THE RUBIN–STARK CONJECTURE FOR IMAGINARY ABELIAN FIELDS OF ODD PRIME POWER CONDUCTOR

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Abstract. We build upon ideas developed in [9], as well as results of Greither on a strong form of Brumer's Conjecture ([2]–[4]), and prove Rubin's integral version of Stark's Conjecture for imaginary abelian extensions of $\mathbf{Q}$ of odd prime power conductor.

INTRODUCTION

In the present paper, we prove Rubin's integral version (Conjecture B, [10], §2.1) of Stark's general conjecture (see Conjecture 5.1 in [12], Chap. I), for abelian extensions of type $K/\mathbf{Q}$, where K is an abelian, imaginary number field of odd prime power conductor ℓ^n. These extensions belong to the class of "nice" CM extensions of totally real number fields, introduced by Greither in [4]. In [9], we used results obtained by Greither in [4] on the "odd part" of a strong form of Brumer's Conjecture, and settled Rubin's Conjecture up to a power of 2, for general "nice" extensions. In this paper, we restrict ourselves to the subclass of "nice" extensions $K/\mathbf{Q}$ described above and, by using ideas developed in [9], as well as results of Greither ([2] and [3]) on the 2–part of the strong Brumer conjecture, we completely settle Rubin's Conjecture in this case.

Unlike in [9], where we attack the "odd" part of Rubin's Conjecture "one prime at a time", all the arguments in this paper are global in nature. This is made possible by the crucial observation that, for general abelian CM extensions of totally real number fields K/k, one can state a conjecture equivalent to Rubin's, over the ring $\mathbf{Z}[\mathrm{Gal}(K/k)]/(1+j)$, rather than $\mathbf{Z}[\mathrm{Gal}(K/k)]$ itself (see Conjecture B_- in §2 below), where j is the complex conjugation automorphism of K. As it turns out, for $K/\mathbf{Q}$ as above, many arithmetically meaningful $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$ –modules which, in general, are not of finite projective dimension over $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]$, have projective dimension at most 1 over $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]/(1+j)$, after tensoring with $\mathbf{Z}[\mathrm{Gal}(K/\mathbf{Q})]/(1+j)$. We take full advantage of this fact and prove Conjecture B_- in the context described above, and conclude that the equivalent Conjecture B of Rubin is also true.

The paper is organized as follows. In §1, we introduce the notations and main definitions, state Rubin's Conjecture B and list some of its functoriality properties. In §2, we state the equivalent Conjecture B_- for general abelian CM extensions of totally real number fields. In §3, we describe results of Greither on the strong Brumer Conjecture which are of relevance in this context, and draw several conclusions which are used in the subsequent sections. In §4, we prove Conjecture B_- for

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extensions $K/\mathbf{Q}$ as described above, under certain additional hypotheses. In §5, we use the results of §4 and some functoriality properties stated in §1 to prove Rubin's Conjecture B for the class of extensions $K/\mathbf{Q}$ under consideration.

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1. PRELIMINARY CONSIDERATIONS

1.1 Notation. In what follows, $\mathbf{Z}_{(p)}$ will denote the localization of $\mathbf{Z}$ at its prime ideal $p\mathbf{Z}$, for all prime numbers p. Let us assume that M is an arbitrary $\mathbf{Z}$–module. Then $\widetilde{M}$, will denote the quotient of M by its submodule of $\mathbf{Z}$–torsion points. If A is an arbitrary (commutative) ring, $AM := A \otimes_{\mathbf{Z}} M$. In particular, for any prime number p, $M_{(p)} := \mathbf{Z}_{(p)}M$. If G is a finite, abelian group, then $\widehat{G}$ denotes the set of its irreducible, complex–valued characters. For any $\chi \in \widehat{G}$, $e_\chi := 1/|G| \sum_{\sigma\in G} \chi(\sigma^{-1}) \cdot \sigma$ is the idempotent associated to χ in the group–ring $\mathbf{C}[G]$. Also, if M is a $\mathbf{Z}[G]$–module, $\wedge^r_G M := \wedge^r_{\mathbf{Z}[G]} M$.

Let us assume that K/k is a finite, abelian extension of number fields, of Galois group $G := \mathrm{Gal}(K/k)$. Let S be a finite set of primes in k, which contains the infinite primes as well as the ones which ramify in K/k, and let S_K be the set of primes in K, sitting above primes in S. Then, U_S and A_S will denote respectively the group of units and the ideal–class group of the ring of S_K–integers O_S of K. If T is an additional finite, non–empty set of primes in k, disjoint from S, then $U_{S,T}$ denotes the subgroup of U_S consisting of S–units congruent to 1 modulo every prime in T_K, and $A_{S,T}$ denotes the (S,T)–ideal-class group of K, as defined in [10],§1.1. For a finite prime w in K, we denote by $K(w)$ its residue field. Then, as pointed out in [10], §1.1, we have an exact–sequence of $\mathbf{Z}[G]$–modules

$$(0) \qquad 0 \longrightarrow U_{S,T} \longrightarrow U_S \xrightarrow{\mathrm{res}_T} \Delta_T \longrightarrow A_{S,T} \longrightarrow A_S \longrightarrow 0\,,$$

where $\Delta_T := \oplus_{w\in T_K} K(w)^\times$, and $\mathrm{res}_T(x) := (x \bmod w;\ w \in T_K)$, for all $x \in U_S$. As in [10], to any S, T and χ as above, one can associate a meromorphic L–function $L_S(s,\chi)$ of complex variable s, holomorphic away from $s = 1$, and an everywhere holomorphic L–function $L_{S,T}(s,\chi) := \prod_{v\in T}(1 - \chi(\sigma_v^{-1})\mathrm{N}v^{1-s}) \cdot L_S(s,\chi)$, where $\mathrm{N}v = |k(v)|$, and σ_v is the Frobenius morphism associated to v in G, for all primes v in T. As shown in [12], Chapitre I, §3, the orders of vanishing at $s = 0$ of these L–functions, $r_{\chi,S} := \mathrm{ord}_{s=0}L_S(s,\chi) = \mathrm{ord}_{s=0}L_{S,T}(s,\chi)$, are given by

$$(1) \qquad r_{\chi,S} := \begin{cases} \mathrm{card}\{v \in S \mid \chi|_{G_v} = \mathbf{1}_{G_v}\}, \text{ if } \chi \neq \mathbf{1}_G \\ \mathrm{card}(S) - 1, \text{ if } \chi = \mathbf{1}_G \end{cases},$$

where G_v is the decomposition group of v relative to K/k. Now, we combine the L–functions above, to obtain the associated Stickelberger functions

$$\Theta_S(s) := \sum_{\chi\in\widehat{G}} L_S(s,\chi^{-1}) \cdot e_\chi\,, \quad \Theta_{S,T}(s) := \sum_{\chi\in\widehat{G}} L_{S,T}(s,\chi^{-1}) \cdot e_\chi\,.$$

These are viewed as $\mathbf{C}[G]$–valued, meromorphic functions, of complex variable s. For any $\mathbf{Z}[G]$–module M with trivial $\mathbf{Z}$–torsion and any positive integer r, we let

$$M_{r,S} := \{x \in M \mid e_\chi \cdot x = 0 \text{ in } \mathbf{C}M\,, \forall \chi \in \widehat{G}\,, \text{ such that } r_{\chi,S} \neq r\}\,.$$

1.2 Rubin's Conjecture. Let K/k, S and T be as in §1.1 and let r be a positive integer. Assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies the following group of extended hypotheses.

$$(\mathrm{H})\qquad \begin{cases} S \text{ contains all the infinite primes of } k. \\ S \text{ contains all the primes which ramify in } K/k. \\ S \text{ contains at least } r \text{ primes which split completely in } K/k. \\ |S| \geq r+1. \\ T \neq \emptyset\,, \quad S\cap T = \emptyset\,, \quad U_{S,T}\cap \mu_K = \{1\}\,. \end{cases}$$

Under these extra–hypohteses, equalities (1) above imply right away that $r_{\chi,S} \geq r$, for all $\chi \in \widehat{G}$. Let

$$\Theta_{S,T}^{(r)}(0) := \lim_{s\to 0} s^{-r}\Theta_{S,T}(s)\,,$$

viewed as an element in $\mathbf{C}\,[G]$. In this context, the main purpose of Stark's Conjecture and its integral version due to Rubin, is an interpretation of the (analytically defined) $\Theta_{S,T}^{(r)}(0)$ in terms of the arithmetic of K/k.

