TORSION SUBGROUPS OF BRAUER GROUPS AND EXTENSIONS OF CONSTANT LOCAL DEGREE FOR GLOBAL FUNCTION FIELDS

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Abstract. We show that, for all characteristic p global fields k and natural numbers n coprime to the order of the non–p–part of the Picard group $\mathrm{Pic}^0(k)$ of k, there exists an abelian extension L/k whose local degree at every prime of k is equal to n. This answers in the affirmative in this context a question recently posed by Kisilevsky and Sonn. As a consequence, we show that, for all n and k as above, the n–torsion subgroup $\mathrm{Br}_n(k)$ of the Brauer group $\mathrm{Br}(k)$ of k is algebraic, answering a question of Aldjaeff and Sonn in this context.

Introduction

Let k be a field and n a strictly positive integer. We denote by $\mathrm{Br}_n(k)$ the n–torsion subgroup of the Brauer group $\mathrm{Br}(k)$ of k. In [AS], the authors raise the question whether the group $\mathrm{Br}_n(k)$ is algebraic, i.e. equal to the kernel $\mathrm{Br}(L/k)$ of the restriction map $\mathrm{Br}(k) \longrightarrow \mathrm{Br}(L)$, for an algebraic separable extension L/k. For arbitrary fields k, the answer to this question is in general "No" (see [AS] for an example). However, the question is still open in the case where the field k is a global field, i.e. either a number field or a characteristic p–function field with a finite constant field. In [AS] and [KS], the authors answer this question in the affirmative in certain instances where k is a number field. In this paper, we are concerned with this question in the situation where k is a global function field. It turns out that, for a given global function field k and a given integer n, the group $\mathrm{Br}_n(k)$ is algebraic if there exists a Galois extension L/k of local degree equal to n at every prime v of k (see Proposition 1.5 below). Consequently, in this paper we focus mainly on the construction of such Galois extensions, which is an interesting problem in its own right. We use global and local class–field theoretical methods to show that, given n and k as above, there exists an abelian Galois extension L/k whose local degrees are all equal to n provided that n is coprime to the order of the non–p part of the order of the Picard group of k (see Theorem 5.1 below). The proof is divided into two major cases: the case where n is a p–power (see §3) and the case where n is coprime to p (see §4). Our approach to the coprime–to–p case was inspired by the constructions of Kisilevsky and Sonn in the case where k is a number field whose class–number is coprime to n (see [KS]). The proof of the p–power case uses methods and results which are more specific to characteristic p global fields.

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1. Notations, definitions, and setup

1.1. Global function fields. Throughout this paper, p denotes a fixed prime number, $q := p^\nu$ denotes a fixed power of p, with $\nu \in \mathbb{N}_{\geq 1}$, and $\mathbb{F}_q$ denotes the field of q elements. Let k be a characteristic p global field (i.e. a finite extension of a rational field $\mathbb{F}_p(T)$ of variable T) of exact field of constants $\mathbb{F}_q$, meaning that $\mathbb{F}_q$ is contained and algebraically closed in k. We denote by J_k de group of idèles of k and by $\mathrm{Pic}^0(k)$ the Picard group of equivalence classes of degree 0 divisors in k. In the sequel, we refer to the equivalence classes of valuations of k as the primes of k. They are in a canonical one–to–one correspondence to the non–zero prime ideals of various Dedekind subrings of k. For every prime v in k, we let k_v denote the completion of k with respect to v. As usual, we let O_v, $\mathfrak{m}_v$, U_v and $U_v^{(1)} := 1 + \mathfrak{m}_v$ denote the valuation ring, its maximal ideal, the group of units, and group of principal units in k_v, respectively. The group $U_v^{(1)}$ comes endowed with the canonical filtration $\{U_v^{(i)}\}_{i\geq 1}$, where $U_v^{(i)} := 1 + \mathfrak{m}_v^i$, for all $i \geq 1$. The residue field $\mathbb{F}_q(v) := O_v/\mathfrak{m}_v$ is a finite extension of $\mathbb{F}_q$, and its degree $d_v := [\mathbb{F}_q(v) : \mathbb{F}_q]$ is called the degree of v over $\mathbb{F}_q$. If π_v is a uniformizer for v, then there are canonical ring isomorphisms

$$O_v \xrightarrow{\sim} \mathbb{F}_{q^{d_v}}[[\pi_v]]\,, \qquad k_v \xrightarrow{\sim} \mathbb{F}_{q^{d_v}}((\pi_v))\,,$$

and group isomorphisms

$$U_v/U_v^{(1)} \xrightarrow{\sim} \mathbb{F}_{q^{d_v}}^\times\,, \qquad U_v \xrightarrow{\sim} U_v^{(1)} \times \mathbb{F}_{q^{d_v}}^\times\,, \qquad U_v^{(i)}/U_v^{(i+1)} \xrightarrow{\sim} \mathbb{F}_{q^{d_v}}^+\,,$$

for all $i \in \mathbb{N}_{\geq 1}$. Consequently, $U_v^{(1)}$ is a pro–p group and therefore a (compact) topological $\mathbb{Z}_p$–module. Its $\mathbb{Z}_p$–module structure is described by the following proposition, whose proof can be found in [I] (see Proposition 2.8, page 25).

