ON THE RUBIN–STARK CONJECTURE FOR A SPECIAL CLASS OF CM EXTENSIONS OF TOTALLY REAL NUMBER FIELDS

Cristian D. Popescu[1]

Abstract. We use Greither's recent results on Brumer's Conjecture to prove Rubin's integral version of Stark's Conjecture, up to a power of 2, for an infinite class of CM extensions of totally real number fields, called "nice extensions". As a consequence, we show that the Brumer–Stark Conjecture is true for "nice extensions", up to a power of 2.

INTRODUCTION

In [12], Rubin formulates an integral version of Stark's general conjecture (see the Main Conjecture in [15]), in the case of abelian L–functions of arbitrary order of vanishing at $s = 0$. The Brumer-Stark Conjecture can be viewed as a particular case of Rubin's statement, restricted to L–functions of order of vanishing 1 at $s = 0$. Brumer's Conjecture is a less precise (and therefore weaker) version of the Brumer-Stark Conjecture. It roughly states that, for an abelian extension K/k of number fields, of Galois group G, the associated Stickelberger ideal is contained in the $\mathbf{Z}[G]$–annihilator of the ideal-class group A_K of K. Brumer's Conjecture is an attempt to extend the classical theorem of Stickelberger, dealing with abelian extensions of $\mathbf{Q}$, to abelian extensions of general base fields.

Although its function field analogue has been settled for quite some time (see [15, Chapitre V] and [6]), Brumer's Conjecture is far from being proved in the number field case. In [5], Greither uses techniques introduced by Wiles in [16] to settle a strong form of Brumer's Conjecture, up to a power of 2, for a special class of CM extensions of totally real fields, which he calls "nice extensions".

In this paper, we use Greither's results to prove Rubin's Conjecture, up to a power of 2, for "nice extensions". By restricting this result to L–functions of order of vanishing 1 at $s = 0$, we prove that the Brumer–Stark Conjecture is true, up to a power of 2, for "nice extensions".

Further results of Greither [4] have recently helped us settle the 2–part of and therefore give a proof for the full conjecture of Rubin for the particular case of "nice extensions" $K/\mathbf{Q}$, where K is an imaginary abelian field of odd prime power conductor. These results will be treated in detail in [11].

The paper is organized as follows. In §1, we introduce the notations and give several general definitions. In §2, we state Rubin's Conjecture and study some of its functoriality properties. In §3, we state the conjectures of Brumer and Brumer–Stark and provide links between these statements and Rubin's Conjecture. In §4,

1991 *Mathematics Subject Classification.* 11R42, 11R58, 11R27.

[1]Research on this project was partially supported by NSF grants DMS–9801267 and DMS–0196340

we prove Rubin's Conjecture, up to a power of 2, for general "nice extensions" (see Theorem 4.2.3). As a consequence, we prove the Brumer-Stark Conjecture, up to a power of 2, for general "nice extensions" (see Corollary 4.2.4).

Acknowledgement. We would like to thank Cornelius Greither for kindly sharing with us his results on "nice extensions" prior to publication, as well as promptly answering all our questions regarding Fitting ideals.

1. PRELIMINARY CONSIDERATIONS AND NOTATIONS

Throughout this paper, K/k will denote a a finite, abelian extension of number fields, of Galois group $G = G(K/k)$. We will denote by μ_K the group of roots of unity in K. w_K and $\mathcal{A}(K/k)$ are the order and respectively the $\mathbf{Z}[G]$–annihilator of μ_K. A_K denotes the ideal–class group of K.

For a prime w of K, we write K_w for the completion of K at w, and $|\cdot|_w : K_w \longrightarrow \mathbf{R}^+ \cup \{0\}$ for the w–absolute value, normalized so that

$$|x|_w = \begin{cases} \pm x \text{ (the usual absolute value)}, & \text{if } K_w = \mathbf{R} \\ x\bar{x}, & \text{if } K_w = \mathbf{C} \\ (\mathbf{N}w)^{-\mathrm{ord}_w(x)}, & \text{if } K_w \text{ is nonarchimedian.} \end{cases}$$

Here $\mathbf{N}w$ denotes the cardinality of the residue field $K(w)$ at w. For a prime v in k, which does not ramify in K/k, σ_v denotes its Frobenius automorphism in G.

Let $\widehat{G}$ be the set of complex valued, irreducible characters of G. For every $\chi \in \widehat{G}$, let $e_\chi = 1/|G| \sum_{\sigma \in G} \chi(\sigma) \cdot \sigma^{-1}$ be the corresponding idempotent in the group–ring $\mathbf{C}[G]$. If M is a $\mathbf{Z}[G]$–module and R is a commutative ring, then $RM := R \otimes_{\mathbf{Z}} M$, $M^* := \mathrm{Hom}_{\mathbf{Z}[G]}(M, \mathbf{Z}[G])$. For every $\chi \in \widehat{G}$, $(\mathbf{C}M)^\chi$ will denote the χ–component of the $\mathbf{C}[G]$–module $\mathbf{C}M$.

Now, let us assume that S is a finite set of primes in k, containing at least all the primes which ramify in K/k and all the infinite primes. Let S_K be the set of primes in K, sitting above primes in S. Then O_S will denote the ring of S_K–integers in K, U_S is the group of S_K–units in K (i.e $U_S := O_S^\times$), and A_S is the ideal–class group of O_S. If T is an auxiliary, nonempty, finite set of primes in k, disjoint from S, we will denote by $U_{S,T}$ the subgroup of finite index in U_S, consisting of elements congruent to 1 modulo every prime in T_K. $A_{S,T}$ denotes the quotient of the group of fractional O_S–ideals of K by the subgroup of principal O_S–ideals which have a generator congruent to 1 modulo every prime in T_K. We have the following exact sequence of $\mathbf{Z}[G]$–modules (see §1.1 in [12]).

$$0 \longrightarrow U_{S,T} \longrightarrow U_S \xrightarrow{j} \bigoplus_{w \in T_K} K(w)^\times \xrightarrow{\xi} A_{S,T} \xrightarrow{\pi} A_S \longrightarrow 0 \tag{1}$$

Here, $j(x) := (x \bmod w;\, w \in T_K)$, for all $x \in U_S$, and $\xi((x_w\,; w \in T_K))$ is the ideal–class of fO_S in $A_{S,T}$, for some $f \in K^\times$ satisfying the equality $j(f) = (x_w\,; w \in T_K)$ in $\oplus_w K(w)^\times$. Finally, $\pi(\widehat{\mathfrak{a}})$ is the class of the O_S–fractional ideal $\mathfrak{a}$ in A_S, for all $\widehat{\mathfrak{a}} \in A_{S,T}$.

For every $\chi \in \widehat{G}$, let $L_{K/k,S}(s,\chi)$ be the Artin L–function associated to χ, with Euler factors at primes in S removed. If $\chi \neq \mathbf{1}_G$, $L_{K/k,S}(s,\chi)$ is holomorphic at every $s \in \mathbf{C}$, while for $\chi = \mathbf{1}_G$, $L_{K/k,S}(s,\chi)$ is holomorphic everywhere except for $s = 1$, where it has a pole of order 1. For $\mathrm{Re}(s) > 1$, one has a product expansion $L_{K/k,S}(s,\chi) = \prod_{v \notin S} (1 - \mathbf{N}v^{-s} \cdot \chi(\sigma_v))^{-1}$, which is uniformly convergent on compact sets. For a set T as above, if one multiplies $L_{K/k,S}(s,\chi)$ by the complex-analytic function $\delta_T(s,\chi) := \prod_{v \in T} (1 - \mathbf{N}v^{1-s} \cdot \chi(\sigma_v))$, one obtains the so called (S,T)–modified L-function associated to χ, given by $L_{K/k,S,T}(s,\chi) := \delta_T(s,\chi) \cdot L_{K/k,S}(s,\chi)$. Since $\delta_T(s,\chi)$ is holomorphic everywhere and $\delta_T(1, \mathbf{1}_G) = 0$, $L_{K/k,S,T}(s,\chi)$ is holomorphic everywhere, for all $\chi \in \widehat{G}$. It is also very important to notice that, for any given χ, since $\delta_T(0,\chi) \neq 0$, the orders of vanishing at $s = 0$ of $L_{K/k,S}(s,\chi)$ and $L_{K/k,S,T}(s,\chi)$ are the same.

For fixed K/k, S and T as above, and for every $\chi \in \widehat{G}$, let $r_{\chi,S}$ be the (common) order of vanishing of $L_{K/k,S}(s,\chi)$ and $L_{K/k,S,T}(s,\chi)$ at $s = 0$. As Tate shows in Chapitre 0 of [15],

$$r_{\chi,S} = \dim_{\mathbf{C}} (\mathbf{C}U_S)^{\chi} = \begin{cases} \mathrm{card}\{v \in S \mid \chi|_{G_v} = \mathbf{1}_{G_v}\}, \text{ if } \chi \neq \mathbf{1}_G \\ \mathrm{card}(S) - 1, \text{ if } \chi = \mathbf{1}_G \end{cases}, \tag{2}$$

where G_v is the decomposition group of v relative to K/k.

