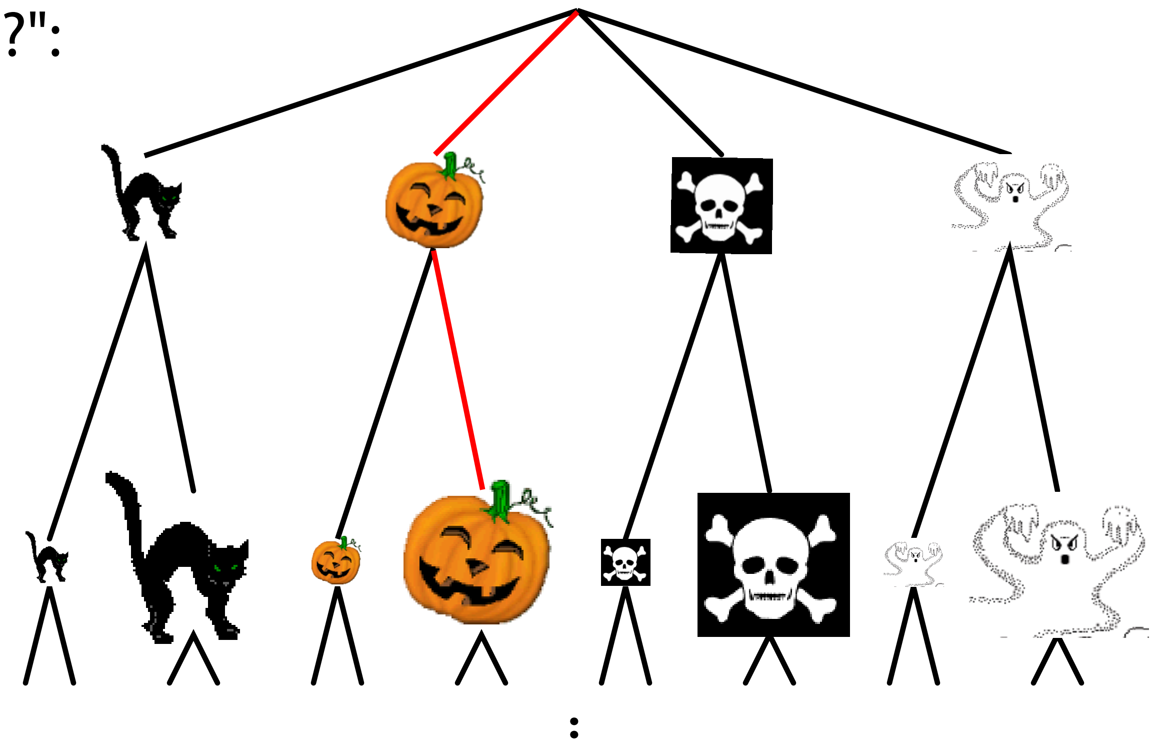

Player 2 responds:

Then **YES/NO** questions like,
"Bigger than a breadbox?":

Player 2 responds:

...

Player 1 wins if s/he uses 20 or fewer questions to determine what Player 2 is thinking.

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Search problems

So problem is to identify one out of $N = 2^n$ things w/ fewest questions.

Optimal strategy: each question should maximize amount of information received, independently of response.

Formalize this by defining:

$$\text{entropy } S = \text{lack of information}$$
$$= -\sum_{\text{possibilities } x} \text{prob}(x) \log \text{prob}(x)$$

Initially,

$$S_0 = -N\,(1/N) \log(1/N)$$
$$= -\log(1/N) = \log N\,,$$

assuming uniform distribution of Player 2's choices.

Optimal YES/NO questions

Reduce entropy as much as possible w/ each question.

Suppose fraction α of possibilities get response **YES**; then after question:

$$S_1 = -\alpha \sum_{\alpha N \text{ possibilities}} [1/\alpha N] \log[1/\alpha N] - (1-\alpha) \sum_{(1-\alpha)N \text{ possibilities}} [1/(1-\alpha)N] \log[1/(1-\alpha)N]$$

$$= \alpha \log \alpha N + (1-\alpha) \log (1-\alpha)N.$$

$$\frac{dS_1}{d\alpha} = \log \alpha N + \alpha[1/\alpha N]N - \log (1-\alpha)N + (1-\alpha)[1/(1-\alpha)N](-N) = 0,$$

which implies extremum at $\alpha = 1/2$, actually a minimum since at this value of α,

$$S_1 = (1/2)\log(N/2) + (1/2)\log(N/2) = \log(N/2) < \log N.$$

So optimal question partitions the possibilities evenly.

Optimal classical strategy

Similarly, questions with multiple responses (*e.g.*, "Animal, vegetable, or mineral?") ideally should partition possibilities evenly (assuming uniform prior distribution).

So optimal reduction of entropy is:

$$\log N \xrightarrow{Q_1} \log(N/4) \xrightarrow{Q_2} \log(N/8) \xrightarrow{Q_3} \ldots \xrightarrow{Q_{n-1}} \log(N/2^n) = 0.$$

If all questions were allowed to have 4 responses, optimal questions would reduce entropy like:

$$\log N \xrightarrow{Q_1} \log(N/4) \xrightarrow{Q_2} \log(N/4^2) \xrightarrow{Q_3} \ldots \xrightarrow{Q_{n/2}} \log(N/4^{n/2}) = 0.$$

That is, in worst case Player 1 will require at least $\log_4 N = n/2$ questions.

Classical computer

Standardize by using numerical labels, written as *n*-bit strings, for possibilities.

Allow every question to have 4 possible responses:

Player 1 names an *n*-bit string *x*.