Proposition 1.1. *As a topological $\mathbb{Z}_p$–module, $U_v^{(1)}$ is isomorphic to a product of countably many copies of $\mathbb{Z}_p$.*

For a given k as above and a given $m \in \mathbb{N}_{\geq 1}$, we denote by k_m the constant field extension of degree m of k, i.e. the field compositum $k \cdot \mathbb{F}_{q^m}$ over $\mathbb{F}_q$, inside a fixed separable closure of k. The following is a direct consequence of class–field theory.

Lemma 1.2.

(1) *The extension k_m/k is unramified everywhere.*
(2) *A prime v in k splits in a product of* $\gcd(m, d_v)$ *distinct primes in k_m/k.*

In what follows, $\mathcal{P}_k$ will denote the set of primes of the global function field k.

Definition 1.3. A separable extension L/k of k is called of finite (respectively constant, equal to n) local degree if $[L_w : k_v]$ is finite (respectively equal to n), for all $v \in \mathcal{P}_k$ and all $w \in \mathcal{P}_L$, such that w divides v.

If L/k is a Galois extension then, for each $v \in \mathcal{P}_k$, the local degree $[L_w : k_v]$ does not depend on $w \in \mathcal{P}_L$ dividing v, and will be denoted by $[L : k]_v$ in what follows.

1.2 Brauer groups of global function fields. In what follows, if k is an arbitrary field, we denote by $\mathrm{Br}(k)$ its Brauer group. For the definitions, examples and properties of Brauer groups which are relevant in the present context, the reader can consult [S]. The Brauer group $\mathrm{Br}(k)$ is a commutative, torsion group, and there exists a canonical group isomorphism

$$\mathrm{H}^2(G(\overline{k}/k), \overline{k}^\times) \xrightarrow[\sim]{j_k} \mathrm{Br}(k)\,,$$

where $\overline{k}$ denotes a separable closure of k and $\mathrm{H}^i(G(\overline{k}/k), M)$ is the i–th Galois cohomology group with coefficients in the $G(\overline{k}/k)$–module M. If L/k is an algebraic separable field extension, then the restriction map $\mathrm{res}_{k/L}$ at the level of Galois–cohomology groups induces a restriction map at the level of Brauer groups, also denoted by $\mathrm{res}_{k/L}$, such that the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{H}^2(G(\overline{k}/k), \overline{k}^\times) & \xrightarrow[\sim]{j_k} & \mathrm{Br}(k) \\ \downarrow{\scriptstyle \mathrm{res}_{k/L}} & & \downarrow{\scriptstyle \mathrm{res}_{k/L}} \\ \mathrm{H}^2(G(\overline{L}/L), \overline{L}^\times) & \xrightarrow[\sim]{j_L} & \mathrm{Br}(L)\,. \end{array} \tag{1}$$

The kernel of $\mathrm{res}_{k/L}$ is denoted by $\mathrm{Br}(L/k)$ and it is a subgroup of $\mathrm{Br}(k)$ called the L/k–relative Brauer group.

Definition 1.4. A subgroup H of $\mathrm{Br}(k)$ is called *algebraic* if there exists a separable algebraic extension L/k, such that $H = \mathrm{Br}(L/k)$.

For any $m \in \mathbb{Z}_{\geq 1}$, we denote by $\mathrm{Br}_m(k)$ the m–torsion subgroup of $\mathrm{Br}(k)$. In [AS], the authors raise the question whether, given a field k and $m \in \mathbb{Z}_{\geq 1}$, the group $\mathrm{Br}_m(k)$ is algebraic. For arbitrary fields k, the answer to this question is in general "No" (see [AS] for an example). However, the question is open in the case where the field k is a global field, i.e. either a number field or a characteristic p–function field with a finite constant field. In [AS] and [KS] the authors answer this question in the affirmative in certain instances where k is a number field. In this paper, we are concerned with this question in the case where k is a global function field.

Proposition 1.5. *Let k be a characteristic p global field, let $m \in \mathbb{Z}_{\geq 1}$, and let L/k be a Galois extension, such that $[L:k]_v = m$, for all $v \in \mathcal{P}_k$. Then, we have*

$$\mathrm{Br}(L/k) = \mathrm{Br}_m(k)\,.$$

In particular, $Br_m(k)$ is algebraic.

