SPECIAL VALUES OF ABELIAN L-FUNCTIONS AT $S = 0$

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Abstract. In [12], Stark formulated his far-reaching refined conjecture on the first derivative of abelian (imprimitive) L–functions of order of vanishing $r = 1$ at $s = 0$. In [10], Rubin extended Stark's refined conjecture to describe the r-th derivative of abelian (imprimitive) L-functions of order of vanishing r at $s = 0$, for arbitrary values r. However, in both Stark's and Rubin's setups, the order of vanishing is imposed upon the imprimitive L–functions in question somewhat artificially, by requiring that the Euler factors corresponding to r distinct completely split primes have been removed from the Euler product expressions of these L–functions. In this paper, we formulate and provide evidence in support of a conjecture in the spirit of and extending the Rubin-Stark Conjectures to the most general (abelian) setting: arbitrary order of vanishing abelian imprimitive L-functions, regardless of their type of imprimitivity. The second author's conversations with Harold Stark and David Dummit (especially regarding the order of vanishing 1 setting) were instrumental in formulating this generalization.

1. Introduction and notation

In a series of papers published in the 1970s and early 1980s, culminating in [12], Stark developed the programme which is now widely known as "Stark's conjectures." The purpose is to extract information on arithmetic invariants of global field extensions K/k from special values of the associated Artin L-functions. Stark's original *(refined) integral* conjecture [12] predicted an arithmetic formula for the first derivative of an abelian S–imprimitive L-function at $s = 0$ under the presence in the set S of primes whose Euler factors "are missing" of a distinguished prime v_0 which splits completely in K/k. In [10], Rubin presented a conjecture which extended Stark's to the r^{th} derivative under the presence of r splitting primes in S. In [6], the second author introduced a modification of Rubin's conjecture which behaved more naturally under "base change". Previous work by Dummit, Hayes, Sands, and Tangedal (see *e.g.* [2], [3]) and their discussions with Stark lead Stark in 2001 to proposing the *extended first order abelian Stark question*—an extension of Stark's original integral conjecture which dropped the requirement of the distinguished splitting prime v_0. This question was investigated by Erickson in [5]. The aim of this work is to formulate and provide evidence for a conjecture in the spirit of and extending the Rubin-Stark Conjectures to the most general (abelian) setting: arbitrary order of vanishing abelian imprimitive L-functions, regardless of their type of imprimitivity. The second author is responsible for the statement of the conjecture, which was subsequently investigated by the first in [4]. The conjecture is developed in §§2-3. In §4, we study its various functoriality properties as well as its links to the Rubin-Stark conjectures. In §5, we provide some evidence in its support.

1991 *Mathematics Subject Classification.* 11R42, 11R58, 11R27.
Key words and phrases. Global L-functions, Regulators, Units, Class Groups.
The second author was partially supported by NSF Grant DMS-0600905.

Fix a finite, abelian extension of global fields K/k, and let G be its Galois group. Let $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$. For every $\chi \in \widehat{G}$, $\mathbf{e}_\chi$ is the corresponding idempotent in the group algebra $\mathbb{C}[G]$. Let S and T denote finite sets of places of k. The sets of all places in K dividing places in S and T will be denoted S_K and T_K respectively. Let $U_{S,T}$ denote the group consisting of all the S_K-units in K which are congruent to 1 modulo every prime in T_K. Let $U_{S,T}^* = \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])$ be the dual group of $U_{S,T}$. For every prime $v \in S$ we fix once and for all a prime $w(v) \in S_K$ sitting above v and denote by G_v its corresponding decomposition group in K/k (which is independent of the choice of $w(v)$ because G is abelian). For every place w in K, $| \cdot |_w$ denotes the absolute value associated to w, normalized in the canonical way (so that the product formula holds for K.) We call a pair (S, T) *appropriate* for K/k if S and T are finite, nonempty, disjoint sets of places of k such that

$$(H) \quad \left\{ \begin{array}{l} (1)\ S \text{ contains all the archimedian and all the } K/k\text{-ramified primes.} \\ (2)\ U_{S,T} \text{ has no } \mathbb{Z}\text{-torsion.} \end{array} \right\}$$

For any set E, $|E|$ will denote the cardinality of E. If r is a nonnegative integer, $\wp(E)$ will denote the power set of E, and $\wp_r(E)$ will denote the set of subsets of E of exact cardinality r. If A is a $\mathbb{Z}$-module, $\mathbb{C}A := \mathbb{C} \otimes_{\mathbb{Z}} A$, and $\mathbb{Q}A := \mathbb{Q} \otimes_{\mathbb{Z}} A$.

2. The L-functions, covering sets, and orders of vanishing

For any $\chi \in \widehat{G}$, let

$$L_{S,T}(\chi, s) := \prod_{v \in T} \left(1 - \chi(\sigma_v^{-1}) \mathbf{N}v^{1-s}\right) \cdot L_S(\chi, s)$$

where $L_S(\chi, s)$ is the usual S-incomplete ($\mathbb{C}$-valued, meromorphic) Artin L-function attached to χ. As in [10] and [8], we define the G-equivariant (S, T)-modified L-function by

$$\Theta_{S,T} : \mathbb{C} \longrightarrow \mathbb{C}[G], \quad \Theta_{S,T}(s) := \sum_{\chi \in \widehat{G}} L_{S,T}(\chi, s) \cdot \mathbf{e}_{\chi^{-1}} .$$

This function takes values in $\mathbb{C}[G]$ and is holomorphic everywhere in $\mathbb{C}$. For every natural number r, we let $\Theta_{K/k,S,T,r} = \frac{1}{r!}\Theta^{(r)}_{K/k,S,T}(0)$ denote the r–th Taylor coefficient of $\Theta_{S,T}(s)$ at $s = 0$. The main goal of Stark's conjectural programme is to extract the arithmetic information encoded in the first non-vanishing element in this infinite list of Taylor coefficients (the so-called leading term of $\Theta_{S,T}(s)$ at $s = 0$.)

For every character $\chi \in \widehat{G}$, we let

$$r_S(\chi) = \mathrm{ord}_{s=0} L_{S,T}(\chi, s)$$

be the order of vanishing of the corresponding L-function at $s = 0$. This is a nonnegative integer, which is easily seen to be independent of T. It is well known that

$$r_S(\chi) = \left\{ \begin{array}{ll} \mathrm{card}\{v \in S \,|\, \chi(G_v) = \{1\}\}, & \text{if } \chi \neq \mathbf{1}_G \ ; \\ \mathrm{card}\, S - 1, & \text{if } \chi = \mathbf{1}_G. \end{array} \right.$$

(see, *e.g.*, [13, Proposition 3.4]). In what follows, we let $r_S(K/k) := \min_\chi r_S(\chi)$.

Let S' be a subset of S, Π be a subset of $\widehat{G}$ and r be a nonnegative integer. In light of the above equalities, we give the following.

Definition 2.1. *We say that S' is an r-cover for Π if the following two conditions are satisfied:*
(1) For all $\chi \in \Pi$, there exist (at least) r distinct primes $v \in S'$, such that $\chi(G_v) = \{1\}$.
(2) If the trivial character $\mathbf{1}_G$ belongs to Π, then $|S'| \geq r + 1$.

Note that if S is an r-cover for $\widehat{G}$, then $r_S(\chi) \geq r_S(K/k) \geq r$, for all $\chi \in \widehat{G}$. In particular, this happens if, for example, S contains r distinct primes which split completely in K/k and $|S| \geq r+1$ (which is precisely the hypothesis in Rubin's conjecture [10], for arbitrary r, or Stark's refined conjecture [12], in the case where $r = 1$.)

Lemma 2.2. *If S is an r-cover for $\widehat{G}$ and $|S| = r + 1$, then S contains at least r primes which split completely in K/k (i.e. S has to satisfy the hypotheses in Rubin's conjecture.)*

Proof. Using the factorization of the $(S,T)-$zeta function of K into L-functions and counting the orders of zeros at $s = 0$ yields

$$\sum_{\substack{\chi \in \widehat{G} \\ \chi \neq \mathbf{1}_G}} r_S(\chi) = \sum_{v \in S} (g_v - 1) \tag{1}$$

where $g_v = |G|/|G_v|$ is the number of primes of K above v. Now, since $r \leq r_S(\chi)$ for all χ, $r(|G| - 1) \leq \sum_{v \in S}(g_v - 1)$. Suppose, for the sake of contradiction, that S contained two primes v_1, v_2 which did not split completely in K/k. That is to say $g_{v_1}, g_{v_2} < |G|$, and hence (as g_v divides $|G|$), $\frac{g_{v_i}-1}{|G|-1} < \frac{1}{2}$ for $i = 1, 2$. Then we have

$$r \leq \sum_{v \in S} \frac{g_v - 1}{|G| - 1} < \frac{1}{2} + \frac{1}{2} + (r - 1) = r,$$

a contradiction. Therefore such v_1 and v_2 do not exist, and at least r primes of S split in K/k. □

For every r, we let $\widehat{G}_{r,S} := \{\chi \in \widehat{G} \mid r_S(\chi) = r\}$. Note that $\mathbf{1}_G \in \widehat{G}_{r,S}$ if and only if $|S| = r + 1$.