Let $\delta_T(s) := \sum_{\chi \in \widehat{G}} \delta_T(s,\chi) e_{\chi^{-1}}$, for all $s \in \mathbf{C}$. The S–Stickelberger and respectively (S,T)–Stickelberger functions are defined by

$$\Theta_{K/k,S}(s) := \sum_{\chi \in \widehat{G}} L_{K/k,S}(s,\chi) \cdot e_{\chi^{-1}}$$

$$\Theta_{K/k,S,T}(s) := \delta_T(s) \cdot \Theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S,T}(s,\chi) \cdot e_{\chi^{-1}} .$$

We think of $\Theta_{K/k,S}$ (respectively $\Theta_{K/k,S,T}$) as a complex meromorphic (respectively holomorphic) function, holomorphic at $s = 0$ and taking values in $\mathbf{C}[G]$. The value $\Theta_{K/k,S}(0)$ satisfies the following most remarkable integrality property, proved independently by Deligne–Ribet [3] and Barsky–Cassou-Nogues [1].

Theorem 1.1. *If $\alpha \in \mathcal{A}(K/k)$, then $\alpha \cdot \Theta_{K/k,S}(0) \in \mathbf{Z}[G]$.*

The next corollary explains why one prefers working with the (S,T)–modified instead of the S–modified L–functions and it also provides a motivation for the hypotheses in Rubin's Conjecture (see §2 below).

Corollary 1.2. *If S and T are two sets of primes as above, and $U_{S,T} \cap \mu_K = \{1\}$, then $\Theta_{S,T}(0) \in \mathbf{Z}[G]$.*

Proof. Clearly, $\delta_T(0) \cdot x \in U_{S,T}$, for all $x \in U_S$ and, in particular, for all $x \in \mu_K$. Under the assumption $U_{S,T} \cap \mu_K = \{1\}$, this implies that $\delta_T(0) \in \mathcal{A}(K/k)$. The desired result now follows from Theorem 1.1. □

2. RUBIN'S CONJECTURE

Let K/k, S and T be as in §1, and let $r \geq 0$ be an integer. Let us assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies the following extended set of hypotheses:

$$\text{(H)} \qquad \begin{cases} S \text{ contains all the infinite primes of } k. \\ S \text{ contains all the primes which ramify in } K/k. \\ S \text{ contains at least } r \text{ primes which split completely in } K/k. \\ |S| \geq r+1. \\ T \neq \emptyset\,, \quad S \cap T = \emptyset\,, \quad U_{S,T} \cap \mu_K = \{1\}\,. \end{cases}$$

Under hypotheses (H), (2) above implies that, for any $\chi \in \widehat{G}$, we have $r_{\chi,S} \geq r$, and therefore $\Theta^{(r)}_{K/k,S,T}(0) := \lim_{s \longrightarrow 0} s^{-r}\Theta_{K/k,S,T}(s)$ makes sense in $\mathbf{C}\,[G]$. If R is a subring of $\mathbf{C}$ and M an $R\,[G]$–module without R–torsion, let

$$M_{r,S} = \{x \in M \mid e_\chi \cdot x = 0 \text{ in } \mathbf{C} \otimes_R M, \forall \chi \in \widehat{G} \text{ such that } r_{\chi,S} > r\}\,.$$

If $r \geq 1$, let us choose an r-tuple $V = (v_1, \ldots, v_r)$ of r distinct primes in S which split completely in K/k, and fix $W = (w_1, \ldots, w_r)$, where w_i is a prime in K lying above v_i, for any $1 \leq i \leq r$. For every $\mathbf{Z}\,[G]$–module M, let $\wedge^r_G M$ denote its r–th exterior power over $\mathbf{Z}\,[G]$. One can define a regulator map

$$\mathbf{C} \wedge^r_G U_{S,T} \xrightarrow{R_W} \mathbf{C}\,[G]\ ,$$

by letting $R_W(u_1 \wedge \cdots \wedge u_r) = \det_{1 \leq i,j \leq r}\left(-\sum_{\sigma \in G} \log |u_j|_{w_i^\sigma} \cdot \sigma\right)$, for $u_1, \ldots, u_r \in U_{S,T}$, and then extending to $\mathbf{C} \wedge^r_G U_{S,T}$ by $\mathbf{C}$–linearity.

If $r = 0$, one defines R_W to be the identity endomorphism of $\mathbf{C} \wedge^0_G U_{S,T} = \mathbf{C}\,[G]$.

Remark 1. As pointed out in Remark 2, §1.6 of [9], R_W is a $\mathbf{C}\,[G]$–morphism, which induces an isomorphism

$$R_W|_{(\mathbf{C}\wedge^r_G U_{S,T})_{r,S}} : (\mathbf{C} \wedge^r_G U_{S,T})_{r,S} \xrightarrow{\sim} \mathbf{C}\,[G]_{r,S}\,.$$

For every $(r-1)$–tuple $(\phi_1, \ldots, \phi_{r-1}) \in \{U^*_{S,T}\}^{r-1}$, one can view each ϕ_i as an element of $\mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}U_{S,T}, \mathbf{C}\,[G]) = \mathbf{C}U^*_{S,T}$, and define a $\mathbf{C}\,[G]$–morphism

$$\mathbf{C} \wedge^r_G U_{S,T} \xrightarrow{\phi_1 \wedge \cdots \wedge \phi_{r-1}} \mathbf{C}U_{S,T}\ ,$$

such that, for every $u_1, \ldots, u_r \in \mathbf{C}U_{S,T}$, one has

$$\phi_1 \wedge \cdots \wedge \phi_{r-1}(u_1 \wedge \cdots \wedge u_r) := \sum_{1 \leq k \leq r} (-1)^{k+1} \det_{\substack{1 \leq i \leq r-1 \\ 1 \leq j \leq r \\ j \neq k}} (\phi_i(u_j)) \cdot u_k\,.$$

In order to simplify notations, in the equality above, and often throughout this paper, we write the internal operation on $\mathbf{C}U_{S,T}$ additively rather than multiplicatively.

Definition 2.1. *Assuming that $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H), let $\Lambda_{S,T}$ be the $\mathbf{Z}\,[G]$–submodule of $\mathbf{Q} \wedge^r_G U_{S,T}$, defined by*

$$\Lambda_{S,T} = \begin{cases} \{\varepsilon \in (\mathbf{Q} \wedge^r_G U_{S,T})_{r,S} \mid \Phi(\varepsilon) \in U_{S,T},\, \forall \Phi \in (U^*_{S,T})^{r-1}\},\ \textit{if } r \geq 1\,, \\ \mathbf{Z}\,[G]_{0,S},\ \textit{if } r = 0\,. \end{cases}$$

In [12], Rubin states the following conjecture, extended to the case of global fields of arbitrary characteristic in [7].

Conjecture $\mathrm{B}(K/k,\, S,\, T,\, r)$ **(Rubin).** *Let us assume that the data* $(K/k, S, T, r)$ *satisfies hypotheses (H). Then, for any choice of* V *and* W*, there exists a unique* $\varepsilon_{S,T,W} \in \Lambda_{S,T}$*, such that* $R_W(\varepsilon_{S,T,W}) = \Theta^{(r)}_{K/k,S,T}(0)$*.*

Remark 2. Let us note that, under the hypotheses above, there always exists a unique element $\varepsilon_{S,T,W}$ in $(\mathbf{C} \wedge^r_G U_{S,T})_{r,S}$, satisfying the above regulator condition (see Remark 1 above, and notice that $\Theta^{(r)}_{K/k,S,T}(0) \in \mathbf{C}\,[G]_{r,S}$). In particular, the uniqueness in conjecture B is automatic.

Remark 3. It is not difficult to show that, given $(K/k,\, S,\, T,\, r)$, if conjecture $\mathrm{B}(K/k,\, S,\, T,\, r)$ is true for a choice of V and W then it is true for any other choice. In fact, one can show that if one has "too much freedom" in choosing V (in the sense that S contains at least $r+1$ primes which split completely in K/k), then $\mathrm{B}(K/k,\, S,\, T,\, r)$ is trivially true in virtue of the S–class number formula. (See [12], §3.1 for a proof of conjecture B in this case.) Also, once V is chosen, the element $\varepsilon_{S,T,W}$ depends in a very simple way of the choice of W. For these reasons, we will suppress V and W from our future notations and denote $\varepsilon_{S,T,W}$ simply $\varepsilon_{S,T}$.

Remark 4. For $r = 0$, Definition 2.1 gives $\Lambda_{S,T} = \mathbf{Z}\,[G]_{0,S}$. Therefore, Corollary 1.2 shows that $\mathrm{B}(K/k, S, T, 0)$ is true.

For $r = 1$, Definition 2.1 gives $\Lambda_{S,T} = (U_{S,T})_{1,S}$. As Proposition 3.4 below shows, under certain hypotheses, Conjecture $\mathrm{B}(K/k, S, T, 1)$, for a fixed K/k, and varying S and T, is equivalent to the Brumer–Stark Conjecture.