Proof. We need a few standard facts about Brauer groups of global function fields. The reader can consult [S] for the proofs. If E is a global function field, then there are canonical group morphisms

$$\mathrm{Br}(E) \xrightarrow{\mathrm{res}_{E/E_w}} \mathrm{Br}(E_w) \xrightarrow[\sim]{\mathrm{inv}_{E_w}} \mathbb{Q}/\mathbb{Z}\,,$$

for all $w \in \mathcal{P}_E$. The composition $\mathrm{inv}_w := \mathrm{res}_{E/E_W} \circ \mathrm{inv}_{E_W}$ is called the local invariant map at w. For every $x \in \mathrm{Br}(E)$, one has $\mathrm{inv}_w(x) = 0$, for all but finitely many $w \in \mathcal{P}_E$. The Brauer group $\mathrm{Br}(E)$ fits in an exact sequence

$$0 \to \mathrm{Br}(E) \xrightarrow{\oplus_{w\in\mathcal{P}_E}\mathrm{inv}_w} \bigoplus_{w\in\mathcal{P}_E} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0\,, \tag{2}$$

where the right–most non–trivial map sends every element of $\bigoplus_{w\in\mathcal{P}_E}\mathbb{Q}/\mathbb{Z}$ into the sum of its components. If E/k is a finite extension of global function fields, then for all $w\in\mathcal{P}_E$ and $v\in\mathcal{P}_k$, such that w divides v, one has commutative diagrams

$$\begin{array}{ccc} \mathrm{Br}(k_v) & \xrightarrow[\sim]{\mathrm{inv}_{k_v}} & \mathbb{Q}/\mathbb{Z} \\ \big\downarrow{\scriptstyle \mathrm{res}_{k_v/E_w}} & & \big\downarrow{\scriptstyle [E_w:k_v]} \\ \mathrm{Br}(E_w) & \xrightarrow[\sim]{\mathrm{inv}_{E_w}} & \mathbb{Q}/\mathbb{Z} \end{array} \tag{3}$$

These lead to the following morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(k) & \xrightarrow{\oplus_v \mathrm{inv}_v} & \bigoplus_v \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{res}_{k/E}} & & \big\downarrow{\scriptstyle \oplus_v(\oplus_{w\mid v}[E_w:k_v])} & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{Br}(E) & \xrightarrow{\oplus_w \mathrm{inv}_w} & \bigoplus_v(\bigoplus_{w\mid v}\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}\ , \tag{4}$$

where v runs through $\mathcal{P}_k$ and w runs through $\mathcal{P}_E$. Now, we return to the proof of Proposition 1.5. Let x be an element of $\mathrm{Br}_m(k)$, ant let S_x be the finite subset of $\mathcal{P}_k$ consisting of all those $v\in\mathcal{P}_k$, such that $\mathrm{inv}_v(x)\neq 0$. Since $[L:k]_v=m$, for all $v\in\mathcal{P}_k$, and S_x is finite, Krasner's Lemma (see [L], II, §2) implies the existence of a finite Galois extension E/k, with $E\subseteq L$, such that $[E:k]_v=m$, for all $v\in S_x$. Since $x\in Br_m(k)$, the exact sequence (2) for $E:=k$ implies that $\mathrm{inv}_v(x)\in(1/m)\mathbb{Z}/\mathbb{Z}$, for all $v\in\mathcal{P}_k$. Consequently, the commutative diagram (3) implies that

$$\mathrm{inv}_w(\mathrm{res}_{k/E}(x))=[E:k]_v\cdot\mathrm{inv}_v(x)=0\,,$$

for all $w\in\mathcal{P}_E$ and $v\in\mathcal{P}_k$ sitting below w. Consequently, the commutative diagram (4) implies that $\mathrm{res}_{k/E}(x)=0$. Therefore, we have $x\in\mathrm{Br}(E/k)$. However, (1) implies that $\mathrm{Br}(E/k)\subseteq\mathrm{Br}(L/k)$. Consequently, we have an inclusion $\mathrm{Br}_m(k)\subseteq\mathrm{Br}(L/k)$.