For every r–tuple $(\phi_1,\dots,\phi_r) \in \mathrm{Hom}_{\mathbf{Z}[G]}\left(U_{S,T}\,,\, \mathbf{Z}\,[G]\right)^r$, one can view each ϕ_i as an element of $\mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}U_{S,T}, \mathbf{C}\,[G])$, and define a $\mathbf{C}\,[G]$–morphism

$$\mathbf{C}\wedge_G^r U_{S,T} \xrightarrow{\phi_1\wedge\dots\wedge\phi_r} \mathbf{C}\,[G]\ ,$$

such that, for every $u_1,\dots,u_r \in U_{S,T}$, one has

$$\phi_1\wedge\dots\wedge\phi_r(u_1\wedge\dots\wedge u_r) := \det_{1\leq i,j\leq r}\left(\phi_i(u_j)\right)\ .$$

Definition 1.2.1. *Assuming that* $(K/k,\, S,\, T,\, r)$ *satisfies hypotheses (H), let* $\Lambda_{S,T}$ *be the* $\mathbf{Z}\,[G]$*–submodule of* $\mathbf{Q}\wedge_G^r U_{S,T}$*, defined by*

$$\Lambda_{S,T} = \begin{cases} \{\varepsilon \in \left(\mathbf{Q}\wedge_G^r U_{S,T}\right)_{r,S} \mid (\phi_1\wedge\dots\wedge\phi_r)(\varepsilon) \in \mathbf{Z}\,[G]\ , \\ \qquad\qquad \forall \phi_1,\dots,\phi_r \in \mathrm{Hom}_{\mathbf{Z}[G]}\left(U_{S,T}\,,\,\mathbf{Z}\,[G]\right)\},\ \textit{if } r\geq 1\,, \\ \mathbf{Z}\,[G]_{0,S},\ \textit{if } r=0\,. \end{cases}$$

If $r \geq 1$, let us fix an r–tuple $V := (v_1, v_2, \dots, v_r)$ of r–distinct primes in S, which split completely in K/k, and let $W := (w_1, w_2, \dots, w_r)$, with w_i prime in K, sitting above v_i, for all $i = 1,\dots,r$. We associate to every w_i an element $\lambda_{w_i} \in \mathrm{Hom}_{\mathbf{C}[G]}\left(U_{S,T}\,,\,\mathbf{C}\,[G]\right)$, defined as

$$\lambda_{w_i}(u) := -\sum_{\sigma\in G}\log|u^{\sigma^{-1}}|_{w_i}\cdot\sigma\,,$$

for all $u \in U_{S,T}$. Here, for a prime w in K, $|\cdot|_w$ denotes the absolute value associated to w, normalized as in [10], §1.1. Rubin's $\mathbf{C}\,[G]$–linear regulator

$$R_W : \mathbf{C}\wedge_G^r U_{S,T} \xrightarrow{R_W} \mathbf{C}\,[G]\ ,$$

is defined by

$$R_W := \begin{cases} \lambda_{w_1}\wedge\dots\wedge\lambda_{w_r}\ ,\ \text{if } r\geq 1 \\ \mathbf{1}_{\mathbf{C}\wedge_G^0 U_{S,T}} = \mathbf{1}_{\mathbf{C}[G]}\ ,\ \text{if } r=0 \end{cases}$$

Remark 1. As pointed out in [7, §1.6, Remark 2], R_W induces an isomorphism

$$R_W|_{\left(\mathbf{C}\wedge_G^r U_{S,T}\right)_{r,S}} : \left(\mathbf{C}\wedge_G^r U_{S,T}\right)_{r,S} \xrightarrow{\sim} \mathbf{C}\,[G]_{r,S}\,.$$

In [10], §2.1, Rubin states the following.

Conjecture B$(K/k,\, S,\, T,\, r)$ **(Rubin).** *Let us assume that the data* $(K/k, S, T, r)$ *satisfies hypotheses (H). Then, for any choice of* V *and* W*, there exists a unique* $\varepsilon_{S,T,W} \in \Lambda_{S,T}$*, such that* $R_W(\varepsilon_{S,T,W}) = \Theta^{(r)}_{K/k,S,T}(0)$.

Remark 2. As pointed out in Remark 2, §5.1 of [7], if $(K/k,\, S,\, T,\, r)$ is given, then the truth of Rubin's Conjecture does not depend on the particular choice of V and W. This is why we will suppress W from the notation $\varepsilon_{S,T,W}$. Also, Remark 1 above shows that the uniqueness in the conjecture above is automatic.

In [9], we prove the following functoriality properties for Conjecture B.

Lemma 1.2.2. *Let us assume that the set of data* $(K/k,\, S,\, T,\, r)$ *satisfies hypotheses (H). Then, the following hold true.*

(i) *If* $S \subseteq S'$ *and* $(K/k, S', T, r)$ *satisfies hypotheses (H), then*

$$\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longrightarrow \mathrm{B}(K/k, S', T, r)\,.$$

(ii) *Let* $S' := S \cup \{v_{r+1}, \dots, v_{r'}\}$*, with* $v_{r+1}, \dots, v_{r'}$ *distinct primes in* k*, which do not belong to* S *and split completely in* K/k*. If* $T \cap S' = \emptyset$*, then*

$$\mathrm{B}(K/k, S', T, r') \Longrightarrow \mathrm{B}\,(K/k,\, S,\, T,\, r)\ .$$

(iii) *Let* T' *be a finite set of primes in* k*, such that* $T \subseteq T'$ *and* $S \cap T' = \emptyset$*. Then*

$$\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longrightarrow \mathrm{B}(K/k, S, T', r)\,.$$

Proof. See Proposition 2.3(i), (ii), (v), in §3 of [9]. □

2. AN EQUIVALENT CONJECTURE

Throughout this section, K/k will denote an abelian extension of Galois group G, such that k is a totally real and K is a CM number field. We will denote by j the (unique) automorphism of K induced by complex conjugation. Obviously, since k is totally real, j is an element (of order 2) in G. The main goal of this section is to formulate a conjecture $\mathrm{B}_-\,(K/k, S, T, r)$, equivalent to $\mathrm{B}(K/k, S, T, r)$, but stated in terms of the ring $\mathbf{Z}\,[G]_- := \mathbf{Z}\,[G]\,/(1+j)$ and modules over it, rather than the ring $\mathbf{Z}\,[G]$ itself.

2.1 The ring R_-**.** As in [9], §4.1, let $R := \mathbf{Z}\,[G]$ and $R_- := \mathbf{Z}\,[G]\,/(1+j)$. For every R–module M, let

$$M_- := M/(1+j)M\ (\xrightarrow{\sim} M \otimes_R R_-)\,, \qquad M^- := \{x \in M \mid (1+j)\cdot x = 0\}\,,$$

and let $\pi_M : M \twoheadrightarrow M_-$ be the canonical projection. In particular, let $\pi := \pi_R$. Obviously, both M^- and M_- come endowed with natural R_-–module structures.

Lemma 2.1.1. *Let H be the subgroup of G generated by j, and let M be an R–module. Let us assume that M is H–cohomologically trivial. Then,*

(1) *One has an equality, respectively isomorphism of R_-–modules*

$$M^- = (1-j)M\,, \qquad M_- \xrightarrow[(1-j)_M]{\sim} M^-\,,$$

where $(1-j)_M$ takes $(x \mod (1+j)M)$ into $(1-j)\cdot x$.