In what follows, if R is a subring of $\mathbf{Q}$, we denote by $R\mathrm{B}(K/k,\, S,\, T,\, r)$ the statement obtained if one replaces $\Lambda_{S,T}$ by $R\Lambda_{S,T} := R \otimes_{\mathbf{Z}} \Lambda_{S,T}$ in Rubin's Conjecture, for the set of data $(K/k,\, S,\, T,\, r)$. In this paper, R will typically be either $\mathbf{Z}_{(p)}$ (the localization of $\mathbf{Z}$ at a prime p), $\mathbf{Z}[1/2]$, $\mathbf{Z}[1/|G|]$, or $\mathbf{Q}$.

Remark 5. The uniqueness property emphasized in Remark 2 above shows that, for a given set of data $(K/k,\, S,\, T,\, r)$, satisfying hypotheses (H), one has the following equivalences.

$$\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\,, \text{ for all primes } p.$$
$$\mathbf{Z}[1/2]\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\,, \text{ for all primes } p \neq 2.$$
$$\mathbf{Z}[1/|G|]\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\,, \text{ for all primes } p \nmid |G|.$$

Definition 2.2. *Let* $S \subseteq S'$ *be two finite set of primes in* k*, containing all the infinite primes as well as the primes which ramify in* K/k*. Let* T *be a finite, nonempty set of primes in* k*, such that* $S' \cap T = \emptyset$*. Then,* $A_{S'/S,T}$ *denotes the subgroup of* $A_{S,T}$ *generated by the set of ideal–classes associated to primes in* K *sitting above primes in* S'*.*

One obviously has an exact sequence of $\mathbf{Z}\,[G]$–modules

$$0 \to A_{S'/S,T} \to A_{S,T} \to A_{S',T} \to 0\,. \tag{3}$$

Throughout this paper, if R denotes a Noetherian ring and M a finitely generated R–module, then $\mathrm{Fitt}_R(M)$ denotes the first Fitting ideal of M. For definition and properties of Fitting ideals needed for our purposes, the reader can consult [7], §1.4. The next proposition shows how Rubin's Conjectures depends on S, T, and r.

Proposition 2.3. *Let p be a prime number, and assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). Then, the following hold true.*

(i) *If $S \subseteq S'$ and $(K/k, S', T, r)$ satisfies hypotheses (H), then*

$$\mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r)\,.$$

(ii) *Let $S' := S \cup \{v_{r+1}, \ldots, v_{r'}\}$, with $v_{r+1}, \ldots, v_{r'}$ distinct primes in k, which do not belong to S and split completely in K/k. If $T \cap S' = \emptyset$, then*

$$\mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r') \Longrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\ .$$

(iii) *Under the assumptions and notations of* (ii), *let $\varepsilon_{S,T}$ be the unique element $\varepsilon_{S,T} \in \left(\mathbf{C} \wedge_G^r U_{S,T}\right)_{r,S}$, satisfying $R_W(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0)$. Then*

$$\varepsilon_{S,T} \in \mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{S'/S,T} \otimes \mathbf{Z}_{(p)}) \cdot \mathbf{Z}_{(p)}\Lambda_{S,T} \Longrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r')\,.$$

(iv) *Let S_0 be a finite set of primes in k, containing all the infinite primes as well as those which ramify in K/k. Let $S' = S_0 \cup \{v_1', \ldots, v_{r'}'\}$, $S'' = S_0 \cup \{v_1'', \ldots, v_{r''}''\}$, with $v_1', \ldots, v_{r'}'$ (respectively $v_1'', \ldots, v_{r''}''$) distinct, completely split in K/k, and not in S_0. Assume that $(K/k, S', T, r')$ and $(K/k, S'', T, r'')$satisfy hypotheses (H). Then, if $A_{S'/S_0,T} \otimes \mathbf{Z}_{(p)} = A_{S''/S_0,T} \otimes \mathbf{Z}_{(p)}$, one has an equivalence*

$$\mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r') \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, S'', T, r'')\,.$$

(v) *Let T' be a finite set of primes in k, such that $T \subseteq T'$ and $S \cap T' = \emptyset$. Then*

$$\mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r) \Longrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, S, T', r)\,.$$

Proof. For the proofs of (i), (ii), (iii) see §5.1 in [12]. Please note that Rubin denotes $A_{S'/S,T}$ by $A_{S,S'}$. For the proof of (v), see [9], Proposition 5.3.1 .

We will now prove (iv). Let $\Sigma := S' \cup S''$ and $\sigma := \mathrm{card}(\Sigma \setminus S_0)$. Then, the set of data $(K/k, \Sigma, T, \sigma)$ satisfies hypotheses (H). The equality $A_{S'/S_0,T} \otimes \mathbf{Z}_{(p)} = A_{S''/S_0,T} \otimes \mathbf{Z}_{(p)}$, combined with exact sequence (3) above, shows that $A_{\Sigma/S',T} \otimes \mathbf{Z}_{(p)} = A_{\Sigma/S'',T} \otimes \mathbf{Z}_{(p)} = 0$. This implies that

$$\mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{\Sigma/S',T} \otimes \mathbf{Z}_{(p)}) = \mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{\Sigma/S'',T} \otimes \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}\,[G]\ .$$

These equalities, combined with (ii) and (iii), shows that

$$\mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r') \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, \Sigma, T, \sigma) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}(K/k, S'', T, r'')\,,$$

which concludes the proof of (iv). □

Corollary 2.4. *Let S_0 be a finite set of primes in k, containing all the infinite primes as well as those which ramify in K/k. Let $S = S_0 \cup \{v_1, \ldots, v_r\}$, with $v_1, \ldots, v_r$ distinct, not in S_0, and completely split in K/k. Let T be a finite set of primes in k, such that $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). Then*

$$\Theta_{S_0,T}(0) \in \mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{S_0,T} \otimes \mathbf{Z}_{(p)}) \cdot \mathbf{Z}_{(p)}\,[G]_{0,S_0} \Longrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\ .$$

Proof. Tchebotarev's density theorem allows us to pick $v_{r+1}, \dots, v_{r'}$, mutually distinct primes in k, which are completely split in K/k, do not belong to S, and, if $S' := S \cup \{v_{r+1}, \dots, v_{r'}\}$, then $A_{S',T} \otimes \mathbf{Z}_{(p)} = 0$. The exact sequence (3) implies that

$$A_{S'/S_0,T} \otimes \mathbf{Z}_{(p)} = A_{S_0,T} \otimes \mathbf{Z}_{(p)} .$$

Assume now that $\Theta_{S_0,T}(0) \in \mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{S_0,T} \otimes \mathbf{Z}_{(p)}) \cdot \mathbf{Z}_{(p)}\,[G]_{0,S_0}$. This implies that $\mathbf{Z}_{(p)}\mathrm{B}(K/k, S_0, T, 0)$ is true and that the unique element $\varepsilon_{S_0,T} \in \left(\mathbf{C} \wedge_G^0 U_{S_0,T}\right)_{0,S_0}$, satisfying $R_W(\varepsilon_{S_0,T}) = \Theta^{(0)}_{S_0,T}(0)$, belongs in fact to $\mathrm{Fitt}_{\mathbf{Z}_{(p)}[G]}(A_{S'/S_0,T} \otimes \mathbf{Z}_{(p)}) \cdot \Lambda_{S_0,T}$. Now, we use Proposition 2.3 (iii) to conclude that $\mathbf{Z}_{(p)}\mathrm{B}(K/k, S', T, r')$ is true. Proposition 2.3 (ii) implies that $\mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)$ is true. □

3. THE CONJECTURES OF BRUMER AND BRUMER-STARK. LINKS TO RUBIN'S CONJECTURE

Although the conjectures we are about to state can be formulated for global fields of arbitrary characteristic, we will restrict ourselves to the number field case. A more detailed discussion of these conjectures can be found in [15] and [14]. As in the previous sections, K/k is an abelian extension of number fields, of Galois group G. Let S_0 be a finite set of primes in K/k, containing the infinite primes, as well as those which ramify in K/k.

Conjecture $\mathrm{Br}(K/k, S_0)$ **(Brumer).** *Under the above assumptions, one has an inclusion of* $\mathbf{Z}\,[G]$*–ideals*

$$\mathcal{A}(K/k) \cdot \Theta_{S_0}(0) \subseteq \mathrm{Ann}_{\mathbf{Z}[G]}(A_K)\,.$$

As the reader will realize right away, this conjecture aims at generalizing the classical theorem of Stickelberger (which is precisely the statement above, for $K/k = \mathbf{Q}(\zeta_n)/\mathbf{Q}$) to general abelian extensions of number fields. The left–hand side of the inclusion above is the S_0–Stickelberger ideal associated to K/k.

Remark 1. For every subring R of $\mathbf{Q}$, we denote by $R\mathrm{Br}(K/k, S_0)$ the statement

$$(R \otimes_{\mathbf{Z}} \mathcal{A}(K/k)) \cdot \Theta_{S_0}(0) \subseteq \mathrm{Ann}_{R[G]}\,(A_K \otimes_{\mathbf{Z}} R)\;.$$

Obviously, one has the following equivalences

$$\mathrm{Br}(K/k, S_0) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{Br}(K/k, S_0)\,,\ \text{for all primes } p.$$
$$\mathbf{Z}[1/2]\mathrm{Br}(K/k, S_0) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{Br}(K/k, S_0)\,,\ \text{for all odd primes } p.$$

In what follows, if S is a finite set of primes in k, containing S_0, we denote by $\widetilde{u}$ the image of $u \in U_S$ via the canonical (non–injective) group morphism $U_S \longrightarrow \mathbf{Q}U_S$. Also, if M is a subgroup of U_S, we denote by $\widetilde{M}$ the image of M via this morphism. Obviously, one has a canonical group isomorphism $\widetilde{M} \xrightarrow{\sim} M/M \cap \mu_K$.