Now, let $x\in\mathrm{Br}(L/k)$. The commutative diagram (1) implies that there exists E/k finite, Galois, with $E\subseteq L$, such that $x\in\mathrm{Br}(E/k)$. The commutative diagram (3) shows that

$$\mathrm{inv}_w(\mathrm{res}_{k/E}(x))=[E:k]_v\cdot\mathrm{inv}_v(x)=0\,,$$

for all $w\in\mathcal{P}_E$ and $v\in\mathcal{P}_k$ sitting below w. However, since $[E:k]_v$ is a divisor of $[L:k]_v=m$, for all $v\in\mathcal{P}_k$, the last equality shows that

$$\mathrm{inv}_v(x)\in(1/m)\mathbb{Z}/\mathbb{Z}\,,$$

for all $v\in\mathcal{P}_k$. Now, via the exact sequence (2) for $E:=k$, this implies that $x\in\mathrm{Br}_m(k)$, which concludes the proof of Proposition 1.5. □

2. Reduction Steps

Proposition 1.5 above shows that the algebraicity of $\mathrm{Br}_n(k)$ for a given global function field k and a given $n\in\mathbb{N}_{\geq 1}$ is a consequence of the existence of a separable algebraic extension L/k with constant local degrees equal to n. The existence of such extensions L/k is an interesting question in its own right. The remainder of this paper focuses mainly on this question. The following lemmas provide two reduction steps which will turn out to be very useful in the process of constructing extensions L/k of finite constant local degrees.

Lemma 2.1. *Assume that* $n = \ell_1^{a_1} \cdot \ldots \cdot \ell_r^{a_r}$*, with* $\ell_1, \ldots, \ell_r$ *distinct primes numbers and* $a_1, \ldots, a_r \in \mathbb{N}_{\geq 1}$*. If* L_i/k *are Galois extensions such that* $[L_i : k]_v = \ell_i^{a_i}$*, for all* $v \in \mathcal{P}_k$*, and all* $i = 1, \ldots, r$*, then their compositum* $L := L_1 \cdot \ldots \cdot L_r$ *is a Galois extension satisfying* $[L : k]_v = n$*, for all* $v \in \mathcal{P}_k$*.*

Proof. This is a direct consequence of the fact that local degrees are multiplicative in towers of fields. □

Lemma 2.1 allows us to reduce the proof of the main theorem to the case where n is a prime power.

Lemma 2.2. *If* k/k' *is a finite separable extension of local degrees coprime to* n *and* L'/k' *is a Galois extension of local degrees* $[L' : k']_{v'} = n$*, for all* $v' \in \mathcal{P}_{k'}$*, then the field compositum* $L := L' \cdot k$ *is a Galois extension of* k*, such that* $[L : k]_v = n$*, for all* $v \in \mathcal{P}_k$*.*

Proof. It is very clear that L/k is Galois. Let $v \in \mathcal{P}_k$. Let w be a prime in L, sitting above v, and let w' and v' be its restrictions to L' and k', respectively. Then $L_w = L_{w'} \cdot k_{v'}$ and therefore

$$[L_w : k_v] \cdot [k_v : k'_{v'}] = [L_w : L'_{w'}] \cdot [L'_{w'} : k'_{v'}] \,.$$

Since $[L_w : k_v] \leq [L'_{w'} : k'_{v'}] = n$, and $[k_v : k'_{v'}]$ is coprime to n, the equality above implies that $[L_w : k_v] = n$. □

Lemma 2.2 is especially useful in the case where $[k : k']$ is a finite Galois extension of degree coprime to n. Then, its local degrees will divide its degree and, consequently, will be coprime to n.

3. p–Power constant local degrees

In this section we will prove the following.

Theorem 3.1. *Let* k *be a characteristic* p *function field and* $n := p^m$*, for some* $m \in \mathbb{Z}_{\geq 1}$*. Then, there exists an abelian extension* L/k *whose local degrees* $[L : k]_v$ *are equal to* n*, for all* $v \in \mathcal{P}_k$*.*

We need several preparatory lemmas.

Lemma 3.2. *Let* w *be a fixed prime in* $\mathcal{P}_k$*. Then there exists a finite, non-empty set* $S \subseteq \mathcal{P}_k$*, such that* $w \notin S$ *and satisfying the following equivalent conditions.*

(1) *There exists no non–trivial abelian extension of* k *which is unramified everywhere and completely split at all primes* $v \in S$*.*
(2) *We have an equality* $J_k = k^\times(\prod_{v \in S} k_v^\times \times \prod_{v \notin S} U_v)$*.*

Proof. For any finite, non–empty set $S' \subseteq \mathcal{P}_k$, such $w \notin S'$, if we denote by $K_{S'}$ the maximal abelian extension of k, unramified everywhere and completely split at all primes in S', class–field theory shows that the extension $K_{S'}/k$ is finite and the global Artin reciprocity map induces a group isomorphism

$$J_k/k^\times \cdot (\prod_{v \in S'} k_v^\times \times \prod_{v \notin S'} U_v) \xrightarrow{\sim} G(K_{S'}/k) \,. \tag{5}$$