Lemma 2.3. *Let r be a natural number. Assume that S is an r-cover for $\widehat{G}$. If $S', S'' \subset S$ are r-covers for $\widehat{G}_{r,S}$ (respectively $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$), then their intersection $S' \cap S''$ is also an r-cover for $\widehat{G}_{r,S}$ (respectively $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$).*

Proof. Let $\chi \in \widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$. Then S' and S'' contain two subsets of cardinality r, say $\{v'_1, \ldots, v'_r\}$ and $\{v''_1, \ldots, v''_r\}$, respectively, which are r-covers of $\{\chi\}$ (meaning that $\chi(G_{v'_i}) = \chi(G_{v''_i}) = \{1\}$, for all $i = 1, \ldots, r$.) However, since $\chi \in \widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$, χ is trivial when restricted to the decomposition groups of exactly r primes in S. This shows that $\{v'_1, \ldots, v'_r\} = \{v''_1, \ldots, v''_r\} \subseteq S' \cap S''$. Therefore $S' \cap S''$ is an r–cover of $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$.

If $\mathbf{1}_G \in \widehat{G}_{r,S}$, then $|S| = r + 1$. Therefore, if S' and S'' are r-covers of $\widehat{G}_{r,S}$, then $S' = S'' = S$, so $S' \cap S'' = S$ is also an r-cover of $\widehat{G}_{r,S}$. □

Definition 2.4. *Let r be a natural number. Assume that S is an r-cover of $\widehat{G}$. We let*

$$S_{\min} = \bigcap S',$$

where $S' \subseteq S$ runs over all r-covers for $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$ contained in S. (By the previous lemma, this set is the unique minimal r-cover for $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\}$ and it depends on both S and r.)

Examples. (1) Assume that S contains (at least) r distinct primes which split completely in K/k and that $|S| \geq r+1$. Then S is an r–cover for the entire $\widehat{G}$, as mentioned before. If $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\} \neq \emptyset$, then S has to contain exactly r distinct primes which split completely, say $\{v_1, \dots, v_r\}$. Clearly, in this case we have $S_{\min} = \{v_1, \dots, v_r\}$. On the other hand, if $\widehat{G}_{r,S} \setminus \{\mathbf{1}_G\} = \emptyset$, then S contains more than r primes which split completely and $S_{\min} = \emptyset$. In particular, $\widehat{G}_{r,S} = \{\mathbf{1}_G\}$ if and only if $|S| = r+1$ and all primes in S split completely in K/k.

Also, it is very important to note that if G *is cyclic,* then S is an r–cover for $\widehat{G}$ if and only if G contains (at least) r distinct primes which split completely in K/k and $|S| \geq r+1$. Indeed, this is a consequence of the fact that a generator χ of $\widehat{G}$ (which is a faithful character of G) has to be trivial if restricted to the decomposition groups of at least r distinct primes in S, rendering those groups trivial.

(2) In this example, $r = 1$. Let p and q be two odd prime numbers, satisfying

$$p \equiv q \equiv 1 \mod 4\,, \quad \left(\frac{p}{q}\right) = 1\,.$$

Let $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-p}, \sqrt{-q})$. We ask the reader to check that $S = \{\infty, 2, p, q\}$ is a 1–cover for $\widehat{G}$. Please note that in this case S consists precisely of the primes which ramify and therefore no prime in S splits completely in K/k. In this case, $\widehat{G}_{1,S} = \widehat{G}_{1,S} \setminus \{\mathbf{1}_G\} = \{\chi_p, \chi_q\}$, where χ_p and χ_q are the two nontrivial (quadratic) characters of $G(\mathbb{Q}(\sqrt{-p})/\mathbb{Q})$ and $G(\mathbb{Q}(\sqrt{-q})/\mathbb{Q})$, respectively. It is an easy exercise to show that $S_{\min} = \{p, q\}$, in this case.

(3) In this example, $r = 1$. Let p and q be two odd prime numbers, satisfying

$$p \equiv 1 \mod 4\,, \quad q \equiv 3 \mod 4\,, \quad \left(\frac{p}{q}\right) = 1\,.$$

Let $K' := \mathbb{Q}(\zeta_q)^{D_p}$, where $\zeta_q := e^{2\pi i/q}$ and D_p is the decomposition group associated to p in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. Please note that $\mathbb{Q}(\sqrt{-q}) \subseteq K'$. Let K'^{+} be the maximal real subfield of K'. Let l be an odd prime number, different from p and q and satisfying

$$\left(\frac{l}{K'/\mathbb{Q}}\right) \neq 1\,, \quad \left(\frac{l}{K'^{+}/\mathbb{Q}}\right) = 1\,, \quad \left(\frac{p}{l}\right) = -1$$

(i.e. l splits completely in $K'^{+}/\mathbb{Q}$, but it does not split completely in $K'/\mathbb{Q}$ and $\mathbb{Q}(\sqrt{p})/\mathbb{Q}$.) Let $k := \mathbb{Q}$ and $K := K'(\sqrt{p})$. We ask the reader to check that $S := \{\infty, p, q, l\}$ is a 1–cover for $\widehat{G}$. Obviously, none of the primes in S splits completely in K/k. Let χ be a generator of $\widehat{G(K'/\mathbb{Q})}$ and ψ_p the generator of $\widehat{G(\mathbb{Q}(\sqrt{p})/\mathbb{Q})}$. Then, it is not hard to see that

$$\widehat{G}_{1,S} = \widehat{G}_{1,S} \setminus \{\mathbf{1}_G\} = \{\chi^i \mid i \text{ odd }\} \cup \{\psi_p \cdot \chi^i \mid i \neq 0\}\,,$$

where i runs through the obvious range. Also, one easily shows that $S_{\min} = \{\infty, p, l\}$.

3. Regulator maps, evaluators and lattices. The conjecture.

In this section and what follows, (S, T) is an appropriate pair for the abelian extension of global fields K/k, whose Galois group is denoted by G. Also, we assume that S is an r-cover for $\widehat{G}$, for some (fixed) positive integer r. Throughout the rest of the paper, all exterior powers are viewed over the ring $\mathbb{Z}[G]$, unless otherwise specified. For a $\mathbb{Z}[G]$-module $\mathcal{M}$ with no $\mathbb{Z}$-torsion, let

$$\mathcal{M}_{r,S} := \left\{ m \in \mathcal{M} \mid \mathbf{e}_\chi \cdot m = 0 \text{ in } \mathbb{C}\mathcal{M}, \text{ for all } \chi \notin \widehat{G}_{r,S} \right\} .$$

Note that $\mathbb{C}[G]_{r,S} = \mathbb{C}[G] \cdot \Theta_{K/k,S,T,r}$, by definition. In particular, $\mathbb{C}[G]_{r,S} = 0$ if only if $\Theta_{K/k,S,T,r} = 0$, if and only if S is an $r+1$ cover of $\widehat{G}$, if and only if $\widehat{G}_{r,S} = \emptyset$, if and only if $r_S(K/k) > r$.