(2) *The inverse of the isomorphism $(1-j)_M$ in (1) is given by*

$$\frac{1}{2}\pi_M : M^- \longrightarrow M_-\,,$$

defined as $\frac{1}{2}\pi_M((1-j)x) := \pi_M(x)$, for all x in M.

Proof (sketch). The equality in (1) is a direct consequence of $\widehat{\mathrm{H}}^{-1}(H,M)=0$. The isomorphism in (1) is a consequence of the equalities $M^-=(1-j)M$ and $\widehat{\mathrm{H}}^{0}(H,M)=0$. (2) can be proved by direct calculation. □

In particular, Lemma 2.1.1 can be applied to the following H–cohomologically trivial modules: R, $R\otimes_{\mathbf{Z}} A = A[G]$, where A is an arbitrary subring of $\mathbf{C}$, and $N\otimes_{\mathbf{Z}}\mathbf{Q}$, $N\otimes\mathbf{Z}\mathbf{C}$, $N_{(p)}$, where N is an arbitrary R–module and p is an odd prime.

Corollary 2.1.2. *Let N be an R module and A a subring of $\mathbf{C}$. Then, one has an isomorphism of abelian groups*

$$\begin{array}{ccc}\mathrm{Hom}_{A[G]}\Big((N\otimes A)^-\ ,\ A[G]\Big) & \xrightarrow{\sim} & \mathrm{Hom}_{A[G]_-}\Big((N\otimes A)^-\ ,\ A[G]_-\Big)\\ \phi & \longrightarrow & \phi_-\,,\end{array}$$

where $\phi_- := \frac{1}{2}\pi_{A[G]}\circ\phi$, and $\phi = (1-j)_{A[G]}\circ\phi_-$, for all $A[G]$–linear maps ϕ in $\mathrm{Hom}_{A[G]}\Big((M\otimes A)^-, A[G]\Big)$.

Proof. Let us first remark that $A[G]$–linearity implies that any ϕ as above satisfies $\mathrm{Im}(\phi)\subseteq A[G]^-$. Now, the isomorphism in Corollary 2.1.2 is a consequence of Lemma 2.1.1, applied to $M := A[G]$, and the observation that one always has an equality

$$\mathrm{Hom}_{A[G]}\Big((N\otimes A)^-\ ,\ A[G]_-\Big) = \mathrm{Hom}_{A[G]_-}\Big((N\otimes A)^-\ ,\ A[G]_-\Big)\,.$$

□

Remark 1. In general, the quotient of a group ring by one of its ideals, and in particular R_-, is not a group ring any longer. However, **let us assume that the 2–Sylow subgroup G_2 of G is cyclic.** (This will be the case throughout sections §§3–5 of this paper). Then, R_- happens to be a group ring still. Indeed, let $|G_2| = 2^t$, for some $t\geq 1$. Let $G = G_2\times G'$, where G' has odd order. Then, as Greither remarks in [3], since G is cyclic and consequently G_2 is cyclic of order 2^t, we have isomorphisms

$$\begin{aligned}R_- = \mathbf{Z}[G_2]/(1+j)\,[G'] &\xrightarrow{\sim} \mathbf{Z}[X]/(X^{2^t}-1, X^{2^{t-1}}+1)\,[G']\\ &\xrightarrow{\sim} \mathbf{Z}[\zeta_{2^t}][G']\,.\end{aligned} \tag{2}$$

The first isomorphism above takes a generator of G_2 into the class of X, and the second isomorphism sends the class of X to ζ_{2^t}. The importance of 2 lies in the fact that it implies the following double equivalence, for any R_- module N.

$$\begin{aligned} \text{pd}_{R_-} N \leq 1 &\iff \text{pd}_{R_{(p),-}} N_{(p)} \leq 1\,, \text{ for all } \textbf{odd} \text{ primes } p \\ &\iff N \text{ is } G'\text{–cohomologically trivial}\,. \end{aligned} \tag{3}$$

Indeed, since R_- is of characteristic 0 and integral over $\mathbf{Z}$, $\text{pd}_{R_-} N \leq 1$ if and only if $\text{pd}_{R_{(p),-}} N_{(p)} \leq 1$, for all primes p. However, for all p, $\mathbf{Z}_{(p)}[\zeta_{2^t}]$ is semilocal and Dedekind and therefore a P.I.D. In particular, for $p = 2$, we have

$$R_{(2),-} = \bigoplus_{\chi \in \widehat{G'}/\sim} \mathbf{Z}_{(2)}[\zeta_{2^t}][\chi]\,, \tag{4}$$

where each direct summand is a P.I.D. Therefore, $\text{pd}_{R_{(2),-}}(N_{(2)}) \leq 1$ is automatically satisfied, for any R_-–module N. This takes care of the first equivalence in (3) above. In order to show the second equivalence, one first remarks that, since $|G'|$ is odd, N is G'–cohomologically trivial if and only if $N_{(p)}$ is G'–cohomologically trivial, for all odd primes p. Since $\mathbf{Z}_{(p)}[\zeta_{2^t}]$ is a P.I.D., Proposition 5.2.2 in [7] implies that the G'–cohomological triviality of $N_{(p)}$ is equivalent to $\text{pd}_{R_{(p),-}}(N_{(p)}) \leq 1$.

2.2 Conjecture $\mathrm{B}_-\,(K/k,\, S,\, T,\, r)$**.** We are now ready to state the conjecture over R_-, promised at the beginning of §2. Assume that $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). We will first define an R_-–analogue $\Lambda'_{S,T}$ of Rubin's lattice $\Lambda_{S,T}$.

Definition 2.2.1. *Let $\Lambda'_{S,T}$ be the R_-–submodule of $\left(\mathbf{Q} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S}$, consisting of those elements ε' in $\left(\mathbf{Q} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S}$, such that*

$$\psi_1 \wedge \cdots \wedge \psi_r(\varepsilon') \in R_-\,,$$

for all $\psi_1, \ldots, \psi_r$ in $\text{Hom}_{R_-}\left(U^-_{S,T}\,,\, R_-\right)$.

Remark 1. Let us note that, since any character $\chi \in \widehat{G}$, satisfying $\chi(j) = +1$, also satisfies $r_{\chi,S} > r$ (see (1) in §1), we have an equality

$$\left(\mathbf{Q} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S} = \left(\mathbf{Q} \wedge^r_{R} U_{S,T}\right)_{r,S}\,. \tag{5}$$

For the same reason, we also have $\Theta^{(r)}_{S,T}(0) \in \mathbf{C}\,[G]^-$.

Now, as in Rubin's Conjecture, let us fix ordered r–tuples of primes $V = (v_1, ..., v_r)$ and $W = (w_1, ..., w_r)$. With notations as in §1, let us recall that W gives rise to maps $\lambda_{w_1}, \ldots, \lambda_{w_r}$ in $\text{Hom}_{\mathbf{C}[G]}\left(\mathbf{C}U_{S,T}, \mathbf{C}\,[G]\right)$. By restriction to $\mathbf{C}U^-_{S,T}$, we can think of these maps as belonging to $\text{Hom}_{\mathbf{C}[G]}\left(\mathbf{C}U^-_{S,T}, \mathbf{C}\,[G]\right)$. Corollary 2.1.2 associates to $\lambda_{w_1}, \ldots, \lambda_{w_r}$, in a canonical way, maps $(\lambda_{w_1})_-\,, \ldots\,, (\lambda_{w_r})_-$ in $\text{Hom}_{\mathbf{C}[G]_-}\left(\mathbf{C}U^-_{S,T}, \mathbf{C}\,[G]_-\right)$. By using these, we can define an R_-–analogue R_{W_-} of Rubin's regulator R_W, as follows.

$$R_{W_-} : \mathbf{C} \wedge^r_{R_-} U^-_{S,T} \longrightarrow \mathbf{C}\,[G]_-\,,$$

given by $R_{W_-}(\varepsilon') := ((\lambda_{w_1})_- \wedge \cdots \wedge (\lambda_{w_r})_-)(\varepsilon')$, for all ε' in $\mathbf{C} \wedge^r_{R_-} U^-_{S,T}$. Lemma 2.1.1 helps relate R_{W_-} and R_W restricted to $\mathbf{C} \wedge^r_{R_-} U^-_{S,T} = (\mathbf{C} \wedge^r_R U_{S,T})^-$. Namely, we have equalities

$$(6) \qquad R_{W_-}(\varepsilon') = \frac{1}{2^r}\pi_{\mathbf{C}[G]}\left(R_W(\varepsilon')\right), \quad R_W(\varepsilon') = 2^{r-1}(1-j)_{\mathbf{C}[G]}\left(R_{W_-}(\varepsilon')\right),$$

for all ε' in $\mathbf{C} \wedge^r_{R_-} U^-_{S,T}$.