Consequently, it suffices to find a set S satisfying property (2) in Lemma 3.2 Let $v_1 \in \mathcal{P}_k$, $v_1 \neq w$ and let $S_1 ;= \{v_1\}$. Then the extension K_{S_1}/k is finite. In fact, we have an exact sequence of groups

$$1 \to J_k^{(0)}/k^\times(\prod_{v\in P_k} U_v) \to J_k/k^\times(k_{v_1}^\times \times \prod_{v\neq v_1} U_v) \xrightarrow{\widehat{\deg}} \mathbb{Z}/d_{v_1}\mathbb{Z} \to 1\,,$$

where $J_k^{(0)}$ denotes the group of idèles of total degree 0 and $\widehat{\deg}$ denotes the total degree modulo d_{v_1} map at the level of idèles. Since the left–most non–trivial term of the exact sequence above is canonically isomorphic to $\mathrm{Pic}^0(k)$ and therefore finite, the term in the middle is finite. Consequently, $G(K_{S_1}/k)$ is also finite. For any $v \notin S_1$ the isomorphism (5) for $S' := S_1$ sends any v–uniformizer $\pi_v \in k_v^\times$ into the Frobenius morphism σ_v associated to v in k_v/k. Chebotarev's density theorem allows us to pick primes $v_2, \dots, v_r$ in $\mathcal{P}_k \setminus \{w, v_1\}$, such that $\sigma_{v_2}, \dots, \sigma_{v_r}$ generate the group $G(K_{S_1}/k)$. The isomorphism (5) for $S' := S_1$ shows that this is equivalent to the equality

$$J_k := k^\times(\prod_{v\in S} k_v^\times \times \prod_{v\notin S} U_v)\,,$$

where $S := \{v_1, v_2, \dots, v_r\}$. This concludes the Proof of Lemma 3.2. □

Proposition 3.3. *Let w be a fixed prime in $\mathcal{P}_k$. Let $T \subseteq \mathcal{P}_k$ be a finite set of primes in k, such that $w \notin T$. Let $\alpha \in \mathbb{Z}_{\geq 0}$. Then, there exists an abelian extension K/k, which is totally ramified at w, unramified everywhere else, completely split at all primes v in T, and of Galois group $G(K/k) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\alpha$.*

Proof. The case $\alpha = 0$ is clear. Let us assume that $\alpha \in \mathbb{Z}_{\geq 1}$. For our fixed prime w, let S be a finite set of primes satisfying the conditions in Lemma 3.2. Let $S' := T \cup S$. Note that $w \notin S'$. For all $i \in \mathbb{Z}_{\geq 1}$, let $K_{S',w}^{(i)}$ denote the maximal abelian extension of k of conductor dividing w^i, and completely split at all primes $v \in S'$. By class–field theory, the global Artin reciprocity map induces a group isomorphism

$$J_k/k^\times(\prod_{v\in S'} k_v^\times \times U_w^{(i)} \times \prod_{v\notin S'\cup\{w\}} U_v) \xrightarrow{\sim} G(K_{S',w}^{(i)}/k)\,.$$

However, according to Lemma 3.2, we have $J_k = k^\times(\prod_{v\in S} k_v^\times \times \prod_{v\notin S} U_v)$. This implies that the left–hand–side of the above isomorphism is isomorphic to

$$U_w/\iota_w(U_{S'})\cdot U_w^{(i)}\,,$$

where $\iota_w : k^\times \longrightarrow k_w^\times$ is the canonical inclusion and $U_{S'}$ is the group of S'–units in $k^\times$. Therefore, we obtain a group isomorphism

$$U_w/\iota_w(U_{S'})\cdot U_w^{(i)} \xrightarrow{\sim} G(K_{S',w}^{(i)}/k)\,.$$

Since $U_w = \mathbb{F}_{q^{d_w}}^\times \times U_w^{(1)}$, the isomorphism above shows that there exists a subfield $L_{S',w}^{(i)}$ of $K_{S',w}^{(i)}$, containing k, such that

$$U_w^{(1)}/\iota_w(U_{S'}^{(1)})\cdot U_w^{(i)} \xrightarrow{\sim} G(L_{S',w}^{(i)}/k)\,, \tag{6}$$

where $U_{S'}^{(1)}$ denotes the subgroup (of finite index) of $U_{S'}$ consisting of those S'–units in $k^\times$ which are congruent to 1 modulo w. We will need the following Lemma.