Definition 3.1. *For any* S *as above, let*

$$U^{\mathrm{ab}}_{K/k,S} := \{u \in U_S \mid K(u^{1/w_K})/k \textit{ is abelian }\}\,.$$

Obviously, $U^{\mathrm{ab}}_{K/k,S}$ is a $\mathbf{Z}\,[G]$–submodule of U_S. For every prime v in k, $v \notin S$, let $\delta_v := (1 - \mathbf{N}v \cdot \sigma_v^{-1}) \in \mathbf{Z}\,[G]$. For any finite set T of primes in k, $T \cap S = \emptyset$, let $\delta_T := \prod_{v \in T} \delta_v$. Obviously, $\delta_T = \delta_T(0)$, where $\delta_T(s)$ is the complex-holomorphic function defined in §1. A proof of the following lemma can be found in [15], Chap. IV, §1.

Lemma 3.2. *Fix a set S as above. Then $\mathcal{A}(K/k)$ is generated as a $\mathbf{Z}[G]$–module by its subset $\{\delta_T \,|\, T \in \mathcal{T}_S\}$, where $\mathcal{T}_S = \{T \,|\, T$ finite set of primes in k, $T \cap S = \emptyset$, $U_{S,T} \cap \mu_K = \{1\}\}$.*

Lemma 3.3. *Let S be as above, and let $u \in U_S$. Then, $\widetilde{u} \in \widetilde{U^{\mathrm{ab}}_{K/k,S}}$ if and only if one of the following equivalent conditions is satisfied*

I. *There exists $\{u_\alpha \,|\, \alpha \in \mathcal{A}(K/k)\} \subseteq U_S$, such that*

$$u^\alpha = u_\alpha^{w_K}\,, \text{ for all } \alpha \in \mathcal{A}(K/k)$$
$$u_\alpha^\beta = u_\beta^\alpha\,, \text{ for all } \alpha, \beta \in \mathcal{A}(K/k)\,.$$

II. *For all $T \in \mathcal{T}_S$, there are elements $u_T \in U_{S,T}$, such that*

$$u^{\delta_T} = u_T^{w_K}\,.$$

III. *There exists a finite subset $\mathcal{T}$ of $\mathcal{T}_S$, such that $\{\delta_T \,|\, T \in \mathcal{T}\}$ generates $\mathcal{A}(K/k)$ over $\mathbf{Z}[G]$ and there are elements $u_T \in U_{S,T}$, such that*

$$u^{\delta_T} = u_T^{w_K}\,, \text{ for all } T \in \mathcal{T}\,.$$

Proof. For a proof, see [15], Chapitre IV. □

We are now ready to state an equivalent form of the Conjecture of Brumer–Stark.

Conjecture $\mathrm{BrSt}(K/k, S_0)$ **(Brumer–Stark).** *Assume that K/k and S_0 are as above. Let w be a prime in K, not in $(S_0)_K$, sitting above a prime v in K, which splits completely in K/k. Let $S_v := S_0 \cup \{v\}$. Then, there exists a unique $\widetilde{u}_w \in \{\widetilde{U^{\mathrm{ab}}_{K/k,S_v}}\}_{0,S_0}$, such that*

$$w^{\left(w_K \cdot \Theta_{S_0}(0)\right)} = u_w \cdot O_K\,,$$

as fractional O_K–ideals.

Remark 2. For a prime number p, we denote by $\mathbf{Z}_{(p)}\mathrm{BrSt}(K/k, S_0)$ the statement in Conjecture $\mathrm{BrSt}(K/k, S_0)$, restricted to primes w, whose ideal–class $\widehat{w}$ has order a power of p in A_K. Obviously, one has an equivalence

$$\mathrm{BrSt}(K/k, S_0) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{BrSt}(K/k, S_0)\,, \text{ for all primes } p.$$

In light of this, it is sensible to denote by $\mathbf{Z}[1/2]\mathrm{BrSt}(K/k, S_0)$ the statement in Conjecture $\mathrm{BrSt}(K/k, S_0)$, restricted to primes w, whose ideal–class $\widehat{w}$ has odd order in A_K. The analogue of the second equivalence in Remark 1 above obviously holds true in this case as well.

The following results provide the link between Rubin's Conjecture and the Conjecture of Brumer-Stark (see [12], §2.2 as well).

Proposition 3.4. *Let K/k and S_0 be as above. Then, the following are equivalent.*

1. *Conjecture* $\mathrm{B}(K/k, S_v = S_0 \cup \{v\}, T, 1)$ *holds true, for all primes v in k which do not belong to S_0 and split completely in K/k, and all sets $T \in \mathcal{T}_{S_v}$.*
2. *Conjecture* $\mathrm{BrSt}(K/k, S_0)$ *holds true.*

Proof. First, let us assume that (1) holds true. Let w and v be as in the statement of $\mathrm{BrSt}(K/k, S_0)$. Lemma 3.2 allows us to write $w_K = \sum_{T\in\mathcal{T}} a_T\cdot\delta_T$, where $a_T \in \mathbf{Z}[G]$ and the sum is taken with respect to a finite subset $\mathcal{T}$ of $\mathcal{T}_{S_v}$, such that $\{\delta_T \mid T \in \mathcal{T}\}$ generates $\mathcal{A}(K/k)$ over $\mathbf{Z}[G]$. Equality (2) implies right away that $M_{0,S_0} = M_{1,S_v}$, for all $\mathbf{Z}[G]$-modules M. Since $\mathrm{B}(K/k, S_v, T, 1)$ is true, we can find unique elements $\varepsilon_{S_v,T} \in (U_{S_v,T})_{0,S_0}$, such that

$$R_{\{w\}}(\varepsilon_{S_v,T}) = \Theta'_{S_v,T}(0) = -\log \mathrm{N}w\cdot\Theta_{S_0,T}(0)\,,\ \text{for all } T\in\mathcal{T}\,.$$

The definition of $R_{\{w\}}$ shows that the equalities above are equivalent to the following equalities of fractional O_K–ideals.

$$w^{\delta_T\cdot\Theta_{S_0,T}(0)} = \varepsilon_{S_v,T}\cdot O_K\,,\ \text{for all } T\in\mathcal{T}\,. \tag{4}$$

Now, let us consider the element $u \in \{U_{S_v}\}_{0,S_0}$, defined by

$$u_w := \prod_{T\in\mathcal{T}} \varepsilon_{S_v,T}^{a_T}\,.$$

Equalities (4) clearly imply the following equality of fractional O_K–ideals.

$$w^{w_K\cdot\Theta_{S_0}(0)} = u_w\cdot O_K\,. \tag{5}$$

We also claim that $u_w \in \left(U^{\mathrm{ab}}_{K/k,S_v}\right)_{0,S_0}$. Indeed, we have

$$R_{\{w\}}\left(u_w^{\delta_T}/\varepsilon_{S_v,T}^{w_K}\right) = -\log\mathrm{N}w\cdot(\delta_T w_K\cdot\Theta_{S_0}(0) - w_K\delta_T\cdot\Theta_{S_0}(0)) = 0\,.$$

However, $u_w^{\delta_T}/\varepsilon_{S_v,T}^{w_K} \in (U_{S_v,T})_{0,S_0}$ (see the prof of Corollary 1.2). On the other hand, as Remark 1 in§2 shows, $R_{\{w\}}$ is injective on $(\widetilde{U_{S_v,T}})_{0,S_0}$, which is isomorphic to $(U_{S_v,T})_{0,S_0}$. Therefore, the equalities above imply

$$u_w^{\delta_T} = \varepsilon_{S_v,T}^{w_K}\,,\ \text{for all } T\in\mathcal{T}\,.$$

Now, Lemma 3.3(III), implies that $u_w \in \left(U^{\mathrm{ab}}_{K/k,S_v}\right)_{0,S_0}$. This fact, combined with (5) above, shows that $\mathrm{BrSt}(K/k, S_0)$ holds true.