Now, we introduce and fix an order on the set S. In particular, this induces an order on every subset I of S. If $\widehat{G}_{r,S} = \{\mathbf{1}_G\}$ (and this can happen if and only if S consists precisely of $r+1$ completely split primes – see Example 1 in §2), we let $I(S) := \{v_1, v_2, \dots, v_r\}$, assuming that $v_1 < v_2 < \cdots < v_r < v_{r+1}$ are the elements of the (ordered) set S. For any $I \subseteq S$ of cardinality r, we define a $\mathbb{C}[G]$-linear regulator map $R_I : \mathbb{C} \wedge^r U_{S,T} \to \mathbb{C}[G]$, by setting

$$R_I(u_1 \wedge \ldots \wedge u_r) := \det_{\substack{v \in I \\ 1 \le j \le r}} \left(\frac{1}{|G_v|} \sum_{\sigma \in G} \log \left| u_j^{\sigma^{-1}} \right|_{w(v)} \cdot \sigma \right) ,$$

for all $u_1, \dots, u_r \in U_{S,T}$ and then extending by $\mathbb{C}$-linearity. Finally, we define the regulator map

$$\mathcal{R} := \mathcal{R}_{r,S} := \begin{cases} \sum_{I \in \wp_r(S_{\min})} R_I, & \text{if } \widehat{G}_{r,S} \neq \{\mathbf{1}_G\}; \\ R_{I(S)}, & \text{if } \widehat{G}_{r,S} = \{\mathbf{1}_G\}, \end{cases}$$

where the summation over all the subsets of cardinality r of $S_{\min}$ is by definition equal to 0 if $S_{\min} = \emptyset$. Consequently, $\mathcal{R} = 0$ if $r < r_S(K/k)$ (i.e. if S is an $(r+1)$–cover of $\widehat{G}$.)

Remark 3.1. *Note that the discussion in Example 1 of §2 shows that if S contains r primes which split completely $v_1 < v_2 < \dots < v_r$ and $I(S) = \{v_1, \dots, v_r\}$ if $\widehat{G}_{r,S} = \{\mathbf{1}_G\}$, then the map $\mathcal{R}$ defined above is equal to the regulator map R_W defined by Rubin in* [10]*, for $W = (w(v_1), w(v_2), \dots, w(v_r))$.*

Proposition 3.2. *The map $\mathcal{R}$ gives a $\mathbb{C}[G]$-isomorphism $(\mathbb{C} \wedge^r U_{S,T})_{r,S} \xrightarrow{\cong} (\mathbb{C}[G])_{r,S}$.*

Proof. First of all, let us note that if S contains r primes which split completely, then the proposition above is a direct consequence of [10, Lemma 2.7]. Therefore, in light of Lemma 2.2, we may assume that $|S| > r+1$. Consequently, $\mathbf{1}_G \notin \widehat{G}_{r,S}$ and $\mathcal{R} = \sum_{I \in \wp_r(S_{\min})} R_I$. Also, we may assume that $r = r_S(K/k)$, otherwise the proposition is trivially true, as $\mathcal{R} = 0$ and $(\mathbb{C} \wedge^r U_{S,T})_{r,S} = (\mathbb{C}[G])_{r,S} = 0$. For the proof of Proposition 3.2, we will need some additional definitions and auxiliary lemmas.

Let Y_S be the $\mathbb{Z}[G]$–module of divisors of K supported above S_K (i.e. the free abelian group generated by S_K, endowed with the obvious G–action.) Let X_S be the $\mathbb{Z}[G]$–submodule of Y_S consisting of all divisors of degree 0. We have an exact sequence of $\mathbb{C}[G]$–modules

$$0 \longrightarrow \mathbb{C}X_S \longrightarrow \mathbb{C}Y_S \xrightarrow{\deg} \mathbb{C} \longrightarrow 0 ,$$

where deg is the degree map (extended by $\mathbb{C}$–linearity) and the last non–zero module to the right is endowed with the trivial G–action. Since $\mathbf{1}_G \notin \widehat{G}_{r,S}$, the exact sequence above implies that we have the following equalities

$$(\mathbb{C}X_S)_{r,S} = (\mathbb{C}Y_S)_{r,S} \text{ and } (\mathbb{C} \wedge^r X_S)_{r,S} = (\mathbb{C} \wedge^r Y_S)_{r,S} .$$

Note that $Y_S = \oplus_{v \in S} \mathbb{C}[G] \cdot w(v)$. Also, it is obvious that for all $v \in S$, we have a $\mathbb{C}[G]$–module isomorphism $\mathbb{C}[G] \cdot w(v) \xrightarrow{\cong} \mathbb{C}[G]/\mathcal{I}_{G_v}$, where $\mathcal{I}_{G_v}$ is the relative augmentation ideal associated to

G_v (generated as an ideal of $\mathbb{Z}[G]$ by the set $\{\sigma - 1 \mid \sigma \in G_v\}$.) Consequently, we have

$$\mathbb{C}\wedge^r Y_S = \bigoplus_{I\in\wp_r(S)} \mathbb{C}[G]\cdot W_I \xrightarrow{\cong} \bigoplus_{I\in\wp_r(S)} \mathbb{C}[G]/\mathcal{I}_{D_I}\,. \tag{2}$$

Here, $W_I := w(v_1)\wedge\cdots\wedge w(v_r)$, assuming that $I = \{v_1, v_2, \dots, v_r\}$ and $v_1 < \cdots < v_r$. Also, D_I is the subgroup of G generated by $\cup_{v\in I} G_v$ and $\mathcal{I}_{D_I}$ is its relative augmentation ideal. Note that $\mathcal{I}_{D_I} = \sum_{v\in I}\mathcal{I}_{G_v}$.

It is well known (see [13]) that we have a $\mathbb{C}[G]$–linear isomorphism

$$\mathcal{L}_S : \mathbb{C}U_{S,T} \xrightarrow{\cong} \mathbb{C}X_S\,, \qquad \mathcal{L}_S(u) = \sum_{v\in S}\ell_{w(v)}(u)\cdot w(v)\,,$$

where $\ell_{w(v)}(u) = (1/|G_v|)\sum_{\sigma\in G}\log|u^{\sigma^{-1}}|_{w(v)}\cdot\sigma$, for all $u \in U_{S,T}$. As a direct consequence of the definitions, the link between $\mathcal{L}_S$ and the regulators R_I defined earlier is the following.

$$(\wedge^r\mathcal{L}_S)(z) = \sum_{I\in\wp_r(S)} R_I(z)\cdot W_I\,, \text{ for all } z\in\mathbb{C}\wedge^r U_{S,T}\,, \tag{3}$$

where $\wedge^r\mathcal{L}_S : \mathbb{C}\wedge^r U_{S,T} \longrightarrow \mathbb{C}\wedge^r X_S \subseteq \mathbb{C}\wedge^r Y_S$ is the usual r–th exterior power of $\mathcal{L}_S$.

The following equalities follow from the functional equation and the nonvanishing at $s = 1$ of the corresponding L–functions (see [13]).

$$\dim_{\mathbb{C}}(\mathbf{e}_\chi\cdot\mathbb{C}U_{S,T}) = \dim_{\mathbb{C}}(\mathbf{e}_\chi\cdot\mathbb{C}X_S) = r_S(\chi), \text{ for all } \chi\in\widehat{G}\,.$$

Consequently, since under the current hypotheses $r = r_S(K/k)$ and $\mathbf{1}_G \notin \widehat{G}_{r,S}$, the $\mathbb{C}[G]_{r,S}$–modules $(\mathbb{C}\wedge^r X_S)_{r,S} = (\mathbb{C}\wedge^r Y_S)_{r,S}$ and $(\mathbb{C}\wedge^r U_{S,T})_{r,S}$ are free of rank 1 over $\mathbb{C}[G]_{r,S} = \oplus_{\chi\in\widehat{G}_{r,S}}\mathbb{C}\cdot\mathbf{e}_\chi$. The following Lemma provides a natural basis for the 1–dimensional $\mathbb{C}[G]_{r,S}$–module $(\mathbb{C}\wedge^r X_S)_{r,S}$.