Player 2 responds w/ the Hamming distance from the answer *a* (the number of wrong bits), dist(*x*,*a*) mod 4.

x =	0	1	0	0	1	0	1	1	0	1
a =	0	0	1	1	1	0	0	0	0	1

0	1	1	1	0	0	1	1	0	0

dist(x,a) = 5

0	1

response is 5 mod 4 = 1

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Optimal solution must require at least $n/2$ queries; n queries is easy:

1	1	0	0	1	0	1	1	0	1

1	0

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dist(*x*,*a*) = 5 | 0 | 1 |

response is 5 mod 4 = 1

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| 1 | 1 |

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0	0

Invertible classical computing

Initialize query and response registers to:

0	0	0	0	0	0	0	0	0	0		0	0

Player 1 prepares query:

$\xrightarrow{+x}$	0	1	0	0	1	0	1	1	0	1		0	0

Player 2 responds:

$\xrightarrow{+\text{dist}(x,a)}$	0	1	0	0	1	0	1	1	0	1		0	1

Each of these is an invertible operation, unlike *x* **AND** *y*, for example.

All classical computations can be done w/ (local) invertible operations.

C. H. Bennett, *IBM Journal of Research and Development* **17** (1973) 525.

Invertible computation *via* matrix multiplication

Possible states of computer are, say, $(n+2)$-bit strings; have 2^{n+2} of them.

Operations convert one bit string into another; can think of as matrix multiplication:

$$\underbrace{\overset{\begin{matrix}00 & 01 & 10 & 11\end{matrix}}{\begin{bmatrix}0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}}_{T}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{matrix}00\\01\\10\\11\end{matrix} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{matrix}00\\01\\10\\11\end{matrix}$$

Invertible operations are represented by permutation matrices, *i.e.*, those w/ exactly one 1 in each row and column.

Quantum evolution

Evolution in quantum mechanics is represented by multiplication by unitary matrices.

U is unitary if and only if $U(U^*)^{\mathrm{T}} = I = (U^*)^{\mathrm{T}}U$.

Notice that permutation matrices are unitary:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So evolution in quantum mechanics generalizes invertible classical evolution.

Quantum states in quantum computers

States of quantum systems are linear combinations of classical states.

So a single quantum bit (qubit) can be in any state:

$$a_0 \boxed{0} + a_1 \boxed{1}, \text{ where } |a_0|^2 + |a_1|^2 = 1.$$

The reason for the condition on the coefficients it that when the qubit is measured (relative to this basis), it is always either $\boxed{0}$ or $\boxed{1}$, with probabilities $|a_0|^2$ and $|a_1|^2$, respectively.

Two qubits, like the response register, can be in any state of the form:

$$a_{00} \boxed{0\,|\,0} + a_{01} \boxed{0\,|\,1} + a_{10} \boxed{1\,|\,0} + a_{11} \boxed{1\,|\,1},$$

where again the sum of the squared norms of the coefficients is 1.

Unitary operations in quantum computers

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \; X \boxed{0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{1}\,.$$

$$H = 2^{-1/2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \; H \boxed{0} = \left(\boxed{0} + \boxed{1}\right)/2^{1/2}\,.$$

$$T \boxed{0\,|\,0} = \boxed{0\,|\,1}\,.$$

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} / \, 2\,;$$

$$F \boxed{0\,|\,1} = \left(\boxed{0\,|\,0} + i \boxed{0\,|\,1} - \boxed{1\,|\,0} - i \boxed{1\,|\,1}\right)/2.$$

Tensor products of matrices

$$H \otimes I = \left[\begin{array}{c|c} I & I \\ \hline I & -I \end{array}\right] / 2^{1/2} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] / 2^{1/2}$$

$$H \otimes H = \left[\begin{array}{c|c} H & H \\ \hline H & -H \end{array}\right] / 2^{1/2} = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] / 2$$

A quantum algorithm (M. Hunziker and D. A. Meyer, UCSD preprint (2001).)

Initialize query and response registers to:

$$\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\quad\boxed{0}\boxed{0}$$

Player 1 prepares query:

$$\xrightarrow{H^{\otimes n}\otimes FT} (\boxed{0}+\boxed{1})\otimes\cdots\otimes(\boxed{0}+\boxed{1})/2^{n/2}$$

$$\otimes(\boxed{0}\boxed{0}+i\,\boxed{0}\boxed{1}-\boxed{1}\boxed{0}-i\,\boxed{1}\boxed{1})/2$$

$$=(\text{all } n\text{-bit strings } \boxed{\mathbf{x}})/2^{n/2}$$

$$\otimes(\boxed{0}\boxed{0}+i\,\boxed{0}\boxed{1}-\boxed{1}\boxed{0}-i\,\boxed{1}\boxed{1})/2$$

Player 2 responds to the quantum query:

$$\xrightarrow{+\mathrm{dist}(x,a)} \sum(-i)^{\mathrm{dist}(x,a)}\,\boxed{\mathbf{x}}\,/\,2^{n/2}\otimes FT\,\boxed{0}\boxed{0}$$

Only 1 quantum question!

Define a new unitary matrix

$$G = \begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} / 2^{1/2},$$

which Player 1 uses to "interpret" the response:

$$\xrightarrow{G^{\otimes n} \otimes I_4} \sum (-\mathrm{i})^{\mathrm{dist}(x,a)} (\mathrm{i})^{\mathrm{dist}(y,x)} \boxed{\mathtt{y}} / 2^n \otimes FT \boxed{0}\boxed{0} = \boxed{\mathtt{a}} \otimes FT \boxed{0}\boxed{0}.$$

Here the sum is over x and y. The 2^n terms which have $y = a$ each contribute $1/2^n$; the rest exactly cancel.

Thus "20 questions" becomes "1 question" with a quantum computer.