Conjecture $\mathrm{B}_-(K/k,\, S,\, T,\, r)$**.** *Let us assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). Then, there exists a unique $\varepsilon'_{S,T}$ in $\Lambda'_{S,T}$, such that*

$$R_{W_-}(\varepsilon'_{S,T}) = \frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta^{(r)}_{S,T}(0)\right),$$

viewed as an equality in $\mathbf{C}[G]_-$.

Remark 2. Relations (5) and (6) above and Remark 1 in §1.2 show that R_{W_-} restricted to $\left(\mathbf{C} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S}$ is injective, and therefore, as it is the case with Rubin's Conjecture, the uniqueness in Conjecture B_- is automatic.

2.3 The Equivalence. The aim of this subsection is to prove that Conjectures B and B_- are equivalent.

Proposition 2.3.1. *Let us assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). Then, we have an equivalence*

$$\mathrm{B}(K/k,\, S,\, T,\, r) \iff \mathrm{B}_-(K/k,\, S,\, T,\, r).$$

Moreover, the unique elements $\varepsilon_{S,T}$ and $\varepsilon'_{S,T}$ whose existence is respectively predicted by the statements above, satisfy

$$\varepsilon_{S,T} = \frac{1}{2^{r-1}}\varepsilon'_{S,T}.$$

Proof. Let us assume first that $\mathrm{B}(K/k,\, S,\, T,\, r)$ is true, and let $\varepsilon_{S,T}$ be the unique element in $\Lambda_{S,T}$, such that

$$R_W(\varepsilon_{S,T}) = \Theta^{(r)}_{S,T}(0).$$

Let $\varepsilon'_{S,T} := 2^{r-1} \cdot \varepsilon_{S,T}$. Remark 1 in §2.2 shows that

$$\varepsilon'_{S,T} \in (\mathbf{Q} \wedge^r_R U_{S,T})_{r,S} = \left(\mathbf{Q} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S}.$$

Also, (6) above shows that

$$R_{W_-}(\varepsilon'_{S,T}) = \frac{1}{2^r}\pi_{\mathbf{C}[G]}\left(2^{r-1}R_W\varepsilon_{S,T}\right) = \frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta^{(r)}_{S,T}(0)\right).$$

Let $\psi_1, \dots, \psi_r$ be elements in $\mathrm{Hom}_{R_-}\left(U^-_{S,T}, R_-\right)$. Corollary 2.1.2 shows that there exist unique $\phi_1, \dots, \phi_r$ in $\mathrm{Hom}_R(U_{S,T}, R)$, such that

$$\psi_i = (\phi_i)_- , \quad \psi_i = \frac{1}{2}\pi \circ \phi_i \, , \text{ for all } i = 1, \dots, r \ .$$

Consequently, we have an equality

$$\psi_1 \wedge \dots \psi_r(\varepsilon'_{S,T}) = \frac{1}{2}\pi\left(\phi_1 \wedge \dots \wedge \phi_r(\varepsilon_{S,T})\right) .$$

However, since $\varepsilon_{S,T} \in \Lambda_{S,T} \subseteq (\mathbf{Q} \wedge^r_R U_{S,T})^-$ we have $\phi_1 \wedge \dots \wedge \phi_r(\varepsilon_{S,T}) \in R^-$. Therefore,

$$\psi_1 \wedge \dots \wedge \psi_r\left(\varepsilon'_{S,T}\right) \in \frac{1}{2}\pi\left(R^-\right) = R_- \, ,$$

which concludes the proof of Conjecture $\mathrm{B}_-\,(K/k,\, S,\, T,\, r)$.

Conversely, let us now assume that $\mathrm{B}_-\,(K/k,\, S,\, T,\, r)$ is true, and let $\varepsilon'_{S,T}$ be the unique element in $\Lambda'_{S,T}$, satisfying

$$R_{W_-}(\varepsilon'_{S,T}) = \frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta^{(r)}_{S,T}(0)\right) .$$

Let $\varepsilon_{S,T} = 1/2^{r-1} \cdot \varepsilon'_{S,T}$. Clearly, $\varepsilon_{S,T}$ is in $(\mathbf{Q} \wedge^r_R U_{S,T})_{r,S}$. Now, we combine the second equality in (6) with Lemma 2.1.1(2), and obtain

$$\begin{aligned} R_W(\varepsilon_{S,T}) &= \frac{1}{2^{r-1}} \cdot 2^{r-1}(1-j)_{\mathbf{C}[G]} R_{W_-}(\varepsilon'_{S,T}) \\ &= ((1-j)_{\mathbf{C}[G]} \circ \frac{1}{2}\pi_{\mathbf{C}[G]})\left(\Theta^{(r)}_{S,T}(0)\right) = \Theta^{(r)}_{S,T}(0) . \end{aligned}$$

Let $\phi_1, \dots, \phi_r$ be elements of $\mathrm{Hom}_R(U_{S,T}, R)$. Then, we have

$$\begin{aligned} \phi_1 \wedge \dots \wedge \phi_r(\varepsilon_{S,T}) &= (1-j)^{(r)}_{\mathbf{C}[G]}\left((\phi_1)_- \wedge \dots \wedge (\phi_r)_-(\varepsilon_{S,T})\right) \\ &= 2^{r-1}(1-j)_{\mathbf{C}[G]}\frac{1}{2^{r-1}} \cdot (\phi_1)_- \wedge \dots \wedge (\phi_r)_-(\varepsilon'_{S,T}) \\ &= (1-j) \cdot (\phi_1)_- \wedge \dots \wedge (\phi_r)_-(\varepsilon'_{S,T}) . \end{aligned}$$

Since $\varepsilon'_{S,T} \in \Lambda'_{S,T}$, the last equality shows that

$$\phi_1 \wedge \dots \wedge \phi_r(\varepsilon_{S,T}) \in (1-j)R_- = R^- \, ,$$

which shows that $\varepsilon_{S,T} \in \Lambda_{S,T}$, concluding the proof of $\mathrm{B}(K/k,\, S,\, T,\, r)$. □

3. GREITHER'S RESULTS. CONSEQUENCES.

Throughout this section, we fix an odd prime number ℓ, an integer $n \geq 1$. K denotes an imaginary subfield K of the full cyclotomic field $L := \mathbf{Q}(\zeta_{\ell^n})$. Let $S_{00} := \{\ell, \infty\}$, where ∞ is the (unique) infinite prime in $\mathbf{Q}$. Let $G := \mathrm{Gal}(K/\mathbf{Q})$ and let j be the element in G corresponding to complex conjugation. Since K is a CM field, the results of §2 apply to this situation. In particular, since G is cyclic, so is its 2–Sylow subgroup G_2. Therefore, Remark 1 of §2.1 also applies to the case under current discussion. Also, as showed in [9], §4.1, Remark 1(a), for any K as above, the extension $K/\mathbf{Q}$ is **"nice"**, in Greither's terminology (see [4] and [9] for the definition of a "nice" extension).