Lemma 3.3. *Assume that S is an r–cover for $\widehat{G}$, $|S| > r+1$ and $r = r_S(K/k)$. Then*

1. *A set $S_0 \subseteq S$ is an r–cover for $\widehat{G}_{r,S}$ if and only if*
$$(\mathbb{C}\wedge^r X_S)_{r,S} = (\mathbb{C}\wedge^r Y_S)_{r,S} = \mathbb{C}[G]_{r,S}\cdot W_{S_0}\,,$$
where $W_{S_0} := \sum_{I\in\wp_r(S_0)} W_I$.
2. *In particular, $e_{r,S}\cdot W_{S_{\min}}$ is a $\mathbb{C}[G]_{r,S}$–basis for $(\mathbb{C}\wedge^r X_S)_{r,S}$, where $e_{r,S} := \sum_{\chi\in\widehat{G}_{r,S}}\mathbf{e}_\chi$.*

Proof. Since $(\mathbb{C}\wedge^r Y_S)_{r,S}$ is free of rank one over $\mathbb{C}[G]_{r,S}$, the element $e_{r,S}\cdot W_{S_0}$ is a basis of this space if and only if we have $\mathbf{e}_\chi\cdot W_{S_0}\neq 0$, for all $\chi\in\widehat{G}_{r,S}$. However, based on (3), this happens if and only if, for all $\chi\in\widehat{G}_{r,S}$, there exists an $I_\chi\in\wp_r(S_0)$, such that $\mathbf{e}_\chi\cdot W_{I_\chi}\neq 0$. On the other hand, this last non–equality happens if and only if $\mathbf{e}_\chi\cdot\mathcal{I}_{D_{I_\chi}} = 0$, which happens if and only if $\mathbf{e}_\chi\cdot I_{G_v} = 0$, for all all $v\in I_\chi$. Finally, this happens if and only if $\chi(G_v) = \{1\}$, for all $v\in I_\chi$, which means that I_χ is an r–cover for $\{\chi\}$ and consequently that S_0 is an r–cover for $\widehat{G}_{r,S}$. This concludes the proof of Lemma 3.3. □

Lemma 3.4. *Under the hypotheses of Lemma 3.3, if $z\in(\mathbb{C}\wedge^r U_{S,T})_{r,S}$, then*

$$(\wedge^r\mathcal{L}_S)(z) = \mathcal{R}(z)\cdot W_{S_{\min}}\,.$$

Proof. Lemma 3.3 implies that for any z as above, we have

$$(\wedge^r \mathcal{L}_S)(z) \in \mathbb{C}[G]_{r,S} \cdot W_{S_{\min}} \subseteq \oplus_{I \in \wp_r(S_{\min})} \mathbb{C}[G]_{r,S} \cdot W_I\,.$$

Consequently, (3) shows that $(\wedge^r \mathcal{L}_S)(z) = \sum_{I \in \wp_r(S_{\min})} R_I(z) \cdot W_I$. However, note that since G_v fixes $w(v)$, we have $\ell_{w(v)}(z) \in N_{G_v} \cdot \mathbb{C}[G]_{r,S}$, for all $v \in S$, where $N_{G_v} = \sum_{\sigma \in G_v} \sigma$. Consequently,

$$R_I(z) \in \prod_{v \in I} N_{G_v} \cdot \mathbb{C}[G]_{r,S}\,, \text{ for all } I \in \wp_r(S)\,.$$

Now, it suffices to show that if $\alpha_I \in \prod_{v\in I} N_{G_v} \cdot \mathbb{C}[G]_{r,S}$, for all $I \in \wp_r(S_{\min})$, then

$$\sum \alpha_I \cdot W_I = (\sum \alpha_I) \cdot (\sum W_I) = (\sum \alpha_I) \cdot W_{S_{\min}}\,,$$

where I runs through $\wp_r(S_{\min})$ and the second equality above is obvious by the definition of $W_{S_{\min}}$. The first equality above can be shown character-by-character. Indeed due to the fact that $\alpha_I \in \mathbb{C}[G]_{r,S}$, for all I, it suffices to show that

$$\mathbf{e}_\chi \cdot (\sum \alpha_I \cdot W_I) = \mathbf{e}_\chi \cdot (\sum \alpha_I) \cdot (\sum W_I)\,, \text{ for all } \chi \in \widehat{G}_{r,S}\,.$$

Let $\chi \in \widehat{G}_{r,S}$. Since $r_S(\chi) = r$ and $\chi \neq \mathbf{1}_G$, there exists a unique $I_\chi \in \wp_r(S_{\min})$, such that $\chi(G_v) = \{1\}$, for all $v \in I_\chi$. Obviously, the set I_χ also satisfies $\chi(G_v) \neq \{1\}$, for all $v \notin I_\chi$. This implies right away that $\chi(N_{G_v}) = 0$, for all $v \notin I_\chi$ and also that $\chi(\mathcal{I}_{D_I}) \neq 0$, for all $I \neq I_\chi$. Consequently, we have $\mathbf{e}_\chi \cdot \alpha_I = 0$ and $\mathbf{e}_\chi \cdot W_I = 0$, for all $I \neq I_\chi$ (recall (3)). Therefore, both sides of the last displayed equality are equal to $\mathbf{e}_\chi \cdot \alpha_{I_\chi} \cdot W_{I_\chi}$. This concludes the proof of the Lemma. □

Now, Proposition 3.2 is a direct consequence of Lemma 3.3, Lemma 3.4 and of the fact that $(\wedge^r \mathcal{L}_S)$ induces (by restriction) an isomorphism from $(\mathbb{C} \wedge^r U_{S,T})_{r,S}$ to $(\mathbb{C} \wedge^r X_S)_{r,S}$.

□

In light of Proposition 3.2, we can make the following definition.

Definition 3.5. *Assuming that (S,T) is appropriate for K/k and S is an r–cover for $\widehat{G}$, we let*

$$\varepsilon_{K/k,S,T,r} := \mathcal{R}^{-1}\left(\Theta_{K/k,S,T,r}\right)$$

Remark 3.6. *Sometimes we refer to $\varepsilon_{K/k,S,T,r}$ as an (L–function) evaluator, because evaluating the regulator $\mathcal{R}$ against it gives the special value $\Theta_{K/k,S,T,r}$ at $s = 0$ of the equivariant L-function $\Theta_{K/k,S,T}(s)$. Note that $\varepsilon_{K/k,S,T,r} = 0$ if and only if $r_S(K/k) > r$. Also, note that, if S contains r primes which split completely, then $\varepsilon_{K/k,S,T,r}$ is precisely the evaluator $\varepsilon_{K/k,S,T}$ defined by Rubin in* [10] *(a direct consequence of the fact that under these hypotheses, our regulator and Rubin's coincide, as remarked earlier.)*

As in [10], we define $\mathbb{C}[G]$–linear pairing $\mathbb{C} \wedge^r U^*_{S,T} \times \mathbb{C} \wedge^r U_{S,T} \to \mathbb{C}[G]$ by setting

$$(\phi_1 \wedge \cdots \wedge \phi_r)(u_1 \wedge \cdots \wedge u_r) = \det_{1 \leq i,j \leq r}(\phi_i(u_j)).$$

for all $\phi_1, \dots, \phi_r \in U^*_{S,T}$ and all $u_1, \dots, u_r \in U_{S,T}$, and then extending by $\mathbb{C}$-linearity. Following [10] we also define the following $\mathbb{Z}[G]$–submodule of finite rank (lattice) of $(\mathbb{Q} \wedge^r U_{S,T})_{r,S}$.

Definition 3.7.

$$\Lambda_{S,T,r} := \left\{ z \in (\mathbb{Q} \wedge^r U_{S,T})_{r,S} \;\middle|\; \phi(z) \in \mathbb{Z}[G] \text{ for all } \phi \in \wedge^r U^*_{S,T} \right\}.$$

We are now ready to formulate our extension of the Rubin-Stark conjecture.

Conjecture 3.8. $\widetilde{B}(K/k,S,T,r)$. *Assume that (S,T) is an appropriate pair for K/k and that S is an r–cover for $\widehat{G}$. Then*

$$\varepsilon_{K/k,S,T,r}\in\Lambda_{S,T,r}\,.$$

Remark 3.9. *Note that if S contains r primes which split completely in K/k (e.g. if $|S|=r+1$, see Lemma 2.2) then, in light of Remark 2, the conjecture above is equivalent to Rubin's Conjecture $B(K/k,S,T,r)$ (see* [10]*.) In particular, in the exceptional case where $\widehat{G}_{r,S}=\{\mathbf{1}_G\}$, the conjecture is true as a direct consequence of Dirichlet's S–class number formula for k, as proved by Rubin in* [10, Proposition 3.1]*. Also, note that if $r_S(K/k)>r$, the conjecture above is trivially true, with $\varepsilon_{K/k,S,T,r}=0$.*

We conclude this section with a couple of very useful formulas for the regulator $\mathcal{R}$.