Lemma 3.4. *For a multiplicative abelian group X we will denote by $\mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}}(X)$ the dimension of the $\mathbb{Z}/p\mathbb{Z}$–vector space X/X^p. The following hold true.*

(1) $\lim_{i\to\infty}\mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}}(U_w^{(1)}/U_w^{(i)}) = \infty$.
(2) *For i sufficiently large we have*

$$\mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}}(U_w^{(1)}/\iota_w(U_{S'}^{(1)})\cdot U_w^{(i)}) \geq \alpha\,.$$

Proof of Lemma 3.4. According to Proposition 1.1, we have an isomorphism of topological $\mathbb{Z}_p$–modules

$$U_w^{(1)} \xrightarrow[\sim]{j} \mathbb{Z}_p^{\aleph_0}\,.$$

Comparing bases of open neighborhoods for the left-hand–side and right–hand–side of the above isomorphism, respectively, we can conclude that for all $t\in\mathbb{Z}_{\geq 1}$, there exists $i_t\in\mathbb{Z}_{\geq 1}$, such that

$$j(U_w^{(i_t)}) \subseteq (p\mathbb{Z}_p)^t\times\mathbb{Z}_p\times\mathbb{Z}_p\times\cdots\,.$$

This shows that j induces a surjective group morphism

$$U_w^{(1)}/U_w^{(i_t)} \longrightarrow (\mathbb{Z}/p\mathbb{Z})^t \longrightarrow 0\,.$$

This implies that the $\mathbb{Z}/p\mathbb{Z}$–rank of the left–hand–side is at least t, which concludes the proof of part (1). In order to prove part (2), let us observe that, for all i, we have an exact sequence of groups

$$\iota_w(U_{S'}^{(1)}) \longrightarrow U_w^{(1)}/U_w^{(i)} \longrightarrow U_w^{(1)}/\iota_w(U_{S'}^{(1)})\cdot U_w^{(i)} \longrightarrow 1\,.$$

The function field analogue of Dirichlet's S'–unit theorem implies that

$$\mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}}(\iota_w(U_{S'}^{(1)})) \leq \mathrm{rk}_{\mathbb{Z}/p\mathbb{Z}}(U_{S'}^{(1)}) = \mathrm{card}(S')-1\,.$$

In fact, since Leopoldt's Conjecture is known to hold true in function fields (see [K]), one can replace the "$\leq$" sign above with the "$=$" sign. The above inequality, combined with the last exact sequence and Lemma 3.4(1), concludes the proof of Lemma 3.4(2). □

Now, we return to the proof of Proposition 3.3. We combine isomorphism (6) and Lemma 3.4(2) to conclude that, for i sufficiently large, there exists a field $K\subseteq L_{S',w}^{(i)}$ containing k, such that

$$G(K/k) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^\alpha\,.$$

The extension K/k satisfies all the required properties. □

Proof of Theorem 3.1. Let us fix an ordering on $\mathcal{P}_k$, $\mathcal{P}_k=\{v_1,v_2,\dots\}$. The extension L/k will be obtained as the compositum of countably many finite abelian extensions $\{K_i/k\}_{i\geq 1}$, which will be constructed inductively below.

Step 1. We apply Proposition 3.3 for $w := v_1$, $T = \emptyset$, and $\alpha := m$, and let K_1/k be a fixed abelian extension of k, which is totally ramified at v_1, unramified everywhere else, such that

$$G(K_1/k) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^m .$$

Step 2. Since v_2 is unramified in K_1/k and $G(K_1/k)$ has exponent p, the decomposition group G_{v_2} of v_2 in K_1/k is cyclic isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{(m-\gamma_2)}$, where $m-1 \leq \gamma_2 \leq m$. We apply Proposition 3.3 for $w := v_2$, $T = T_1 := \{v_1\}$, and $\alpha := \gamma_2$, and let K_2/k be a fixed abelian extension of k, which is totally ramified at v_2, unramified everywhere else, completely split at all the primes in T_1, such that

$$G(K_2/k) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^{\gamma_2} .$$

Now, we assume that we have constructed $K_1/k, \dots, K_r/k$, for $r \geq 2$, and we describe the construction of K_{r+1}/k.

Step (r+1). Let $\mathcal{K}_r$ be the compositum $K_1 \cdot K_2 \cdot \ldots \cdot K_r$. Since $G(K_i/k)$ is a finite, abelian group of exponent p, for all $i \leq r$, $G(\mathcal{K}_r/k)$ is finite, abelian of exponent p. Consequently, since K_i/k is unramified at v_{r+1}, for all $i \leq r$, and therefore $\mathcal{K}_r/k$ is unramified at v_{r+1}, the decomposition group $G_{v_{r+1}}$ of v_{r+1} in $\mathcal{K}_r/k$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{(m-\gamma_{r+1})}$, where $m-1 \leq \gamma_{r+1} \leq m$. We apply Proposition 3.3 for $w := v_{r+1}$, $T = T_r := \{v_1, \dots, v_r\}$, and $\alpha := \gamma_{r+1}$, and let K_{r+1}/k be a fixed abelian extension of k, which is totally ramified at v_{r+1}, unramified everywhere else, completely split at all the primes in T_r, such that

$$G(K_{r+1}/k) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^{\gamma_{r+1}} .$$