Before we begin discussing Greither's results, we would like to warn the reader that our notations differ from those of [3] and [4]. In particular, Greither denotes by A_K^- the minus ideal–class group of the CM field K, i.e. the quotient of A_K by the image of the ideal–class group A_{K^+} of the maximal real subfield K^+ of K, via the canonical map $i : A_{K^+} \longrightarrow A_K$. However, since K/K^+ is totally ramified at least one prime, the norm map $\mathrm{N}_{K/K^+} : A_K \longrightarrow A_{K^+}$ is surjective. This shows that $\mathrm{Im}(i) = (1+j)A_K$. Consequently, Greither's minus class–group A_K^- is in fact, in our notations, equal to $A_{K,-} := A_K/(1+j)A_K$.

For any field K as above, equalities (1) above, combined with a well–known result of Deligne–Ribet (see Theorem 1.3 in [7]), imply that we have

$$\mathcal{A}(K/k) \cdot \Theta_{S_{00}}(0) \subseteq R^- ,$$

and therefore $\frac{1}{2}\pi\left(\mathcal{A}(K/k) \cdot \Theta_{S_{00}}(0)\right) \subseteq R_-$. The main results in [2], [3], and [4] (see Theorem A in [2], proof of Theorem 2.4 in [3], and Theorem 4.11 in [4]) lead to the following.

Theorem 3.1 (Greither). *Under the above assumptions and notations, we have*

$$\frac{1}{2}\pi\left(\mathcal{A}(K/k) \cdot \Theta_{S_{00}}(0)\right) = \mathrm{Fitt}_{R_-}\left(A_{K,-}\right) .$$

Proof (sketch). Let I and J denote the R_-–ideal on the left–hand side and respectively right–hand side of the equality in the theorem. Then, proving $I = J$ is equivalent to proving $I_{(p)} = J_{(p)}$, for all prime numbers p. Indeed, since R_- is an integral extension of $\mathbf{Z}$, every maximal ideal $\mathfrak{m}$ of R_- lies over a maximal ideal $p\mathbf{Z}$ of $\mathbf{Z}$. So, $I_{(p)} = J_{(p)}$, for all prime numbers p, implies equalities $I_\mathfrak{m} = J_\mathfrak{m}$ for the localizations of I and J at all maximal ideals $\mathfrak{m}$ of R_-. This implies $I = J$.

First, let us assume that p *is an odd prime.* Greither shows the equality $I_{(p)} = J_{(p)}$ in [3] (see the Third and Fourth Steps of the Proof of Theorem 3.4), under the extra–assumption that the prime ℓ is what he calls "suitable". However, in light of the results of [4], this assumption is unnecessary. Indeed, as pointed out above, our extension $K/\mathbf{Q}$ is "nice". Theorem 4.11 in [4] (see also Theorem 4.19 in[9] for a rewriting of Greither's result in our notations) shows that, for a nice extension and p odd, we have

$$I_{(p)} = \mathrm{Fitt}_{R_{(p),-}}\left((A_K \otimes \mathbf{Z}_p)^-\right).$$

However, Lemma 2.1.1 shows that, for p–odd we have an $R_{(p),-}$–isomorphism

$$\left(A_K \otimes \mathbf{Z}_{(p)}\right)^- \xrightarrow{\sim} \left(A_K \otimes \mathbf{Z}_{(p)}\right)_- \quad (= A_{K,-} \otimes \mathbf{Z}_{(p)}) .$$

Therefore, the equality above implies $I_{(p)} = J_{(p)}$, for all odd p.

Secondly, let us assume that $p = 2$. The Second Step of the proof of Theorem 3.4 in [3] shows the equality $I_{(2)} = J_{(2)}$, under no extra–assumptions on ℓ. This follows directly from Theorem A and Remark (a), §1 of [2], as consequence of the fact that G_2 is cyclic and $R_{(2),-}$ is a direct sum of P.I.D.'s. (see (4) in Remark 1, §2.1 above). □

Lemma 3.2. *For $K/\mathbf{Q}$ as above, the following hold true.*

(1) $\mathrm{pd}_{R_-}(\mu_K) \leq 1$.
(2) $\pi(\mathcal{A}(K/\mathbf{Q})) = \mathrm{Ann}_{R_-}(\mu_K)$.
(3) $\pi(\mathcal{A}(K/\mathbf{Q}))$ *is an invertible R_-–ideal.*

Proof. (1) In a nice extension, in particular in $K/\mathbf{Q}$, $\mu_K \otimes \mathbf{Z}_{(p)}$ is G–cohomologically trivial, for all odd primes p (see [9], Corollary 4.1.4(2)). Obviously, this is equivalent to the G'–cohomologically triviality of μ_K, where G' is the product of all the odd–order Sylow–subgroups of G. Remark 1 in §2.1 shows that this is equivalent to $\mathrm{pd}_{R_-}(\mu_K) \leq 1$.

(2) Since μ_K is R–cyclic, we have an exact sequence of R–modules

$$0 \longrightarrow \mathcal{A}(K/\mathbf{Q}) \xrightarrow{i} R \longrightarrow \mu_K \longrightarrow 0\,.$$

If we tensor the above sequence with R_- over R, and take into account that $\mu_{K,-} = \mu_K$, we obtain an exact sequence of R_-–modules

$$\mathcal{A}(K/\mathbf{Q})_- \xrightarrow{i\otimes \mathbf{1}_{R_-}} R_- \longrightarrow \mu_K \longrightarrow 0\,.$$

However, we clearly have $\pi(\mathcal{A}(K/\mathbf{Q})) = \mathrm{Im}(i \otimes \mathbf{1}_{R_-})$. Since μ_K is R_-–cyclic, this equality, combined with the last exact sequence implies (2) in the Lemma.

(3) According to (2), we have an exact sequence of R_-–modules

$$0 \longrightarrow \pi(\mathcal{A}(K/\mathbf{Q})) \longrightarrow R_- \longrightarrow \mu_K \longrightarrow 0\,.$$

Since $\mathrm{pd}_{R_-}(\mu_K) \leq 1$, this exact sequence implies that $\mathrm{pd}_{R_-}(\pi(\mathcal{A}(K/\mathbf{Q}))) = 0$, i.e. $\pi(\mathcal{A}(K/\mathbf{Q}))$ is a projective R_-–ideal, i.e. $\pi(\mathcal{A}(K/\mathbf{Q}))$ is an invertible R_-–ideal. □

In what follows, we will denote by $\pi(\mathcal{A}(K/\mathbf{Q}))^{-1}$ the inverse of the R_- ideal $\pi(\mathcal{A}(K/\mathbf{Q}))$. $\pi(\mathcal{A}(K/\mathbf{Q}))^{-1}$ is viewed as a fractional R_-–ideal contained in the total ring of fractions $Q(R_-) = \mathbf{Q} \otimes_{\mathbf{Z}} R_-$ of R_- and, as usual, it consists of all $x \in Q(R_-)$, such that $x \cdot \pi(\mathcal{A}(K/\mathbf{Q})) \subseteq R_-$. Lemma 3.2(3) is equivalent to either of the following equalities.

(7)
$$\pi(\mathcal{A}(K/\mathbf{Q}))\cdot\pi(\mathcal{A}(K/\mathbf{Q}))^{-1} = R_-\,,\ \mathrm{Hom}_{R_-}(\pi(\mathcal{A}(K/\mathbf{Q}))\,,\, R_-) = \pi(\mathcal{A}(K/\mathbf{Q}))^{-1}\,.$$

The reader is referred to [5], §1.4, for the properties of invertible ideals needed for our purposes.

Remark 1. Let $f : A \longrightarrow A'$ be a morphism a commutative, Noetherian rings and let M be a finitely generated A–module. Then, the definition of Fitting ideals gives an equality

$$\mathrm{Fitt}_{A'}(M \otimes_A A') = f(\mathrm{Fitt}_A(M)) \cdot A'\,.$$

If one applies this observation to the ring morphism $\pi : R \longrightarrow R_-$ and the R–module A_K, Theorem 3.1, combined with the first equality in (7) above, gives

$$\frac{1}{2}\pi_{\mathbf{C}[G]}(\Theta_{S_{00}}(0)) \in \pi(\mathcal{A}(K/\mathbf{Q}))^{-1} \cdot \pi(\mathrm{Fitt}_R(A_K))\,, \tag{8}$$

viewed inside $Q(R_-) = \mathbf{Q}[G]_-$.