Now, let us assume that $\mathrm{BrSt}(K/k, S_0)$ holds true. Let v, w and T be as in Proposition 3.4 (1). Let $\widetilde{u}_w$ be the unique element in $\{\widetilde{U^{\mathrm{ab}}_{K/k,S_v}}\}_{0,S_0}$, such that

$$w^{\left(w_K\cdot\Theta_{S_0}(0)\right)} = u_w\cdot O_K\,, \tag{6}$$

Then, Lemma 3.2 (III) implies that there exists a unique element $\varepsilon_{S_v,T}$ in $\Lambda_{S_v,T} = (U_{S_v,T})_{0,S_0}$, such that $\varepsilon_{S_v,T}^{w_K} = u_w^{\delta_T}$. Equality (6) above implies that

$$R_{\{w\}}(\varepsilon_{S_v,T}) = \delta_T\cdot(-\log\mathrm{N}w\cdot\Theta_{S_0}(0) = -\log\mathrm{N}w\cdot\Theta_{S_0,T}(0) = \Theta'_{S_0,T}(0)\,.$$

This shows that $\mathrm{B}(K/k, S_v = S_0\cup\{v\}, T, 1)$ holds true. □

Proposition 3.5. *Let K/k and S_0 be as above, and let p be a prime number. Then, the following are equivalent.*

(1) *Conjecture* $\mathbf{Z}_{(p)}\mathrm{B}(K/k, S_v = S_0 \cup \{v\}, T, 1)$ *holds true, for all primes v in k which do not belong to S_0 and split completely in K/k, and all sets $T \in \mathcal{T}_{S_v}$.*
(2) *Conjecture* $\mathbf{Z}_{(p)}\mathrm{BrSt}(K/k, S_0)$ *holds true.*

Proof. Similar to the proof of Proposition 3.4. Left to the reader. □

Remark 3. (see [15], Chapitre IV, as well) Let us notice that $\mathrm{BrSt}(K/k, S_0)$ is a strengthening of $\mathrm{Br}(K/k, S_0)$. Indeed, let us assume that $\mathrm{BrSt}(K/k, S_0)$ is true. Let $\mathfrak{w}$ be an element in A_K. Tchebotarev's density theorem allows us to take a prime w in K, sitting above v in k, such that v splits completely in K/k, v is not in S_0, and $\widehat{w} = \mathfrak{w}$ in A_K. Let $T \in \mathcal{T}_{S_v}$. Proposition 3.4 shows that $\mathrm{B}(K/k, S_v, T, 1)$ holds true. As the arguments in the second part of Proposition 3.4 show, this implies that there exists $\varepsilon_{S_v,T} \in (U_{S_v,T})_{0,S_0}$, such that

$$\mathfrak{w}^{\left(\delta_T \cdot \Theta_{S_0}(0)\right)} = \widehat{\varepsilon_{S_v,T} \cdot O_K} = 0\,,$$

as ideal–classes in A_K. This shows that $\delta_T \cdot \Theta_{S_0}(0)$ is in $\mathrm{Ann}_{\mathbf{Z}[G]}(A_K)$, for all $T \in \mathcal{T}_{S_v}$. Lemma 3.2 now implies that, indeed, $\mathrm{Br}(K/k, S_0)$ is true.

4. CONJECTURE $\mathbf{Z}[1/2]\mathrm{B}(K/k, S, T, r)$ FOR "NICE" EXTENSIONS

The goal of this section is to prove Rubin's Conjecture, up to a power of 2 (i.e. statement $\mathbf{Z}[1/2]\mathrm{B}(K/k, S, T, r)$), for a special class of abelian extensions introduced by Greither in [5] and called "nice" extensions.

4.1. "Nice" extensions. Greither's Theorem. In what follows, we restrict ourselves to abelian extensions K/k of Galois group G, where k is a totally real and K is a CM number field. As usual, K^+ denotes the maximal totally real subfield of K. Obviously, $k \subseteq K^+$. Since K is CM, $\mathrm{Gal}(K/K^+)$ is of order two, generated by the (unique) automorphism j of K induced by the complex conjugation on $\mathbf{C}$. A character $\chi \in \widehat{G}$ is called *odd* (respectively *even*) if $\chi(j) = -1$ (respectively $\chi(j) = +1$.) We will denote by K^{cl} the Galois closure of K over $\mathbf{Q}$. It is easy to check that K^{cl} is also a CM–field. The following definitions are due to Greither.

Definition 4.1.1. *Let $\mathfrak{p}$ be a finite prime in k of residual characteristic p. Then, $\mathfrak{p}$ is called* **critical** *for K/k, if one of the following conditions is satisfied.*

(1) *$\mathfrak{p}$ is ramified in K/k.*
(2) $K^{\mathrm{cl}} \subseteq (K^{\mathrm{cl}})^+(\zeta_p)$.

Definition 4.1.2. *Under the above notations and assumptions, the extension K/k is called* **"nice"***, if the following conditions are simultaneously satisfied.*

(1) *For all critical primes $\mathfrak{p}$ in k, the decomposition group $G_{\mathfrak{p}}$ of $\mathfrak{p}$ in K/k contains j.*
(2) *For all odd primes p, one has* $\gcd(|\mu_K \otimes \mathbf{Z}_{(p)}|, [K : k(\mu_K \otimes \mathbf{Z}_{(p)})]) = 1$, *where $\mu_K \otimes \mathbf{Z}_{(p)}$ is (canonically) identified with the group of p–power roots of unity in K.*

We remind the reader that a $\mathbf{Z}[G]$–module M is called G–cohomologically trivial if $\widehat{\mathrm{H}}^i(H, M) = 0$, for all subgroups H of G and all $i \in \mathbf{Z}$. (Here, $\widehat{\mathrm{H}}^i(H, *)$ denote the

Tate–cohomology functors associated to H.) In [9], §5.4, we introduce the class of **admissible** Galois extensions of global fields K/k, characterized by the fact that μ_K is $Gal(K/k)$–cohomologically trivial. Also, following [9], for a rational prime p, we call K/k p–**admissible** if $\mu_K \otimes \mathbf{Z}_{(p)}$ is $Gal(K/k)$–cohomologically trivial. Since $\mu_K = \oplus_p(\mu_K \otimes \mathbf{Z}_{(p)})$, K/k is admissible if and only if it is p–admissible for all p. In [9, Lemma 5.4.4] we provide the following criterion for p–admissibility.

Lemma 4.1.3. *Let K/k be a Galois extension of number fields, of Galois group G, and let p be a prime number.*

(1) *If p is odd, then K/k is p–admissible if and only if*

$$\gcd(|\mu_K \otimes \mathbf{Z}_{(p)}|,\, [K : k(\mu_K \otimes \mathbf{Z}_{(p)})]) = 1\,.$$

(2) *K/k is 2–admissible if and only if*

$$2 \nmid [K : k(\mu_K \otimes \mathbf{Z}_{(2)})] \ \textit{and} \ \left\{ \begin{array}{c} \textit{if } \mu_K \otimes \mathbf{Z}_{(2)} \neq \mu_k \otimes \mathbf{Z}_{(2)}\,, \ \textit{then} \\ k \cap \mathbf{Q}(\mu_K \otimes \mathbf{Z}_{(2)}) \ \textit{is not a (totally) real field} \end{array} \right\}.$$

An immediate consequence of this Lemma is the following.

Corollary 4.1.4. *An abelian extension of number fields K/k, of Galois group G, with K CM and k totally real, is nice if and only if the following conditions are simultaneously satisfied.*

(1) *For all critical primes $\mathfrak{p}$ in k, the decomposition group $G_{\mathfrak{p}}$ of $\mathfrak{p}$ in K/k contains j.*
(2) *For all odd rational primes p, $\mu_K \otimes \mathbf{Z}_{(p)}$ is G–cohomologically trivial.*

Remark 1. **(a)** Clearly, all abelian extensions of type $K/\mathbf{Q}$, where K is an imaginary number field of prime power conductor ℓ^n, are nice. In particular, extensions of type $\mathbf{Q}(\zeta_{\ell^n})/\mathbf{Q}$, with ℓ prime, are nice. Indeed, in these cases, since $K^{\rm cl} = K$, the only critical prime is ℓ. Since $G_\ell = G$, condition (1) in Definition 4.1.2 above is clearly satisfied. Condition (2) is trivially satisfied for $\mathbf{Q}(\zeta_{\ell^n})/\mathbf{Q}$. Let us now check condition (2) for K a general (imaginary) subfield of $\mathbf{Q}\,(\zeta_{\ell^n})$. The only nontrivial case is checking ℓ–admissibility, for $\ell \neq 2$ and $\mu_K \otimes \mathbf{Z}_{(\ell)} \neq \{1\}$. But in this case K is a subfield of $\mathbf{Q}(\zeta_{\ell^n})$, containing $\mathbf{Q}(\zeta_\ell)$. Therefore, $K = \mathbf{Q}\,(\zeta_{\ell^m})$, with $1 \leq m \leq n$, and K is clearly ℓ–admissible.

(b) We leave it as an exercise for the interested reader to check the following.

(1) If k totally real and ℓ is a prime number, such that ℓ is not ramified in $k/\mathbf{Q}$, then $k(\zeta_{\ell^n})/k$ is nice, for all $n \geq 1$.

(2) Let k be a Galois extension of $\mathbf{Q}$ of odd degree. Let $\mathfrak{a}$ be an ideal in O_k, stable under the action of $\mathrm{Gal}(k/\mathbf{Q})$. Let $\mathrm{Supp}(\mathfrak{a})$ denote the set of residual characteristics of all the primes dividing $\mathfrak{a}$. Let $k\,(\mathfrak{a})$ denote the maximal abelian extension of k, of conductor dividing $\mathfrak{a}$, and of degree coprime to $2\prod' p$, where the product is taken over all primes p in $\mathrm{Supp}(\mathfrak{a})$. Then, $k(\mathfrak{a})(\sqrt{-\prod' p})/k$ is nice.