Lemma 3.10. *Assume that $|S|>r+1$. Let $\chi\in\widehat{G}_{r,S}$ and $I_\chi\in\wp_r(S)$ be the unique subset, such that $\chi(G_v)=\{1\}$, for all $v\in I_\chi$. Then, for all $z\in(\mathbb{C}\wedge^r U_{S,T})_{r,S}$, we have*

(1) $\mathcal{R}(z)=\sum_{I\in\wp_r(S)}R_I(z)\,,$ *for all* $z\in(\mathbb{C}\wedge^r U_{S,T})_{r,S}\,.$
(2) $\mathcal{R}(\mathbf{e}_\chi\cdot z)=R_{I_\chi}(\mathbf{e}_\chi\cdot z)\,.$

Proof. It suffices to show that if $I\in\wp_r(S)$ and $I\neq I_\chi$, then $R_I(\mathbf{e}_\chi\cdot z)=0$, for all $\chi\in\widehat{G}_{r,S}$. Obviously, since $\chi\neq\mathbf{1}_G$ (as $|S|>r+1$), we also have $\chi(G_v)\neq\{1\}$, for all $v\notin I_\chi$. Therefore, there is a $v\in I\setminus I_\chi$, such that $\chi(G_v)\neq\{1\}$ or, equivalently, $\chi(N_{G_v})=0$. The proof of Lemma 3.4 shows that $R_I(\mathbf{e}_\chi\cdot z)=\mathbf{e}_\chi\cdot R_I(z)\in\prod_{v\in I}\chi(N_{G_v})\cdot\mathbb{C}\mathbf{e}_\chi=0$. This concludes the proof. □

4. Functoriality results

Throughout this section we assume that (S,T) is an appropriate pair for the abelian extension K/k of Galois group G. Also, we assume that S is an r–cover for $\widehat{G}$. In light of Remark 3 above, we will assume that $r=r_S(K/k)$ and $|S|>r+1$. Throughout, we let $\varepsilon_{K/k}:=\varepsilon_{K/k,S,T,r}$. The main goal of this section is to study various functoriality properties of conjecture $\widetilde{B}(K/k,S,T,r)$ as well as its links to Rubin's conjecture for various intermediate field extensions M/k, with $M\subseteq K$. Note that for any such M, the pair (S,T) is appropriate for M/k and S is an r–cover for $\widehat{G(M/k)}$. So conjecture $\widetilde{B}(M/k,S,T,r)$ makes perfect sense. In order to "align" our regulator maps properly, we make the following convention: for every $v\in S$, the chosen prime $w'(v)$ sitting above v in M is precisely the prime sitting below $w(v)$ (recall that $w(v)$ is the chosen prime in K sitting above v.) Since we will be dealing with a variety of top fields K, M etc. (while the bottom field k remains fixed), we distinguish between the various regulators computed at the level of these top fields by incorporating the relevant top field as a superscript in the regulator notation: $\mathcal{R}^K$, $\mathcal{R}^M$, R_I^K, R_I^M, etc.

Fix a field M intermediate to K/k. Let $H=G(K/M)$, viewed as a subgroup of $G=G(K/k)$. We identify $\Gamma:=G/H$ with the Galois group $G(M/k)$. Let $N_H=N_{K/M}:=\sum_{h\in H}h\in\mathbb{C}[G]$ be the algebraic norm attached to H. We abuse notation and denote by $N_H=N_{K/M}:U_{K,S,T}\to U_{M,S,T}$ the norm map at the level of groups of units as well. This induces a $\mathbb{C}[G]$–linear map

$$N_H^{(r)}=N_{K/M}^{(r)}:=\wedge^r N_H:\mathbb{C}\wedge^r U_{K,S,T}\to\mathbb{C}\wedge^r U_{M,S,T}\,.$$

Note that since $\mathbb{C}[G]$ is a semisimple ring, $\mathbb{C}U_{M,S,T}$ is a direct summand of $\mathbb{C}U_{K,S,T}$ (in the category of $\mathbb{C}[G]$–modules), and consequently we have a natural inclusion of $\mathbb{C}[G]$–modules

$$\mathbb{C}\wedge^r U_{M,S,T}\hookrightarrow\mathbb{C}\wedge^r U_{K,S,T}\,.$$

Let $\pi_{K/M}:\mathbb{C}[G]\to\mathbb{C}[\Gamma]$ denote the natural projection and $\pi^*_{K/M}:\mathbb{C}[\Gamma]\to\mathbb{C}[G]$ be the $\mathbb{C}$–linear coprojection map given by

$$\sigma H\mapsto\frac{\sigma N_{K/M}}{|H|}\,,\ \text{for all }\sigma\in G\,.$$

Obviously, $\pi\circ\pi^*(x)=x$ and $\pi^*\circ\pi(x)=(N_H/|H|)\cdot x$. It is important to note that for any $\chi\in\widehat{\Gamma}$,

$$\pi^*_{K/M}(\mathbf{e}_\chi)=\mathbf{e}_{\chi\circ\pi_{K/M}} \tag{4}$$

and

$$\frac{N_{K/M}}{|H|}=\sum_{\substack{\chi\in\widehat{G}\\ \chi|_H=\mathbf{1}_G}}\mathbf{e}_\chi. \tag{5}$$

Also, there is a natural conorm map $N^*_{K/M}:U^*_{K,S,T}\to U^*_{M,S,T}$, given by

$$\phi\mapsto N^*_{K/M}\phi\ \text{where}\ (N^*_{K/M}\phi)(u)=\frac{1}{[K:M]}\pi_{K/M}\circ\phi(u)\,,\ \text{for all }\phi\in U^*_{K,S,T}\,.$$

Remark 4.1. *It is not very difficult to see that since $U_{K,S,T}$ and $U_{M,S,T}$ have no $\mathbb{Z}$–torsion, the conorm map $N^*_{K/M}$ is surjective* [8, Lemma 4.1.2]*. Also, a straightforward calculation shows that*

$$\phi(u)=|H|\cdot\pi^*_{K/M}\circ N^*_{K/M}\phi(u)\,,\ \textit{for all }u\in U_{M,S,T}\ \textit{and all }\phi\in U^*_{K,S,T}\,.$$

Proposition 4.2. *With notations as above, assume that $I\in\wp_r(S)$. Then, the following hold for all $x\in\mathbb{C}\wedge^r U_{M,S,T}$ and $z\in\mathbb{C}\wedge^r U_{K,S,T}$.*

1. $R^K_I(x)=|H|^r\cdot\pi^*_{K/M}\circ R^M_I(x)$ *and* $\mathcal{R}^K(x)=|H|^r\cdot\pi^*_{K/M}\circ\mathcal{R}^M(x)$.
2. $R^M_I(N^{(r)}_{K/M}z)=\pi_{K/M}\circ R^K_I(z)$ *and* $\mathcal{R}^M(N^{(r)}_{K/M}z)=\pi_{K/M}\circ\mathcal{R}^K(z)$

Proof. This is a direct consequence of Lemma 3.10 and the equalities

$$\ell_{w(v)}(x)=|H|\cdot\pi^*_{K/M}\circ\ell_{w'(v)}(x)\,,\ \text{for all }x\in\mathbb{C}U_{M,S,T}$$

$$\ell_{w'(v)}(N_{K/M}x)=\pi_{K/M}\circ\ell_{w(v)}(x)\ \text{for all }x\in\mathbb{C}U_{K,S,T}$$

where $v\in S$, $w(v)$ is the chosen prime in K sitting above v and $w'(v)$ is the prime in M sitting below $w(v)$. The equalities above are consequences of $|u|^{1/|G_v|}_{w(v)}=|u|^{1/|\Gamma_v|}_{w'(v)}$, for all $u\in U_{M,S,T}$. □

Corollary 4.3. *With notations as above, we have*

$$\varepsilon_{M/K,S,T,r}=N^{(r)}_{K/M}(\varepsilon_{K/k,S,T,r})\,.$$

Proof. First, observe that the $\mathbb{C}[G]$–linear map $N^{(r)}_{K/M}$ maps $(\mathbb{C}\wedge^r U_{K,S,T})_{r,S}$ to $(\mathbb{C}\wedge^r U_{M,S,T})_{r,S}$. Now, the corollary is a direct consequence of the injectivity of $\mathcal{R}^M$ when restricted to this latter space, Proposition 4.2, part (2), and the inflation property of Artin L–functions which implies that $\pi_{K/M}(\Theta_{K/k,S,T,r})=\Theta_{M/k,S,T,r}$. □

Lemma 4.4. *For any $z\in\mathbb{C}\wedge^r U_{K,S,T}$, and $\phi_1,\dots,\phi_r\in U^*_{K,S,T}$, we have*

$$\pi_{K/M}(\phi_1\wedge\ldots\wedge\phi_r)(z)=((N^*_{K/M}\phi_1)\wedge\ldots\wedge(N^*_{K/M}\phi_r))(N^{(r)}_{K/M}z).$$