4. THE PROOF OF CONJECTURE $\mathrm{B}_-\,(K/\mathbf{Q},\, S,\, T,\, r)$

We are working under the assumptions and notations of §3. In this section we prove conjecture $\mathrm{B}_-\,(K/\mathbf{Q},\, S,\, T,\, r)$, under extra assumptions on S and T. Although, at first glance, this result seems weaker than the full conjecture B_-, the next section will reveal that it is in fact equivalent to it and, implicitly, equivalent to conjecture B itself. Next, we describe the hypotheses under which we will be working.

Additional hypotheses. Tchebotarev's density theorem allows us to choose finite primes $w_1, w_2, \dots, w_r$, coprime to ℓ, lying above distinct primes $v_1, \cdots, v_r$ in $\mathbf{Q}$, which are completely split in $K/\mathbf{Q}$, and such that the ideal–classes $\widehat{w_1}, \dots, \widehat{w_r}$ generate A_K as an R–module. Let $S := S_{00} \cup \{v_1, \dots, v_r\}$ and let $T = \{v\}$, such that $v \notin S$ and $\mu_K \cap U_{S,T} = \{1\}$. The second condition on T is in fact equivalent in this case to $v \neq 2$ and $v \neq \ell$. Obviously, for S, T, and r as above, $(K/\mathbf{Q},\, S,\, T,\, r)$ satisfies hypotheses (H).

Equalities (1) imply that, under the present hypotheses, a character $\chi \in \widehat{G}$ satisfies $r_{\chi,S} = r$ if and only if $\chi(j) = -1$. This implies that we have an equality

$$\left(\mathbf{Q} \wedge^r_{R_-} U^-_{S,T}\right)_{r,S} = \mathbf{Q} \wedge^r_{R_-} U^-_{S,T} = \mathbf{Q} \wedge^r_{R_-} U^-_{S}\,. \tag{9}$$

For the same reason, we have $\Theta_{S_{00}}(0) \in \mathbf{Q}\,[G]^-$. In what follows, we let $\delta_T := 1 - \mathrm{N}v \cdot \sigma_v^{-1}$. The hypothesis $\mu_K \cap U_{S,T} = \{1\}$ implies that $\delta_T \in \mathcal{A}(K/\mathbf{Q})$, and therefore $\delta_T \cdot \Theta_{S_{00}}(0) \in R^-$. It is easy to see that

$$\Theta^{(r)}_{S,T}(0) = \prod_{i=1}^{r} \log \mathrm{N}v_i \cdot \delta_T \Theta_{S_{00}}(0)\,. \tag{10}$$

Theorem 4.1. *Assume that S, T, and r satisfy the above conditions. Then, Conjecture $\mathrm{B}_-\,(K/\mathbf{Q},\, S,\, T,\, r)$ is true.*

Proof. Relation (8) in §3 above, allows us to write

$$\frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta_{S_{00}}(0)\right) = \sum_i \alpha_i \cdot \pi(\gamma_i)\,, \tag{11}$$

with $\alpha_i \in \pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}$ and $\gamma_i \in \pi\left(\mathrm{Fitt}_R(A_K)\right)$. However, since the ideal–classes $\widehat{w_1}, \dots, \widehat{w_r}$ generate A_K over R, we can assume without loss of generality that $\gamma_i = \det \Gamma^{(i)}$, where $\Gamma^{(i)} = \left(\gamma^{(i)}_{s,t}\right)_{s,t}$ is an $r \times r$–matrix with entries in R, satisfying

$$\sum_{t=1}^{r} \gamma^{(i)}_{s,t} \cdot \widehat{w_t} = 0 \text{ in } A_K, \text{ for all } s = 1, \dots, r\,.$$

The last equalities are equivalent to the existence of S–units $f^{(i)}_s$ in U_S, such that

$$\mathrm{divisor}_{O_K}(f^{(i)}_s) = \sum_t \gamma^{(i)}_{s,t} \cdot w_t\,,$$

for all $s = 1, \dots, r$ and all i. These equalities imply that, for all i, s, t, we have

$$\lambda_{w_t}(f^{(i)}_s) = -\log\left(\mathrm{N}w_i\right) \cdot \gamma^{(i)}_{s,t}\,.$$

Consequently, Corollary 2.1.2 shows that, for all i, s, t, we have

$$(\lambda_{w_t})_-\left((1-j)f_s^{(i)}\right) = \frac{1}{2}\pi_{\mathbf{C}[G]} \circ (1-j)_{\mathbf{C}[G]}\left(\pi(\lambda_{w_t}(f_s^{(i)}))\right) = \\ = -\log \mathrm{N}w_t \cdot \pi(\gamma_{s,t}^{(i)})\,. \tag{12}$$

Let $\varepsilon'_{S,T} := (-1)^r \delta_T \cdot \sum_i \alpha_i \cdot \left((1-j)f_1^{(i)} \wedge \cdots \wedge (1-j)f_r^{(i)}\right)$, viewed as an element of $\mathbf{Q}\wedge^r_{R_-} U_S^- = \mathbf{Q}\wedge^r_{R_-} U_{S,T}^-$. If we combine (11) and (12) above with the definition of R_{W_-} (see §2.2), we obtain

$$R_{W_-}(\varepsilon'_{S,T}) = \left(\prod_{t=1}^{r} \log \mathrm{N}w_t\right) \cdot \delta_T \cdot \frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta_{S_{00}}(0)\right) = \\ = \frac{1}{2}\pi_{\mathbf{C}[G]}\left(\Theta_{S,T}^{(r)}(0)\right)\,. \tag{13}$$

Moreover, we obviously have

$$\varepsilon'_{S,T} \in \delta_T \cdot \pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1} \cdot \widetilde{\wedge^r_{R_-} U_S^-} \subseteq \mathbf{Q}\wedge^r_{R_-} U_{S,T}^-\,. \tag{14}$$

Relations (13) and (14) above show that, if we prove that

$$\delta_T \cdot \pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1} \cdot \widetilde{\wedge^r_{R_-} U_S^-} \subseteq \Lambda'_{S,T}\,, \tag{15}$$

then Theorem 4.1 is indeed true. This last inclusion is precisely the object of Proposition 4.4 below. □

The remainder of this section is solely concerned with the proof of inclusion (15) in the proof of Theorem 4.1. This will be accomplished in Proposition 4.3 below. The ideas involved in the proof are very similar to those in [7], §5.5. We will need some preparations.

Lemma 4.2. *Under the assumptions $T = \{v\}$ and $U_{S,T} \cap \mu_K = \{1\}$, we have an isomorphism of R_-–modules*

$$\mathrm{Ext}^1_{R_-}\left(\Delta_T^-/\mathrm{res}_T(\mu_K)\,,\, R_-\right) \xrightarrow{\sim} R_-/\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}\,.$$

Proof. Let us first remark that δ_T and $\pi(\delta_T)$ are non–zerodivisors in R, respectively R_-. This is a direct consequence of the fact that $\chi(\delta_T) \neq 0$, for all $\chi \in \widehat{G}$, and of isomorphism (2) in §2.1 above. Remark 1 in §5.3 of [7] shows that, in the case $T = \{v\}$, we have an R–module isomorphism

$$f\,:\,\Delta_T \xrightarrow{\sim} R/\delta_T R\,.$$

Since δ_T is a non–zerodivisor of R, the isomorphism above shows that $\mathrm{pd}_R(\Delta_T) \leq 1$. Therefore, Δ_T is G–cohomologically trivial and, consequently, H–cohomologically trivial (see the notations of Lemma 2.1.1). Lemma 2.1.1, combined with the isomorphism above, gives rise to an R_- isomorphism

$$\left(f \otimes_R \mathbf{1}_{R_-}\right) \circ \left(\frac{1}{2}\pi_{\Delta_T}\right)\,:\,\Delta_T^- \xrightarrow{\sim} R_-/\delta_T R_-\,. \tag{16}$$