We will now introduce additional notations, which will remain valid for the rest of this paper. Please note that the notations which follow differ from those in [4] or [5]. Let $R := \mathbf{Z}\,[G]$, $R_- := \mathbf{Z}\,[G]\,/(1+j)$, and $R_+ = \mathbf{Z}\,[G]\,/(1-j)$. For any prime number p, let $R_{(p)} := \mathbf{Z}_{(p)}\,[G]$. By abuse of notation, let

$$\pi : R \twoheadrightarrow R_-\,, \qquad \pi : R_{(p)} \twoheadrightarrow R_{(p),-}$$

be the natural projections. If M is an R–module, let

$$M_- := M \otimes_R R_- \,, \quad M_+ := M \otimes_R R_+ \,,$$

$$M^- := \{x \in M \mid (1+j)\cdot x = 0\}\,, \quad M^+ := \{x \in M \mid (1-j)\cdot x = 0\}\,.$$

Let H be the subgroup of G generated by j. If an R–module N happens to be H–cohomologically trivial, then one clearly has equalities

$$N^- = (1-j)\cdot N\,, \quad N^+ = (1-j)\cdot N\,.$$

For this reason, the equalities above hold true for $N = R$, $N = R_{(p)}$, and $N = M \otimes \mathbf{Z}_{(p)}$, where M is an arbitrary R–module and p is an odd prime.

For an R–module M, since $(1+j) \in \mathrm{Ann}_R(M^-)$, M^- and $\left(M\otimes\mathbf{Z}_{(p)}\right)^-$ come endowed with natural R_-–module and respectively $R_{(p),-}$–module structures. Also, if p is an odd prime, one has a direct sum decomposition

$$\left(M\otimes\mathbf{Z}_{(p)}\right) = \left(M\otimes\mathbf{Z}_{(p)}\right)^- \oplus \left(M\otimes\mathbf{Z}_{(p)}\right)^+ \,.$$

This decomposition, combined with elementary properties of Fitting ideals, gives

$$\mathrm{Fitt}_{R_{(p)}}\left(\left(M\otimes\mathbf{Z}_{(p)}\right)^-\right)^- = \mathrm{Fitt}_{R_{(p)}}\left(M\otimes\mathbf{Z}_{(p)}\right)^- \,. \tag{7}$$

Moreover, since Fitting ideals commute with extensions of scalars (see [)], one has

$$\begin{aligned}\pi^{-1}\left(\mathrm{Fitt}_{R_{(p),-}}\left(\left(M\otimes\mathbf{Z}_{(p)}\right)^-\right)\right) &= \mathrm{Fitt}_{R_{(p)}}\left(\left(M\otimes\mathbf{Z}_{(p)}\right)^-\right) + R_{(p)}^+ \\ &= \mathrm{Fitt}_{R_{(p)}}\left(\left(M\otimes\mathbf{Z}_{(p)}\right)^-\right)^- \oplus R_{(p)}^+ \\ &= \mathrm{Fitt}_{R_{(p)}}\left(M\otimes\mathbf{Z}_{(p)}\right)^- \oplus R_{(p)}^+ \,.\end{aligned} \tag{8}$$

Remark 2. (a) Let p be an odd prime and P the p–Sylow subgroup of G. Let us write $G = P\times\Delta$, where Δ is a subgroup of G, with $p \nmid |\Delta|$. Let $\widehat{\Delta}/\sim$ be the set of equivalence classes of complex-valued characters of Δ, with respect to the usual equivalence relation of characters, given by $\chi\sim\chi'$, if there exists $\sigma\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, such that $\chi = \sigma\circ\chi'$. For any $\chi\in\widehat{\Delta}$, let $\mathbf{Z}_{(p)}[\chi]$ denote the *semi–local principal ideal domain* obtained by adjoining to $\mathbf{Z}_{(p)}$ the values (of an arbitrary representative of the character–class) of χ. One has the well–known direct sum decomposition

$$R_{(p)} \xrightarrow{\sim} \bigoplus_{\chi\in\widehat{\Delta}/\sim} \mathbf{Z}_{(p)}[\chi][P]\,.$$

With respect to the decomposition above, one obviously has

$$R_{(p)}^- \xrightarrow{\sim} \bigoplus_{\substack{\chi\in\widehat{\Delta}/\sim \\ \chi(j)=-1}} \mathbf{Z}_{(p)}[\chi][P]\,, \qquad R_{(p)}^+ \xrightarrow{\sim} \bigoplus_{\substack{\chi\in\widehat{\Delta}/\sim \\ \chi(j)=+1}} \mathbf{Z}_{(p)}[\chi][P]\,.$$

Also, π induces a ring isomorphism between $R^-_{(p)}$ and $R_{(p),-}$. Obviously, $R^+_{(p)}$ and $R_{(p),+}$ are also isomorphic as rings. This shows in particular that $R_{(p)}$, $R^+_{(p)}$, $R^-_{(p)}$, $R_{(p),-}$, and $R_{(p),+}$ are all *semi–local rings* (as direct sums of integral extensions of semi–local rings). Also, since $R_{(p),-}$ is isomorphic to a direct summand of $R_{(p)}$, $R_{(p),-}$ is a *flat* $R_{(p)}$–algebra.

(b) Now, let us consider a finite set S_0 of primes in k, containing all the infinite primes as well as those which ramify in K/k. A celebrated theorem of Minkowsky implies right away that we have $|S_0| \geq 2$. Since for any infinite prime v_∞ in k, one has $j \in G_{v_\infty}$, equality (2) in §1, implies that

$$\chi\left(\Theta_{S_0}(0)\right) = L_{S_0}(0, \chi^{-1}) = 0\,,$$

for all $\chi \in \widehat{\Delta}$, such that $\chi(j) = 1$. According to (b) above, this shows that, for any odd prime number p, we have

$$\big(\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)}\big) \cdot \Theta_{S_0}(0) \subseteq R^-_{(p)}\,.$$

As a direct consequence of the fact that every ramified critical prime in a nice extension has to satisfy condition (1) in Definition 4.1.2, Greither proves the following (see Theorem 2.1 in [5]).

Proposition 4.1.5 (Greither). *If K/k is a nice extension and p is an odd prime, then $\left(A_K \otimes \mathbf{Z}_{(p)}\right)^-$ is G–cohomologically trivial.*

Lemma 4.1.6. *If K/k is a nice extension and p is an odd prime, then*

(1) $\mathrm{pd}_{R_{(p)}}(\mu_K \otimes \mathbf{Z}_{(p)}) \leq 1$.
(2) $\mathrm{pd}_{R_{(p)}}(A_K \otimes \mathbf{Z}_{(p)})^- \leq 1$.

Here, $\mathrm{pd}_{R_{(p)}}(M)$ denotes the $R_{(p)}$–projective dimension of the $R_{(p)}$–module M.

Proof. If $\mathcal{O}$ is a principal ideal domain, an $O[G]$–module M has $O[G]$–projective dimension at most 1 if and only if M is G–cohomologically trivial. (See [9], Proposition 5.2.2.) Therefore, (1) follows from Corollary 4.1.4(2), and (2) follows from Proposition 4.1.5. □

Corollary 4.1.7. *If K/k is a nice extension and p is an odd prime, then*

(1) *$\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)} = a_p \cdot R_{(p)}$, for some non–zerodivisor $a_p \in R_{(p)}$.*
(2) *$\mathrm{Fitt}_{R_{(p)}}\left(\left(A_K \otimes \mathbf{Z}_{(p)}\right)^-\right) = f_p \cdot R_{(p)}$, for some non–zerodivisor $f_p \in R_{(p)}$.*

Proof. In [10] we prove the following result (see [10], Lemma 4.2.6).

Lemma 4.1.8. *Let S denote a commutative, semi–local, Noetherian ring, and let $Q(S)$ be its total ring of fractions. Let M be a finitely generated S–module, such that $M \otimes_S Q(S) = 0$. Assume that $\mathrm{pd}_S(M) \leq 1$. Then, $\mathrm{Fitt}_S(M)$ is a principal ideal, generated by a non-zerodivisor of S.*

The statements in Corollary 4.1.7 now follow by applying the lemma above to $S := R_{(p)}$ and $M := \mu_K \otimes \mathbf{Z}_{(p)}$, respectively $M := (A_K \otimes \mathbf{Z}_{(p)})^-$. Remark 2(a) above shows that, indeed, $S := R_{(p)}$ satisfies the hypotheses in Lemma 4.1.8. Lemma 4.1.6 ensures that both $\mu_K \otimes \mathbf{Z}_{(p)}$ and $(A_K \otimes \mathbf{Z}_{(p)})^-$ have projective dimension at most 1 over $R_{(p)}$. Also, as the reader will notice right away, since $\mu_K \otimes \mathbf{Z}_{(p)}$ is a cyclic $R_{(p)}$–module, $\mathrm{Fitt}_{R_{(p)}}(\mu_K \otimes \mathbf{Z}_{(p)}) = \mathrm{Ann}_{R_{(p)}}(\mu_K \otimes \mathbf{Z}_{(p)}) = \mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)}$. □

Let us denote by S_{00} the set consisting precisely of all the infinite primes in k and all the finite primes in k which ramify in K/k. The main result of [5] (Theorems 4.10 and 4.11) is the following strong form of $\mathbf{Z}[1/2]\mathrm{Br}(K/k, S_0)$, for all finite sets S_0 of primes in k, containing S_{00}.