Proof. By $\mathbb{C}$-linearity, it suffices to assume $z = u_1 \wedge \ldots \wedge u_r$. Then

$$\begin{aligned}
\pi_{K/M}(\phi_1 \wedge \ldots \wedge \phi_r)(u_1 \wedge \ldots \wedge u_r) &= \pi_{K/M} \det(\phi_i(u_j)) \\
&= \det\left(\pi_{K/M} \frac{N_{K/M}}{[K:M]} \phi_i(u_j)\right) \\
&= \det\left(\frac{1}{[K:M]} \pi_{K/M} \phi_i(N_{K/M} u_j)\right) \\
&= \det\Big((N^*_{K/M}\phi_i)(N_{K/M}u_j)\Big) \\
&= ((N^*_{K/M}\phi_1) \wedge \ldots \wedge (N^*_{K/M}\phi_r))(N^{(r)}_{K/M} z)
\end{aligned}$$

as desired. □

Theorem 4.5. *With notations as above, we have*

$$\widetilde{B}(K/k, S, T, r) \Rightarrow \widetilde{B}(M/k, S, T, r)\,.$$

Proof. According to Corollary 4.3, all we have to check is that $N^{(r)}_{K/M}(\varepsilon_{K/k,S,T,r}) \in \Lambda_{M/k,S,T}$, assuming that $\varepsilon_{K/k,S,T,r} \in \Lambda_{K/k,S,T}$. Let $\varphi_1, \ldots, \varphi_r \in U^*_{M,S,T}$. Recalling that that the map $N^*_{K/M}$ is surjective, pick $\phi_1, \ldots, \phi_r \in U^*_{K,S,T}$ such that $N^*_{K/M}\phi_i = \varphi_i$. Using Lemma 4.4, we compute

$$\begin{aligned}
(\varphi_1 \wedge \ldots \wedge \varphi_r)(\varepsilon_{M/k,S,T,r}) &= (\varphi_1 \wedge \ldots \wedge \varphi_r)(N^{(r)}_{K/M}\varepsilon_{K/k,S,T,r}) \\
&= \pi_{K/M}\Big((\phi_1 \wedge \ldots \wedge \phi_r)(\varepsilon_{K/k,S,T,r})\Big) \\
&\in \pi_{K/M}\mathbb{Z}[G] = \mathbb{Z}[\Gamma].
\end{aligned}$$

□

Lemma 4.6. *Assume that (S, T) is appropriate for K/k and S is an r–cover for $\widehat{G}$. Then for any finite sets S' and T', such that $S \subseteq S'$, $T \subseteq T'$ and $S' \cap T' = \emptyset$, we have the following.*

(1) $\varepsilon_{K/k,S',T',r} = \prod_{v \in T' \setminus T}(1 - \mathbf{N}v \cdot \sigma_v^{-1}) \cdot \prod_{v \in S' \setminus S}(1 - \sigma_v^{-1}) \cdot \varepsilon_{K/k,S,T,r}$

(2) $\widetilde{B}(K/k, S, T, r) \Rightarrow \widetilde{B}(K/k, S', T', r)\,.$

Proof. The proof is identical to that of the corresponding Lemma for Rubin's conjecture (see [10].) □

The abelian extension K/k has a number of distinguished subfields. For each character $\chi \in \widehat{G}$, we have K_χ, the fixed field of the kernel of χ. Note that exactly $r_S(\chi)$ primes of S split completely in the extension K_χ/k. Hence for those characters of minimal order of vanishing r we have a Rubin evaluator $\varepsilon_\chi := \varepsilon_{K_\chi/k,S,T,r}$. In what follows, we let $\varepsilon_{K/k} := \varepsilon_{K/k,S,T,r}$.

Proposition 4.7. *With notations as above,*

$$\varepsilon_{K/k} = \sum_{\chi \in \widehat{G}_{r,S}} \frac{1}{|\ker \chi|^r} \mathbf{e}_\chi \varepsilon_\chi. \tag{6}$$

Proof. First, note that both sides in the equality above belong to $(\mathbb{C} \wedge^r U_{K,S,T})_{r,S}$. Since $\mathcal{R}^K$ is injective when restricted to this space, it suffices to show that $\mathcal{R}^K$ applied to the right hand side is equal to $\Theta_{K/k,S,T,r}$. We prove this one character at a time. Fix a $\chi \in \widehat{G}_{r,S}$. Let $\pi := \pi_{K/K_\chi}$ and let $\widetilde{e}_\chi \in \mathbb{C}[G(K_\chi/k)]$ be the idempotent of χ, viewed as a character of $G(K_\chi/k)$. Let z denote the right hand side of the equality in the statement above. First, we observe that $\pi^*(\widetilde{\mathbf{e}_\chi}) = \mathbf{e}_\chi$. Next, we use Proposition 4.2 and the inflation property of Artin L–functions to compute.

$$\begin{aligned}
\mathcal{R}^K(z)\mathbf{e}_\chi &= \frac{1}{|\ker\chi|^r}\mathcal{R}^K(\varepsilon_\chi)\mathbf{e}_\chi \\
&= \pi^*(\mathcal{R}^{K_\chi}(\varepsilon_\chi))\mathbf{e}_\chi \\
&= \pi^*\left(\Theta_{K_\chi/k,S,T,r}\right)\mathbf{e}_\chi \\
&= \pi^*\left(\widetilde{\mathbf{e}_\chi}\cdot\Theta_{K_\chi/k,S,T,r}\right)\mathbf{e}_\chi \\
&= \pi^*\left(\widetilde{\mathbf{e}_\chi}\cdot\frac{1}{r!}L^{(r)}_{K_\chi/k,S,T}(0,\chi^{-1})\right)\mathbf{e}_\chi \\
&= \frac{1}{r!}L^{(r)}_{K/k,S,T}(0,\chi^{-1})\mathbf{e}_\chi \\
&= \Theta_{K/k,S,T,r}\mathbf{e}_\chi.
\end{aligned}$$

This concludes the proof of the Proposition. ☐

Now, we shift viewpoint and instead of looking at the primes split in K_χ, for a given $\chi \in \widehat{G}_{r,S}$, we fix a set of primes and look at the subextension in which they split completely. Recall that $\wp_r(S_{\min})$ denotes the set of all subsets of $S_{\min}$ of cardinality r. For each $I \in \wp_r(S_{\min})$, let $D_I = \langle G_v\rangle_{v\in I}$ (the subgroup of G generated by the decomposition groups of the primes in I) and let $K_I := K^{D_I}$. Note that every $v \in I$ splits completely in K_I/k. Putting $\varepsilon_I := \varepsilon_{K_I/k,S,T,r}$, an alternative description of the evaluator $\varepsilon_{K/k}$ is given by the following.

Proposition 4.8. *With notations as above, we have*

$$\varepsilon_{K/k} = \sum_{I\in\wp_r(S_{\min})}\frac{1}{|D_I|^r}\varepsilon_I\,. \tag{7}$$

Proof. For any I, we compute

$$\begin{aligned}
\tfrac{1}{|D_I|^r}\varepsilon_I &= \frac{1}{|D_I|^r}\frac{N_{D_I}}{|D_I|}\varepsilon_I \\
&= \frac{1}{|D_I|^r}\sum_{\substack{\chi\in\widehat{G}\\ D_I\subseteq\ker\chi}}\mathbf{e}_\chi\varepsilon_I \\
&= \frac{1}{|D_I|^r}\sum_{\substack{\chi\in\widehat{G}\\ D_I\subseteq\ker\chi}}\mathbf{e}_\chi\frac{N^{(r)}_{K_I/K_\chi}}{[\ker\chi : D_I]^r}\varepsilon_I \\
&= \sum_{\substack{\chi\in\widehat{G}\\ D_I\subseteq\ker\chi}}\frac{1}{|\ker\chi|^r}\mathbf{e}_\chi\varepsilon_\chi
\end{aligned}$$

where the first equality holds because elements of D_I fix ε_I, the second is equation (5), the third holds because $\chi(N_{K_I/K_\chi}) = [\ker\chi : D_I]$ for those χ whose kernels contain D_I. Therefore

$$\sum_{I\in\wp_r(S_{\min})}\frac{1}{|D_I|^r}\varepsilon_I = \sum_{\chi\in\widehat{G}}\frac{n_\chi}{|\ker\chi|^r}\mathbf{e}_\chi\varepsilon_\chi,$$

where

$$\begin{aligned} n_\chi &= \operatorname{card}\{I \in \wp_r(S_{\min}) \mid D_I \subseteq \ker \chi\} \\ &= \operatorname{card}\{I \in \wp_r(S_{\min}) \text{ and for all } v \in I, v \text{ splits in } K_\chi\}. \end{aligned}$$