We let L to be the compositum of all the fields $\{K_i\}_{i \geq 1}$ constructed above and claim that $[L:k]_v = n$, for all $v \in \mathcal{P}_k$. Indeed, let $v = v_r$, for some $r \geq 1$. Then $[L:k]_v = e_{v_r}(L/k) \cdot f_{v_r}(L/k)$, where $e_{v_r}(L/k)$ and $f_{v_r}(L/k)$ are the ramification and inertia degrees of v_r in L/k, respectively. Since K_i/k is unramified at v_r, for all $i \neq r$ (see **Step i**, for $i \neq r$) and K_r/k is totally ramified at v_r (see **Step r**), we have

$$e_{v_r}(L/k) = [K_r : k] = p^{\gamma_r} .$$

Since, by the definition of γ_r, we have $f_{v_r}(\mathcal{K}_{r-1}/k) = p^{(m-\gamma_r)}$, and $f_{v_r}(K_r/k) = 1$ (as K_r/k is totally ramified at v_r, cf. **Step r.**), and $f_{v_r}(K_i/k) = 1$, for all $i \geq r+1$ (as K_i/k is completely split at v_r, cf. **Step i.**, for all $i \geq r+1$), we have an equality

$$f_{v_r}(L/k) = f_{v_r}(\mathcal{K}_{r-1}/k) = p^{(m-\gamma_r)} .$$

Consequently, we have an equality

$$[L:k]_v = p^{(m-\gamma_r)} \cdot p^{\gamma_r} = p^m = n ,$$

which concludes the proof of Theorem 3.1. $\square$

4. Constant local degrees coprime to p

For a characteristic p function field k, we denote by h_k the order of its Picard group $\mathrm{Pic}^0(k)$. In this section, we will prove the following.

Theorem 4.1. *Let k be a characteristic p function field and $n \in \mathbb{N}_{\geq 1}$, such that $\gcd(n, p \cdot h_k) = 1$. Then, there exists an abelian extension L/k whose local degrees $[L : k]_v$ are equal to n, for all $v \in \mathcal{P}_k$.*

Before proceeding to the proof of Theorem 4.1, we need a series of preparatory lemmas. First, let us observe that Lemma 2.1 shows that it suffices to prove Theorem 4.1 for n equal to a power of a prime number which does not divide $p \cdot h_k$. Let us fix such a prime number ℓ and let $n := \ell^m$, for a fixed $m \in \mathbb{N}_{\geq 1}$. Let us fix $d \in \mathbb{N}_{\geq 1}$, such that

$$n \mid \frac{q^d - 1}{q - 1}\,.$$

The reader will notice right away that, for example, d could be taken to be the order of the class $\widehat{q}$ of q in $(\mathbb{Z}/nn'\mathbb{Z})^\times$, where $n' := \gcd(n, q-1)$. Let $\mathcal{P}_{k,nd}$ denote the set of primes in k, whose degrees over $\mathbb{F}_q$ are divisible by nd. Lemma 1.2 above shows that $\mathcal{P}_{k,nd}$ coincides with the set of primes in k which split in the constant field extension k_{nd}/k. Chebotarev's density theorem implies that $\mathcal{P}_{k,nd}$ has density $1/nd$ in $\mathcal{P}_k$ and, consequently, is an infinite set. Let us fix $v_\infty \in \mathcal{P}_k$, such that $\gcd(\ell, d_{v_\infty}) = 1$ (i.e. v_∞ is one of the infinitely many primes in $\mathcal{P}_k$ which are inert in k_ℓ/k, according to Lemma 1.2.) Let

$$A_\infty := \{x \in k \mid \mathrm{ord}_v(x) \geq 0\,, \text{ for all } v \in \mathcal{P}_k \setminus \{v_\infty\}\}\,.$$

Then A_∞ is a Dedekind domain whose non–zero prime ideals are in one–to–one correspondence with $\mathcal{P}_k \setminus \{v_\infty\}$, and its (finite) ideal class-group $\mathrm{Pic}(A_\infty)$ fits into an exact sequence

$$1 \longrightarrow \mathrm{Pic}^0(k) \longrightarrow \mathrm{Pic}(A_\infty) \longrightarrow \mathbb{Z}/d_{v_\infty}\mathbb{Z} \longrightarrow 1\,,$$

where the last non–trivial map is the degree modulo d_∞ map applied to ideal–classes of A_∞, viewed as classes of divisors of k whose support does not contain v_∞. Let $h_\infty := \mathrm{card}(\mathrm{Pic}(A_\infty))$. Then $h_\infty = d_{v_\infty} \cdot h_k$ and $\gcd(\ell, h_\infty) = 1$. Let us fix an ordering of the primes in $\mathcal{P}_k$ (i.e. a bijection between P_k and $\mathbb{Z}_{\geq 0}$),

$$\mathcal{P}_k := \{v_0 := v_\infty, v_1, v_2, v_3, \dots\}\,.$$

Proposition 4.2. *Let $w \in \mathcal{P}_{k,nd}$. Then there exists a unique field extension $k(w)_n/k$ with the following properties.*