Lemma 5.4.2 of [7], combined with $\mu_K = \mu_K^-$, shows that one has an equality

$$\mathrm{res}_T\,(\mu_K) = \Delta_T^-\,[\pi\,(\mathcal{A}(K/\mathbf{Q}))]\ ,$$

where $\Delta_T^-\,[\pi\,(\mathcal{A}(K/\mathbf{Q}))]$ denotes the maximal R_-–submodule of Δ_T^- annihilated by $\pi\,(\mathcal{A}(K/\mathbf{Q}))$. If we combine the last equality with isomorphism (16) and the fact that $\pi(\delta_T)$ is a non–zerodivisor in R_-, we obtain the following isomorphism of R_-–modules.

$$\Delta_T^-/\mathrm{res}_T(\mu_K) \xrightarrow{\sim} R_-/\delta_T\pi\,(\mathcal{A}(K/\mathbf{Q}))^{-1}\ . \tag{17}$$

In order to conclude the proof of Lemma 4.2, we will need the following purely algebraic result.

Lemma 4.3. *Let A be a commutative, Noetherian ring with no $\mathbf{Z}$–torsion, such that $A_{(p)} := A\otimes\mathbf{Z}_{(p)}$ is semilocal, for all prime numbers p. Let I be an invertible ideal of A, such that both A/I and I^{-1}/A are finite. Then, one has an isomorphism of A–modules*

$$\mathrm{Ext}_A^1\,(A/I\,,\,A) \xrightarrow{\sim} A/I\,.$$

Proof. If we write the long–exact $\mathrm{Ext}_A^*\,(*\,,\,A)$–sequence associated to

$$0\longrightarrow I\longrightarrow A\longrightarrow A/I\longrightarrow 0\,,$$

we obtain the following exact sequence of A–modules

$$0\longrightarrow A\longrightarrow I^{-1}\longrightarrow \mathrm{Ext}_A^1\,(A/I\,,\,A)\longrightarrow 0\longrightarrow \dots\,.$$

The algebraically inclined reader will remark right away that we have arrived at the exact sequence above by observing that, since A/I is finite and A has no $\mathbf{Z}$–torsion, we have $\mathrm{Hom}_A\,(A/I\,,\,A)=0$, and also that, since I is invertible and therefore A–projective, we have $\mathrm{Ext}_A^1\,(I\,,\,A)=0$. The last exact sequence gives an A–module isomorphism

$$\mathrm{Ext}_A^1\,(A/I\,,\,A) \xrightarrow{\sim} I^{-1}/A\,.$$

Therefore, in order to conclude the proof of Lemma 4.3, we will need to show that $I^{-1}/A \xrightarrow{\sim} A/I$, as A–modules. However, since these A–modules are both finite, this is equivalent to showing that there are $A_{(p)}$–isomorphisms

$$I_{(p)}^{-1}/A_{(p)} \xrightarrow{\sim} A_{(p)}/I_{(p)}\ ,$$

for all prime numbers p. Let us fix a prime number p. Since $A_{(p)}$ is semilocal, exercise 4.13 in [1] shows that the isomorphism above is equivalent to the $A_{\mathfrak{m}}$–isomorphisms

$$I_{\mathfrak{m}}^{-1}/A_{\mathfrak{m}} \xrightarrow{\sim} A_{\mathfrak{m}}/I_{\mathfrak{m}}$$

for all maximal ideals $\mathfrak{m}$ of A, containing p. However, since $A_{\mathfrak{m}}$ is local and $I_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$–projective, $I_{\mathfrak{m}}$ is generated as an $A_{\mathfrak{m}}$–module by a non–zerodivisor $f_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$. It is now immediate that the last isomorphism above is given by multiplication by $f_{\mathfrak{m}}$. This concludes the proof of Lemma 4.3 □

Now, let us observe that Lemma 4.3 is applicable to the ring $A := R_-$ and its ideal $I := \delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}$. Indeed, isomorphism (2) in §2.1 shows that A satisfies all the required properties. Also, since $\pi(\delta_T)$ is a non–zerodivisor in R_- and $\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}$ is invertible (see Lemma 3.2 above), $\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}$ is also invertible. The finiteness of A/I and I^{-1}/A is immediate in this case. Therefore, we have an R_-–isomorphism

$$\mathrm{Ext}^1_{R_-}\left(R_-/\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}, R_-\right) \xrightarrow{\sim} R_-/\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1},$$

which, combined with (17) above, concludes the proof of Lemma 4.2. □

We are now ready to prove inclusion (15) in the proof of Theorem 4.1.

Proposition 4.4. *Under the current assumptions, one has an inclusion*

$$\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}\cdot\widetilde{\wedge^r_{R_-}U_S^-}\subseteq\Lambda'_{S,T}.$$

Proof. Exact sequence (0) in §1.1, combined with the fact that $U_{S,T}\cap\mu_K=\{1\}$, gives rise to the following exact sequence of R_-–modules

$$0\longrightarrow U_{S,T}^-\longrightarrow\widetilde{U_S^-}\xrightarrow{\widetilde{\mathrm{res}_T}}{\Delta_T}^-/\mu_K.$$

Let $\mathcal{I}:=\mathrm{Im}(\widetilde{\mathrm{res}_T})$ and $\mathcal{Q}:=\mathrm{Coker}(\widetilde{\mathrm{res}_T})$. The exact sequence above gives rise to the following short exact sequences of R_-–modules.

$$\begin{aligned}&0\longrightarrow U_{S,T}^-\longrightarrow\widetilde{U_S^-}\longrightarrow\mathcal{I}\longrightarrow 0\\&0\longrightarrow\mathcal{I}\longrightarrow{\Delta_T}^-/\mu_K\longrightarrow\mathcal{Q}\longrightarrow 0.\end{aligned}\tag{19}$$

Now, the fact that R_- is a group ring with coefficients in the Dedekind domain $\mathbf{Z}[\zeta_{2^t}]$ (see isomorphism (2) in §2.1), allows us to conclude that, for any finitely generated R_-–module M, we have the following vanishing results.

(1) $\mathrm{Ext}^n_{R_-}(M, R_-)=0$, for all $n\geq 2$.
(2) If M has no $\mathbf{Z}$–torsion, then $\mathrm{Ext}^n_{R_-}(M, R_-)=0$, for all $n\geq 1$.

See [7], Lemma 5.2.1 for (1) and (2) above. In light of these facts, if we write the long–exact $\mathrm{Ext}^*_{R_-}(*, R_-)$–sequences associated to 19, we obtain the following.

$$0\to\mathrm{Hom}_{R_-}\left(\widetilde{U_S^-}, R_-\right)\to\mathrm{Hom}_{R_-}\left(U_{S,T}^-, R_-\right)\to\mathrm{Ext}^1_{R_-}(\mathcal{I}, R_-)\to 0\to\dots$$
$$0\to\mathrm{Ext}^1_{R_-}(\mathcal{Q}, R_-)\to\mathrm{Ext}^1_{R_-}\left({\Delta_T}^-/\mu_K, R_-\right)\to\mathrm{Ext}^1_{R_-}(\mathcal{I}, R_-)\to 0\to\cdots.$$

Once again, we have taken advantage of the fact that, since R_- has no $\mathbf{Z}$–torsion, $\mathrm{Hom}_{R_-}(M, R_-)=0$, for any finite R_-–module M. If we combine the last two exact sequences with Lemma 4.2, and take into account that $\mathrm{Hom}_{R_-}\left(\widetilde{U_S^-}, R_-\right)=\mathrm{Hom}_{R_-}\left(U_S^-, R_-\right)$, we conclude that there is a surjective morphism of R_--modules

$$R_-/\delta_T\pi\left(\mathcal{A}(K/\mathbf{Q})\right)^{-1}\twoheadrightarrow\mathrm{Hom}_{R_-}\left(U_{S,T}^-, R_-\right)/\mathrm{Hom}_{R_-}\left(U_S^-, R_-\right).\tag{20}$$

In particular, this shows that $\mathrm{Hom}_{R_-}\left(U^-_{S,T}\,,\,R_-\right)/\,\mathrm{Hom}_{R_-}\left(U^-_S\,,\,R_-\right)$ is a cyclic R_-–module, annihilated by $\delta_T\pi\,(\mathcal{A}(K/\mathbf{Q}))^{-1}$. Let us fix $\psi_0\in\mathrm{Hom}_{R_-}\left(U^-_{S,T}\,,\,R_-\right)$, such that its class modulo $\mathrm{Hom}_{R_-}\left(U^-_S\,,\,R_-\right)$ is an R_-–generator for the quotient $\mathrm{Hom}_{R_-}\left(U^-_{S,T}\,,\,R_-\right)/\,\mathrm{Hom}_{R_-}\left(U^-_S\,,\,R_-\right)$.