Theorem 4.1.9 (Greither). *If K/k is a nice extension, then*

$$\pi\left(\left(\mathcal{A}(K/k)\otimes\mathbf{Z}_{(p)}\right)\cdot\Theta_{S_{00}}(0)\right)=\mathrm{Fitt}_{R_{(p),-}}\left(\left(A_K\otimes\mathbf{Z}_{(p)}\right)^-\right),$$

for all odd primes p.

Remark 3. Let us first note that, indeed, the statement above implies conjecture $Z[1/2]\mathrm{Br}(K/k, S_0)$, for all finite sets S_0 of primes in k, containing S_{00}. In order to see this, let us fix such an S_0 and an odd prime p. Then, $\Theta_{S_0}(0)=\prod(1-\sigma_v^{-1})\cdot\Theta_{S_{00}}(0)$, where the product is taken over primes v in $S_{00}\setminus S_0$. Therefore, if we apply π^{-1} to the equality in Theorem 4.1.9 and use (8), we obtain

$$\left(\mathcal{A}(K/k)\otimes\mathbf{Z}_{(p)}\right)\cdot\Theta_{S_0}(0)\subseteq\mathrm{Fitt}_{R_{(p)}}\left(A_K\otimes\mathbf{Z}_{(p)}\right)^-\oplus R_{(p)}^+.$$

However, if we combine the last inclusion with Remark 2(b) above, we obtain

$$\begin{aligned}\left(\mathcal{A}(K/k)\otimes\mathbf{Z}_{(p)}\right)\cdot\Theta_{S_0}(0)&\subseteq\mathrm{Fitt}_{R_{(p)}}\left(A_K\otimes\mathbf{Z}_{(p)}\right)^-\\&\subseteq\mathrm{Fitt}_{R_{(p)}}\left(A_K\otimes\mathbf{Z}_{(p)}\right)\end{aligned}\tag{9}$$

This indeed shows that $\mathbf{Z}_{(p)}\mathrm{Br}(K/k, S_0)$ is true, for all p and S_0 as above. Therefore, $Z[1/2]\mathrm{Br}(K/k, S_0)$ is true.

4.2. The proof of $\mathbf{Z}[1/2]\mathbf{B}(K/k, S, T, r)$. As a consequence of Theorem 4.1.9, we will first prove the following.

Proposition 4.2.1. *Let p be an odd prime, S_0 a finite set of primes in k, containing S_{00}, and T a finite set of primes in k, such that $S_0\cap T=\emptyset$ and $U_{S_0,T}\cap\mu_K=\{1\}$. Then, the following holds true.*

$$\Theta_{S_0,T}(0)\in\left(R_{(p)}\right)_{0,S_0}\cdot\mathrm{Fitt}_{R_{(p)}}\left(A_{S_0,T}\otimes\mathbf{Z}_{(p)}\right).$$

Proof. We will return to the notations of Corollary 4.1.7. We remind the reader that $\Theta_{S_0,T}(0)=\delta_T\cdot\Theta_{S_0}(0)$, where δ_T is defined in §3 (see the paragraph following definition 3.1) and is obviously a non–zerodivisor in R. Please note that, since $\delta_T\in\mathcal{A}(K/k)$ (see the proof of Corollary 3.2) and a_p is a non–zerodivisor $R_{(p)}$–generator of $\mathcal{A}(K/k)\otimes\mathbf{Z}_{(p)}$, the element $\delta_T\cdot a_p^{-1}$ (which a priori only makes sense in $\mathbf{Q}[G]$) is in fact a non–zerodivisor in $R_{(p)}$. The first inclusion in (9) above, combined with equality (7), implies that

$$\Theta_{S_0,T}(0)\in\left(\delta_T\cdot a_p^{-1}\right)\cdot\mathrm{Fitt}_{R_{(p)}}\left(\left(A_K\otimes\mathbf{Z}_{(p)}\right)^-\right)=\left(\delta_T\cdot a_p^{-1}\cdot f_p\right)\cdot R_{(p)}.\tag{10}$$

Now, by definition $\Theta_{S_0,T}(0)\in\left(R_{(p)}\right)_{0,S_0}$ (i.e. $e_\chi\cdot\Theta_{S_0,T}(0)=0$, for all characters $\chi\in\widehat{G}$, such that $r_{\chi,S_0}>0$.) However, since $\delta_T\cdot a_p^{-1}\cdot f_p$ is a non–zerodivisor in

$R_{(p)}$, this means that $e_\chi \cdot (\delta_T \cdot a_p^{-1} \cdot f_p) \neq 0$, for all $\chi \in \widehat{G}$. These facts, combined with (10), show that, in fact we have

(11)

$$\Theta_{S_0,T}(0) \in \left(\delta_T a_p^{-1} f_p\right) \cdot \left(R_{(p)}\right)_{0,S_0} = \left(\delta_T a_p^{-1}\right) \cdot \mathrm{Fitt}_{R_{(p)}}\left(\left(A_K \otimes \mathbf{Z}_{(p)}\right)^-\right) \cdot \left(R_{(p)}\right)_{0,S_0}.$$

Now, let us note that, since for all characters $\chi \in \widehat{G}$, such that $\chi(j) = +1$, we have $r_{\chi,S_0} > 0$ (see Remark 2(b)), Remark 2(a) implies that $\left(R_{(p)}\right)_{0,S_0} \subseteq R_{(p)}^-$. Therefore, if we take into account (7), we have

$$\begin{aligned}\mathrm{Fitt}_{R_{(p)}}\left(\left(A_K \otimes \mathbf{Z}_{(p)}\right)^-\right) \cdot \left(R_{(p)}\right)_{0,S_0} &= \mathrm{Fitt}_{R_{(p)}}\left(\left(A_K \otimes \mathbf{Z}_{(p)}\right)^-\right)^- \cdot \left(R_{(p)}\right)_{0,S_0}\\ &= \mathrm{Fitt}_{R_{(p)}}\left(A_K \otimes \mathbf{Z}_{(p)}\right)^- \cdot \left(R_{(p)}\right)_{0,S_0}\\ &= \mathrm{Fitt}_{R_{(p)}}\left(A_K \otimes \mathbf{Z}_{(p)}\right) \cdot \left(R_{(p)}\right)_{0,S_0}.\end{aligned}$$

The last equality, combined with (11), gives

$$\Theta_{S_0,T}(0) \in \left(\delta_T a_p^{-1}\right) \cdot \mathrm{Fitt}_{R_{(p)}}\left(A_K \otimes \mathbf{Z}_{(p)}\right) \cdot \left(R_{(p)}\right)_{0,S_0}. \tag{12}$$

In order to complete the proof of Proposition 4.2.1, we need to make a remark and prove a lemma, which are valid in a quite general setting.

Remark 1. Assume that K/k is an arbitrary abelian extension of global fields, of Galois group G. Let T be a finite, nonempty set of finite primes in k, unramified in K/k, such that there are no roots of unity in K, congruent to 1 modulo all the primes in T_K. Let $\Delta_T := \bigoplus_{w \in T_K} K(w)^\times$, and view μ_K as a $\mathbf{Z}[G]$–submodule of Δ_T, via (the injective) morphism

$$\xi : \mu_K \longrightarrow \Delta_T,$$

given by $\xi(\zeta) = (\zeta \mod w)_{w \in T_K}$, for all $\zeta \in \mu_K$. Let p be a prime number and assume that $\mu_K \otimes \mathbf{Z}_{(p)}$ is G–cohomologically trivial. Then, the $R_{(p)}$–cyclicity of $\mu_K \otimes \mathbf{Z}_{(p)}$ and Corollary 4.1.7(1) (whose proof only uses the G–cohomological triviality of $\mu_K \otimes \mathbf{Z}_{(p)}$), imply that $\mathrm{Fitt}_{R_{(p)}}\left(\mu_K \otimes \mathbf{Z}_{(p)}\right) = \mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)}$ is an invertible $R_{(p)}$–ideal. Let us denote by $(\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)})^{-1}$ its inverse, viewed as a fractional $R_{(p)}$–ideal in $\mathbf{Q}[G]$. The assumptions made on T imply that $\delta_T \in \mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)}$, and therefore $\delta_T \cdot (\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)})^{-1}$ is an ideal in $R_{(p)}$.