Then $n_\chi = 1$ if and only if $r_S(K/k) = r_S(\chi)$. When $n_\chi > 1$ we have more than r primes which split in K_χ/k and $\varepsilon_\chi = 0$ by Remark 3.9. The proof concludes by Proposition 4.7. □

Proposition 4.9. *Under the above assumptions and notations, we have.*

(1) *If $\widetilde{B}(K_I/k, S, T, r)$ is true for all $I \in \wp_r(S_{\min})$, then*

$$\varepsilon_{K/k} \in \frac{1}{|G|}\Lambda_{S,T},$$

i.e., *$\widetilde{B}(K/k, S, T, r)$ is true up to a factor of $|G| = [K : k]$.*

(2) *If $\widetilde{B}(K_I/k, S, T, r)$ is true up to primes dividing $|G|$ (i.e. $\varepsilon_I \in \mathbb{Z}[1/|G|]\Lambda_{K_I/k,S,T}$), for all $I \in \wp_r(S_{\min})$, then so is $\widetilde{B}(K/k, S, T, r)$ (i.e. $\varepsilon_{K/k} \in \mathbb{Z}[1/|G|]\Lambda_{K/k,S,T}$.)*

Proof. Let $\phi = \phi_1 \wedge \ldots \wedge \phi_r \in \bigwedge^r_{\mathbb{Z}[G]} U^*_{K,S,T}$. Let M be an arbitrary intermediate field for K/k. Note that Lemma 4.4 combined with the fact that $\varepsilon_{M/k}$ is fixed by $H = G(K/M)$ yield

$$\begin{aligned} \phi\left(\frac{1}{[K:M]^r}\varepsilon_{M/k}\right) &= \frac{1}{[K:M]^r}\pi^*_{K/M}\pi_{K/M}\phi(\varepsilon_{M/k}) \\ &= \frac{1}{[K:M]^r}\pi^*_{K/M}((N^*_{K/M})^{(r)}\phi)(N^{(r)}_{K/M}\varepsilon_{K/M}) \\ &= \pi^*_{K/M}((N^*_{K/M})^{(r)}\phi)(\varepsilon_{K/M}) \\ &\in \pi^*_{K/M}\mathbb{Z}[\Gamma]\,. \end{aligned}$$

Under hypotheses (1) and (2), respectively, we have

$$\pi^*_{K/M}\mathbb{Z}[\Gamma] \subseteq \frac{1}{[K:M]}\mathbb{Z}[G] \quad (\subseteq \frac{1}{[K:M]}\mathbb{Z}[1/|G|][G], \text{ respectively.})$$

Now, we apply this computation repeatedly with $M = K_I$ for each $I \in \wp_r(S_{\min})$ to the formula given by Theorem 4.8 to obtain the result. Of course, we finally need to note that $|D_I|$ divides $|G|$, for all $I \in \wp_r(S_{\min})$. □

Remark 4.10. *Note that under the hypotheses of Propositions 4.8 and 4.9, $\widetilde{B}(K_I/k, S, T, r)$ is equivalent to the classical Rubin–Stark conjecture for the same data, for all $I \in \wp_r(S)$ (as all r primes in I split completely in K_I/k, by definition.) So, these propositions relate our conjecture to the classical Rubin-Stark conjecture for various distinguished intermediate extensions M/k.*

In light of the above remark, a consequence of Proposition 4.9 is the following.

Theorem 4.11. *Under the above notations and assumptions, the following hold.*

(1) *If $K/\mathbb{Q}$ is a Galois abelian extension, then*

$$\varepsilon_{K/k} \in \frac{1}{|G|}\mathbb{Z}[1/2]\Lambda_{K/k,S,T}\,.$$

(2) *Further, if K is an imaginary abelian extension of $\mathbb{Q}$ of odd, prime power conductor, then*

$$\varepsilon_{K/k} \in \frac{1}{|G|}\Lambda_{K/\mathbb{Q},S,T}\,.$$

(3) *If K is a characteristic $p > 0$ global field, then*

$$\varepsilon_{K/k} \in \frac{1}{|G|}\Lambda_{K/k,S,T} \,.$$

Proof. In [1] and [9] the Rubin–Stark conjecture for abelian extensions K/k of characteristic p global fields is proved unconditionally. This result, combined with Proposition 4.9, settles (3) above.

In [1], Burns proves the Rubin–Stark conjecture up to an undetermined power of 2 for all abelian extensions of number fields K/k, provided that $K/\mathbb{Q}$ is abelian. This result, combined with Proposition 4.9, settles (1) above.

In [7], (a strong form of) the Rubin–Stark conjecture is proved for extensions $K/\mathbb{Q}$ with K abelian, imaginary and of odd prime power conductor. This leads to (2) above. □

5. Unramified covers and extensions of prime exponent

In this section, we will provide some evidence in support of conjecture $\widetilde{B}(K/k, S, T, r)$. The assumptions and notations are the same as in §4. In particular, (S, T) is an appropriate pair for the abelian extension K/k, whose Galois group is denoted by G. Also, S is an r–cover for $\widehat{G}$ and $|S| > r + 1$.

Theorem 5.1. *Suppose that S has a subset S' which is an r-cover for $\widehat{G}$ consisting of only finite primes that do not ramify in K/k. Let $S_b := S \setminus S'$. If $\widetilde{B}(K_I/k, S_b \cup I, T, r)$ is true for all $I \in \wp_r(S_{\min})$, then $\widetilde{B}(K/k, S, T, r)$ is true.*

Proof. Note that $S_{\min} \subseteq S'$. As S' contains only finite, unramifying primes, S_b still contains all infinite and ramifying primes and hence is appropriate for the extension K/k.

If $|S'| = r$, then $S_{\min} = S'$ contains r primes which split completely in K/k (as S' is an r–cover for $\widehat{G}$) and $K_I = K$ for $I = S_{\min}$, therefore $\widetilde{B}(K/k, S, T, r)$ is true. Therefore, we may assume that $|S'| > r$. For any $I \in \wp_r(S_{\min})$ we may define

$$\eta_I = \prod_{v \in S' \setminus I} \left(1 - \sigma_v^{-1}\right)$$

The Frobenius automorphisms above exist because we are assuming the primes in S' are unramified. The element η_I is relevant in what follows because, as Lemma 4.6 shows, we have

$$\varepsilon_{K_I/k,S,T,r} = \eta_I \cdot \varepsilon_{K_I/k,S_b \cup I,T,r} \tag{8}$$

for all $I \in \wp_r(S_{\min})$. Temporarily, fix some $I \in \wp_r(S_{\min})$. Take $v \in I$ and $\chi \in \widehat{G}$. We claim that

$$\chi((\sigma_v - 1)\eta_I) = 0 \,.$$

Obviously, the claim is true if $\chi(\eta_I) = 0$. But $\chi(\eta_I) \neq 0$ implies that no prime in $S' \setminus I$ splits in K_χ/k. However we know at least r primes of S' have to split in K_χ/k, as S' is an r-cover. Thus all the primes in I split in K_χ/k, so $\chi(\sigma_v) = 1$ and the claim has been shown. Since this holds for all $\chi \in \widehat{G}$ we conclude that $(\sigma_v - 1)\eta_I = 0$, that is, $\sigma_v \cdot \eta_I = \eta_I$. In the case of unramified primes, D_I, the subgroup generated by the decomposition groups of the primes in I, is actually generated by the Frobenius automorphisms, $D_I = \langle \sigma_v \mid v \in I \rangle$. Thus, we have shown that η_I is fixed by D_I. However $\mathbb{Z}[G]$ is a free $\mathbb{Z}[D_I]$–module, so is D_I–cohomologically trivial. Consequently $\widehat{\mathrm{H}}^0(D_I, \mathbb{Z}[G]) = 0$ and $\mathbb{Z}[G]^{D_I} = N_{D_I}\mathbb{Z}[G]$. Therefore

$$\eta_I = N_{D_I} \cdot \eta'_I \text{ for some } \eta'_I \in \mathbb{Z}[G] \,.$$