We will now proceed to proving the inclusion in Proposition 4.4. Let $\alpha\in\delta_T\pi\,(\mathcal{A}(K/\mathbf{Q}))^{-1}$ and $\varepsilon'_S\in\widetilde{\wedge^r_{R_-}U^-_S}$. We will prove that $\alpha\cdot\varepsilon'_S\in\Lambda'_{S,T}$. Let $\psi_1,\ldots,\psi_r\in\mathrm{Hom}_{R_-}\left(U^-_{S,T}\,,\,R_-\right)$. Then, the definition of ψ_0 shows that there exist $\psi'_1,\ldots,\psi'_r\in\mathrm{Hom}_{R_-}\left(U^-_S\,,\,R_-\right)$ and $\gamma_1,\ldots,\gamma_r\in R_-$, such that

$$\psi_i=\psi'_i+\gamma_i\cdot\psi_0\,,$$

for all $i=1,\ldots,r$. This implies that we have an equality

$$(\psi_1\wedge\cdots\wedge\psi_r)(\alpha\cdot\varepsilon'_S)=\sum_{i=1}^{r}\gamma_i\cdot(\psi'_1\wedge\cdots\wedge\overset{i}{\widehat{\alpha\cdot\psi_0}}\wedge\cdots\wedge\psi'_r)(\varepsilon'_S)\,.$$

However, since $\alpha\cdot\psi_0\in\mathrm{Hom}_{R_-}\left(U^-_S\,,\,R_-\right)$ (see (20)), and $\varepsilon'_S\in\widetilde{\wedge^r_{R_-}U^-_S}$, we have

$$(\psi'_1\wedge\cdots\wedge\overset{i}{\widehat{\alpha\cdot\psi_0}}\wedge\cdots\wedge\psi'_r)(\varepsilon'_S)\in R_-\,,$$

for all $i=1,\ldots,r$. Consequently, we have

$$(\psi_1\wedge\cdots\wedge\psi_r)(\alpha\cdot\varepsilon'_S)\in R_-\,,$$

for all $\psi_1,\ldots,\psi_r\in\mathrm{Hom}_{R_-}\left(U^-_{S,T}\,,\,R_-\right)$. This fact, combined with (9) above, shows that, indeed

$$\alpha\cdot\varepsilon'_S\in\Lambda'_{S,T}\,.$$

This concludes the proof of Proposition 4.4. ☐

5. THE PROOF OF CONJECTURE B($K/\mathbf{Q}$, S, T, r)

We are now ready to prove Conjecture B($K/\mathbf{Q}$, S, T, r).

Theorem 5.1. *Let us assume that the set of data ($K/\mathbf{Q}$, S, T, r) satisfies hypotheses (H). Then, Conjecture B($K/\mathbf{Q}$, S, T, r) is true.*

Proof. Let us fix a set of data ($K/\mathbf{Q}$, S, T, r), satisfying hypotheses (H). Let $v_1,v_2,\ldots,v_r$ be r distinct primes in S which split completely in $K/\mathbf{Q}$, and let $S_0:=S\setminus\{v_1,v_2,\ldots,v_r\}$. Then, clearly $S_{00}\subseteq S_0$. Let $S':=S_{00}\cup\{v_1,v_2,\ldots,v_r\}$. Then, the set of data $(K/\mathbf{Q},S',T,r)$ satisfies hypotheses (H) as well. Also, since $T\cap S_{00}=\emptyset$, there exists a prime $v\in T$, such that $v\neq\ell,2$. Let $T_v:=\{v\}$. Since $\mu_K\subseteq\mu_{2\ell^n}$, we obviously have $U_{S',T_v}\cap\mu_K=\{1\}$. Therefore, the set of data $(K/\mathbf{Q},S',T_v,r)$ also satisfies hypotheses (H).

Tchebotarev's density theorem allows us to pick primes $w_{r+1},\ldots,w_{r'}$ in K, such that, the following requirements are simultaneously met.

1. $w_{r+1},\ldots,w_{r'}$ sit above distinct primes $v_{r+1},\ldots,v_{r'}$ in $\mathbf{Q}$, which are not in $S\cup T$, and which split completely in $K/\mathbf{Q}$.
2. The ideal-classes $\widehat{w_{r+1}},\ldots,\widehat{w_{r'}}$ generate A_K as an R–module.

Let $S'' := S' \cup \{v_{r+1}, \dots, v_{r'}\}$. Then, obviously, the sets of data $(K/\mathbf{Q}, S'', T_v, r'')$ and $(K/\mathbf{Q}, S \cup \{v_{r+1}, \dots, v_{r'}\}, T_v, r')$ satisfy hypotheses (H). Also, condition (2) above shows that the set of data $(K/\mathbf{Q}, S'', T_v, r')$ satisfies the additional hypotheses imposed at the beginning of §4. Therefore, Theorem 4.1 shows that Conjecture $\mathrm{B}_-(K/\mathbf{Q}, S'', T_v, r')$ is true. Consequently, Proposition 2.3.1 shows that Conjecture $\mathrm{B}(K/\mathbf{Q}, S'', T_v, r')$ is true. Now, we use Lemma 1.2.2 to derive Conjecture $\mathrm{B}(K/\mathbf{Q}, S, T, r)$ from Conjecture $\mathrm{B}(K/\mathbf{Q}, S'', T_v, r')$. Lemma 1.2.2 (i), (ii), and (iii) respectively, leads to the following implications.

$$\begin{aligned} \mathrm{B}(K/\mathbf{Q}, S'', T_v, r') &\overset{(i)}{\Longrightarrow} \mathrm{B}(K/\mathbf{Q}, S \cup \{v_{r+1}, \dots, v_{r'}\}, T_v, r') \\ &\overset{(ii)}{\Longrightarrow} \mathrm{B}(K/\mathbf{Q}, S, T_v, r) \\ &\overset{(iii)}{\Longrightarrow} \mathrm{B}(K/\mathbf{Q}, S, T, r)\,. \end{aligned}$$

This shows that, indeed, Conjecture $\mathrm{B}(K/\mathbf{Q}, S, T, r)$ is true. □

Final Remark. In [7], we formulate a weaker version of Rubin's Conjecture, depending only on the set $(K/k,\, S,\, r)$ (see Conjecture $\mathrm{C}(K/k,\, S,\, r)$, [7], §2.1.) As shown in [7], Theorem 5.5.1(2), we always have an implication

$$\left\{ \begin{array}{c} \mathrm{B}\,(K/k,\, S,\, T,\, r)\ ,\ \text{for all } T, \\ \text{such that } (K/k,\, S,\, T,\, r) \text{ satisfies (H)} \end{array} \right\} \Longrightarrow \mathrm{C}\,(K/k,\, S,\, r)\,.$$

Consequently, Theorem 5.1 above implies that conjecture $\mathrm{C}(K/\mathbf{Q}, S, r)$ is also true, for imaginary abelian fields K of odd prime power conductor.

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