Lemma 4.1.2. *Under the assumptions and notations of Remark 1, we have*

$$\delta_T \cdot (\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)})^{-1} = \mathrm{Fitt}_{R_{(p)}}\left(\Delta_T/\mu_K \otimes \mathbf{Z}_{(p)}\right).$$

Proof of Lemma 4.2.2. First, as remarked in [9], §5.3, we have $\mathbf{Z}[G]$–isomorphisms

$$\Delta_T \xrightarrow{\sim} \bigoplus_{v \in T}\left(K(w)^\times \otimes_{\mathbf{Z}[G_v]} \mathbf{Z}[G]\right) \xrightarrow{\sim} \bigoplus_{v \in T} \mathbf{Z}[G]/(\delta_v).$$

where for each v in T, one picks an arbitrary w in T_K, sitting above v. Since δ_v is a non–zerodivisor of $\mathbf{Z}[G]$, the isomorphisms above imply on one hand that Δ_T and therefore $\Delta_T \otimes \mathbf{Z}_{(p)}$ are G–cohomologically trivial, and on the other hand that

$$\mathrm{Fitt}_R(\Delta_T) = \left(\prod_{v\in T} \delta_v\right) \cdot R = \delta_T \cdot R\,.$$

We have a short exact sequence of $R_{(p)}$–modules

$$0 \longrightarrow \mu_K \otimes \mathbf{Z}_{(p)} \longrightarrow \Delta_T \otimes \mathbf{Z}_{(p)} \longrightarrow \Delta_T/\mu_K \otimes \mathbf{Z}_{(p)} \longrightarrow 0\,.$$

Since two terms of this sequence are G–cohomologically trivial, so is the third. Therefore, Lemma 4.9 in [5] (see part b) of the proof), implies that we have an equality

$$\mathrm{Fitt}_{R_{(p)}}\left(\mu_K \otimes \mathbf{Z}_{(p)}\right) \cdot \mathrm{Fitt}_{R_{(p)}}\left(\Delta_T/\mu_K \otimes \mathbf{Z}_{(p)}\right) = \mathrm{Fitt}_{R_{(p)}}\left(\Delta_T \otimes \mathbf{Z}_{(p)}\right)\,.$$

Now, we multiply the last equality by $\left(\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)}\right)^{-1}$ (or, equivalently, by $\left(\mathrm{Fitt}_{R_{(p)}}\left(\mu_K \otimes \mathbf{Z}_{(p)}\right)\right)^{-1}$, and obtain

$$\mathrm{Fitt}_{R_{(p)}}\left(\Delta_T/\mu_K \otimes \mathbf{Z}_{(p)}\right) = \delta_T \cdot (\mathcal{A}(K/k) \otimes \mathbf{Z}_{(p)})^{-1}\,.$$

This concludes the proof of Lemma 4.1.2. □

Now, we return to the Proof of Proposition 4.2.1. The definition of A_{S_0} implies that we have a surjective $R_{(p)}$–morphism

$$A_K \otimes \mathbf{Z}_{(p)} \twoheadrightarrow A_{S_0} \otimes \mathbf{Z}_{(p)}\,.$$

This yields the following inclusion of Fitting ideals (see [7], §1.4).

$$\mathrm{Fitt}_{R_{(p)}}\left(A_K \otimes \mathbf{Z}_{(p)}\right) \subseteq \mathrm{Fitt}_{R_{(p)}}\left(A_{S_0} \otimes \mathbf{Z}_{(p)}\right)\,. \tag{13}$$

On the other hand, since $\mu_K \cap U_{S_0,T} = \{1\}$, (1) in §1 gives the following exact sequence of $R_{(p)}$–modules

$$\Delta_T/\mu_K \otimes \mathbf{Z}_{(p)} \longrightarrow A_{S_0,T} \otimes \mathbf{Z}_{(p)} \longrightarrow A_{S_0} \otimes \mathbf{Z}_{(p)} \longrightarrow 0\,.$$

This, combined with the behaviour of Fitting ideals in short exact sequences (see [7], §1.4) and Lemma 4.2.2, yields the following inclusion of $R_{(p)}$–ideals.

$$(\delta_T a_p^{-1}) \cdot \mathrm{Fitt}_{R_{(p)}}\left(A_{S_0} \otimes \mathbf{Z}_{(p)}\right) \subseteq \mathrm{Fitt}_{R_{(p)}}\left(A_{S_0,T} \otimes \mathbf{Z}_{(p)}\right)\,. \tag{14}$$

We now combine (12), (13), and (14) to obtain

$$\Theta_{S_0,T}(0) \in \left(R_{(p)}\right)_{0,S_0} \cdot \mathrm{Fitt}_{R_{(p)}}\left(A_{S_0,T} \otimes \mathbf{Z}_{(p)}\right)\,,$$

which concludes the proof of Proposition 4.2.1. □

We are now ready to state and prove the main theorem of §4.

Theorem 4.2.3. *let K/k be a nice extension and assume that the set of data $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). Then, conjecture* $\mathbf{Z}[1/2]\mathrm{B}(K/k,\, S,\, T,\, r)$ *is true.*

Proof. Let $v_1, v_2, \cdots, v_r$ be r distinct primes in S, which split completely in K/k. Since K is a CM field and k is totally real, none of the infinite primes in k split completely in K/k. Therefore, the set $S_0 := S \setminus \{v_1, v_2, \cdots, v_r\}$ satisfies the hypotheses in Proposition 4.2.1. Proposition 4.2.1 and Corollary 2.4 imply that conjecture $\mathbf{Z}_{(p)}\mathrm{B}(K/k,\, S,\, T,\, r)$ is true, for all odd primes p. Consequently, the second equivalence in Remark 5, §2, shows that conjecture $\mathbf{Z}[1/2]\mathrm{B}(K/k,\, S,\, T,\, r)$ is true. □

Now, as a consequence of the above theorem, we settle the Brumer–Stark Conjecture, up to a power of 2, for nice extensions.

Corollary 4.2.4. *Let K/k be a nice extension. Let S_0 be a finite set of primes in k, containing all the infinite primes of k as well as those which ramify in K/k. Then, conjecture* $\mathbf{Z}[1/2]\mathrm{BrSt}(K/k, S_0)$ *is true.*

Proof. Theorem 4.2.3, for $r = 1$ and sets $S_v := S_0 \cup \{v\}$, with v not in S_0 and completely split in K/k, together with Proposition 3.5, imply the desired result. □

Remark 2. Assume that K/k is a nice extension and $(K/k,\, S,\, T,\, r)$ satisfies hypotheses (H). In [9], we formulate a weaker version of Rubin's Conjecture, depending only on the set $(K/k,\, S,\, r)$ (see Conjecture $\mathrm{C}(K/k,\, S,\, r)$, [9], §2.1.) As shown in [9], Theorem 5.5.1, if the extension K/k is p–admissible for a prime p (i.e $\mu_K \otimes \mathbf{Z}_{(p)}$ is G–cohomologically trivial), then, one has an equivalence

$$\mathbf{Z}_{(p)}\mathrm{C}\,(K/k,\, S,\, r) \Longleftrightarrow \mathbf{Z}_{(p)}\mathrm{B}\,(K/k,\, S,\, T,\, r)\ ,\ \text{ for all } T \in \mathcal{T}_S\,.$$

As remarked in Corollary 4.1.4, nice extensions are p–admissible, for all odd primes p. Therefore, Theorem 4.2 implies that conjecture $\mathbf{Z}[1/2]\mathrm{C}\,(K/k,\, S,\, r)$ (i.e. conjecture $\mathbf{Z}_{(p)}\mathrm{C}(K/k,\, S,\, r)$, for all odd primes p) is also true for nice extensions K/k.

References

[1] Cassou-Nogues, Pierrette, *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p–adiques*, Invent. Math. **51** (1979), 29–59.

[2] Cornacchia, P. and Greither, C., *Fitting ideals of class groups of real fields with prime power conductor*, J. Number Thry. **73** (1998), 459-471.

[3] Deligne, P. and Ribet, K., *Values of abelian L-functionsat negative integers over totally real fields*, Invent. Math. **59** (1980), 227–286.

[4] Greither, C., *The structure of some minus class groups and Chinburg's third cojecture for abelian fields*, Math. Zeitschr. **229** (1998), 107-136.

[5] Greither, C., *Some cases of Brumer's conjecture for abelian CM extensions of totally real fields*, Math. Zeitschr. **233** (2000), 515–534.

[6] Hayes, D.R., *Stickelberger elements in function fields*, Comp. Math. **55** (1985), 209–239.

[7] Popescu, C.D., *On a refined Stark conjecture for function fields*, Comp. Math. **116** (1999), 321–367.

[8] Popescu, C.D., *Gras–type conjectures for function fields*, Comp. Math. **118** (1999), no. 3, 263–290.

[9] Popescu, C.D., *Base change for Stark–type conjectures "over* **Z***"*, to appear in J. Reine Angew. Math (2001).

[10] Popescu, C.D., *Stark's Question and a strong form of Brumer's Conjecture*, to appear in Compositio Math.

[11] Popescu, C.D., *The Rubin–Stark Conjecture for imaginary abelian fields of odd prime power conductor*, in preparation.

[12] Rubin, K., *A Stark conjecture "over* **Z***" for abelian L-functions with multiple zeros*, Annales de L'Institut Fourier **46** (1996), 33–62.
[13] Stark, H., *L-functions at $s = 1$, I, II, III, IV*, Adv. in Math. **7** (1971), 301–343; **17** (1975), 60–92; **22** (1976), 64–84; **35** (1980), 197–235.
[14] Tate, J., *Brumer–Stark–Stickelberger*, Séminaire de Théorie des Nombres, Univ. de Bordeaux (1980–1981), exp. 24.
[15] Tate, J., *Les conjectures de Stark sur les fonctiones L d'Artin en $s = 0$*, Progr. in Math. **47** (1984), Boston Birkhäuser.
[16] Wiles, A., *On a conjecture of Brumer*, Ann. of Math. **131** (1990), 555–565.

Johns Hopkins University, Department of Mathematics, 3400 N. Charles Street, Baltimore, MD, 21218, USA
E-mail address: cpopescu@math.jhu.edu