Let $\phi = \phi_1 \wedge \ldots \wedge \phi_r \in \bigwedge^r U_{S,T}^*$. We need only show that $\phi(\varepsilon_{K/k,S,T}) \in \mathbb{Z}[G]$, as this will imply that $\varepsilon_{K/k,S,T}$ is an element of $\Lambda_{K/k,S,T}$, which is the prediction of the conjecture. We will show first that $\phi(\varepsilon_{K_I/k,S_b\cup I,T,r}) = N_{D_I}^r \beta_I^\phi$ for some $\beta_I^\phi \in \mathbb{Z}[G]$. Indeed, if we use successively Remark 4.1, the hypothesis $\varepsilon_{K_I/k,S_b\cup I,T,r} \in \Lambda_{K_I/k,S_b\cup I,T}$ and the fact that $\mathrm{Im}(\pi^*_{K/K_I}) \subseteq \frac{N_{D_I}}{|D_I|}\mathbb{Z}[G]$, we conclude

$$\phi(\varepsilon_{K_I/k,S_b\cup I,T,r}) = |D_I|^r \pi^*_{K/K_I}(N^{*(r)}_{K/K_I}(\phi)(\varepsilon_{K_I/k,S_b\cup I,T,r})) \in |D_I|^{r-1} N_{D_I}\mathbb{Z}[G] = N_{D_I}^r \mathbb{Z}[G].$$

Now, we use (8) and Proposition 4.8 to compute

$$\begin{aligned}
\phi(\varepsilon_{K/k,S,T}) &= \sum_{I\in\wp_r(S_{\min})} \frac{1}{|D_I|^r}\phi(\varepsilon_{K_I/k,S,T,r}) \\
&= \sum \frac{\eta_I'}{|D_I|^r} N_{D_I}\phi(\varepsilon_{K_I/k,S_b\cup I,T,r}) \\
&= \sum \frac{\eta_I'}{|D_I|^r} N_{D_I}^{r+1}\beta_I^\phi \\
&= \sum \eta_I' N_{D_I}\beta_I^\phi \\
&\in \mathbb{Z}[G],
\end{aligned}$$

concluding the proof. □

Let us turn to a more specific type of abelian extension. The first author has a forthcoming article inspired by [11] regarding conjecture $\widetilde{B}$ in *multiquadratic* extensions, *i.e.*, those of exponent two. Much more can be said in that case because Rubin's conjecture was shown by Rubin to be true for relative quadratic extensions, and he gave an explicit description of the evaluator [10]. Instead, here let us focus on a more general case where the exponent of our field extension is a prime l. That is $G = G(K/k) \cong (\mathbb{Z}/l\mathbb{Z})^m$. Our goal is to prove $\widetilde{B}(K/k,S,T,r)$ under the hypothesis that the standard Rubin-Stark conjecture is true for (cyclic) extensions of degree l. Please recall that if $G = G(K/k)$ is cyclic and S is an r–cover for $\widehat{G}$, then S has to contain r primes which split completely in K/k, so the standard Rubin-Stark conjecture applies (see Example 1, §2).

Theorem 5.2. *Suppose l is a prime number, $G = G(K/k) \cong (\mathbb{Z}/l\mathbb{Z})^m$ and that $\widetilde{B}(M/k,S',T,r)$ is true for every degree l extension M of k contained in K and appropriate $S' \subseteq S$. Let S_{ram} denote the set of finite primes of k that ramify in K/k, and S_∞ denote the set of infinite places of k. If*

$$|S| \geq r + |S_{\mathrm{ram}}| + |S_\infty| + (m-1)l \tag{9}$$

then $\widetilde{B}(K/k,S,T,r)$ is true.

Proof. Take $\chi \in \widehat{G}_{r,S}$. Let $H = \ker\chi$ and $M = K^H$. We know that M/k is a $\mathbb{Z}/l\mathbb{Z}$-extension in which exactly r primes of S split completely. Let S_H consist of these r primes together with S_{ram} and S_∞. Then we have

$$\varepsilon_{M/k,S,T,r} = \left[\prod_{v\in S\setminus S_H}(1-\sigma_v^{-1})\right]\varepsilon_{M/k,S_H,T,r}. \tag{10}$$

where σ_v here represents the Frobenius of v in M/k. (As none of the primes in $S \setminus S_H$ split in M/k, these Frobenius morphisms are nontrivial.) Note that $G(M/k)$ is cyclic with generator σ.

Because $1 - \sigma^t = (1-\sigma)(1+\sigma+\cdots+\sigma^{t-1})$, the product appearing in equation (10) is divisible by $(1-\sigma)^{|S \setminus S_H|}$, and hence by $(1-\sigma)^{(m-1)l}$. The binomial theorem (combined with the cancellation of the first and last terms) implies $(1-\sigma)^l \in l\mathbb{Z}[G(M/k)]$. Hence

$$\varepsilon_{M/k,S,T,r} = l^{m-1}\eta_H \cdot \varepsilon_{M/k,S_H,T,r}. \tag{11}$$

for some $\eta_H \in \mathbb{Z}[G(M/k)]$. Since $[K : M] = l^{m-1}$, we have the necessary factor such that when the computation in the proof of Proposition 4.9 is carried out, indeed $\phi(\varepsilon_{K/k,S,T}) \in \mathbb{Z}[G]$. □

Definition 5.3. *S is a* completely nontrivial *r-cover for K/k if it contains no prime which splits completely in K/k.*

Corollary 5.4. *If $G = G(K/k) \cong (\mathbb{Z}/l\mathbb{Z})^m$, $\widetilde{B}(M/k, S', T, r)$ is true for every degree l extension of k contained in K and appropriate $S' \subseteq S$, S is a completely nontrivial r-cover and*

$$r \geq \frac{1}{l-1}\Big[|S_{\mathrm{ram}}| + |S_\infty| + (m-1)l - 1\Big]$$

then $\widetilde{B}(K/k, S, T)$ is true.

Proof. By hypothesis, none of the primes in S split completely in K/k, so it follows that each decomposition group is nontrivial and $g_v \leq l^{m-1}$. Since S is an r-cover for $\widehat{G}$, for every $\chi \in \widehat{G}$, $r_S(\chi) \geq r$. Substituting these estimates into equation (1) yields $(l^m - 1)r \leq (l^{m-1} - 1)|S|$, or $|S| \geq \frac{l^m - 1}{l^{(m-1)} - 1} r > lr$. Since $|S|$ is an integer, $|S| \geq lr + 1 = r + (l-1)r + 1$, which, by hypothesis is at least $r + |S_{\mathrm{ram}}| + |S_\infty| + (m-1)l$. We are done by the previous Theorem. □

Example. Theorem 5.2 may also be used to give (infinitely) many examples of extensions $K/\mathbb{Q}$ for which $\widetilde{B}(K/\mathbb{Q}, S', T, r)$ can be proved.

First, let us note that Rubin's conjecture is known to hold for $\mathbb{Z}/l\mathbb{Z}$-extensions $K/\mathbb{Q}$, if l is a prime. If $l = 2$, this is proved in [10] (for arbitrary base fields k.) Assume that $l > 2$. Burns proves in [1] the conjecture for $K/\mathbb{Q}$ up to a power of 2 (see the proof of Theorem 4.11 above.) On the other hand, if the base field is $\mathbb{Q}$, Rubin's conjecture is known to hold true up to primes dividing the order of the Galois group $G(K/\mathbb{Q})$ (as a consequence of the fact that cyclotomic units and Gauss sums form Euler Systems, see [8].) This fact combined with Burns's result settles the conjecture for $l > 2$ as well.

Next, let $K/\mathbb{Q}$ be any $(\mathbb{Z}/l\mathbb{Z})^m$ extension, with l prime, and let (S, T) be a pair which is appropriate for $K/\mathbb{Q}$, such that S is an r–cover for $\widehat{G}$ and $r = r_S(K/k)$. Fix a particular character of minimal order of vanishing $\psi \in \widehat{G}_{r,S}$. Put $b = r + |S_{\mathrm{ram}}| + |S_\infty| + (m-1)l - |S|$. If $b \leq 0$ then $\widetilde{B}(K/\mathbb{Q}, S, T, r)$ is already true by Theorem 5.2, so assume $b \geq 1$. Let E be a set of b primes of $\mathbb{Q}$ disjoint from S and T, such that

$$\psi(\sigma_v) \neq 1 \text{ for all } v \in E. \tag{12}$$

Such a set may be chosen by the Tchebotarev Density Theorem. Finally, let $S' = S \cup E$. Because of our assumption (12), no prime in E splits in $K_\psi/\mathbb{Q}$, and $r_{S'}(\psi) = r_S(\psi) = r$. Now

$$|S'| = |S| + b = r + |S'_{\mathrm{ram}}| + |S'_\infty| + (m-1)l.$$

All the hypotheses of Theorem 5.2 are fulfilled, so $\widetilde{B}(K/\mathbb{Q}, S', T, r)$ follows.

Of course, the same idea can be used to construct an infinite class of examples in characteristic $p > 0$, where the full Rubin-Stark conjecture is known to hold (see proof of Theorem 4.11).

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