1. *$k(w)_n/k$ is cyclic of degree n.*
2. *$k(w)_n/k$ is totally ramified at w and unramified at all $v \in \mathcal{P}_k \setminus \{w\}$.*
3. *The prime v_∞ splits completely in $k(w)_n/k$.*

Proof. Let $k(w)$ be the maximal abelian extension of k, of conductor dividing w, in which v_∞ splits completely. If an extension $k(w)_n/k$ with the properties listed above exists then, by class–field theory, $k(w)_n \subseteq k(w)$. Consequently, in order to prove Proposition 4.2 it suffices to show that $k(w)/k$ has a unique intermediate extension

$k(w)_n/k$ satisfying the properties above. In fact, we will show that $k(w)/k$ has a unique intermediate extension of degree n, and then show that this extension satisfies properties 1–3 above. Let H_∞ denote the A_∞–Hilbert class–field of k, i.e. the maximal abelian extension of k, which is unramified everywhere, in which v_∞ splits completely. Class–field theory shows that we have an inclusion $H_\infty \subseteq k(w)$ and an isomorphism of exact sequences of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \dfrac{U_w}{U_w^{(1)}\mathbb{F}_q^\times} & \longrightarrow & \dfrac{J_k}{k^\times\left[k_{v_\infty}^\times U_w^{(1)}\prod_v' U_v\right]} & \longrightarrow & \dfrac{J_k}{k^\times\left[k_{v_\infty}^\times U_w\prod_v' U_v\right]} & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\ 1 & \longrightarrow & G(k(w)/H_\infty) & \longrightarrow & G(k(w)/k) & \longrightarrow & G(H_\infty/k) & \longrightarrow & 1\,, \end{array}$$

where $\prod_v'$ is the product with respect to all primes $v \in \mathcal{P}_k \setminus \{v_\infty, w\}$, the vertical isomorphisms are given by the global Artin reciprocity map, and the horizontal morphism are the canonical inclusion and projection maps at the level of idèle classes and Galois groups, respectively. Consequently, we have the following group isomorphisms

$$G(H_\infty/k) \xrightarrow{\sim} \frac{J_k}{k^\times\left[k_{v_\infty}^\times U_w \times \prod_v' U_v\right]} \xrightarrow{\sim} \mathrm{Pic}(A_\infty)\,,$$
$$G(k(w)/H_\infty) \xrightarrow{\sim} U_w/U_w^{(1)} \cdot \mathbb{F}_q^\times \xrightarrow{\sim} \mathbb{F}_{q^{d_w}}^\times/\mathbb{F}_q^\times\,.$$

Since $\gcd(\ell, h_\infty) = 1$ and $w \in P_{k,nd}$, the isomorphisms above show that the ℓ–Sylow subgroup of $G(k(w)/k)$ is isomorphic to the ℓ–Sylow subgroup of $G(k(w)/H_\infty) \xrightarrow{\sim} \mathbb{F}_{q^{d_w}}^\times/\mathbb{F}_q^\times$, and therefore it is a cyclic group of order divisible by $n = \ell^m$. This implies right away that there exists a unique subfield of $k(w)$ of degree n over k, namely the maximal subfield of $k(w)$ which is fixed by the subgroup

$$G(k(w)/k)^n := \{g^n \mid g \in G(k(w)/k)\}$$

of $G(k(w)/k)$. Let us denote by $k(w)_n$ this subfield. Then, we have the following group isomorphisms,

$$\text{(7)}\quad \mathbb{F}_{q^{d_w}}^\times/(F_{q^{d_w}}^\times)^n \xrightarrow{\sim} U_w/U_w^n \overset{i_{w,n}}{\xrightarrow{\sim}} J_k/J_k^n k^\times[k_{v_\infty}^\times U_w^{(1)} \times \Pi_v' U_v] \overset{\rho_{w,n}}{\xrightarrow{\sim}} G(k(w)_n/k)\,.$$

The first two isomorphisms above are induced by the canonical inclusions $\mathbb{F}_{q^{d_w}}^\times \subseteq U_w$ and $U_w \subseteq J_k$, respectively, and $\rho_{w,n}$ is given by the global Artin reciprocity map. Since $k_{v_\infty}^\times$ and U_v, for all $v \neq v_\infty, w$, are contained in the kernel of the global Artin map $\rho_{w,n}$ in (7), global class–field theory implies that $k(w)_n/k$ is indeed completely split at v_∞ and unramified away from w. Also, the third and second isomorphisms in (7) show that $k(w)_n/k$ is totally ramified at w. This concludes the proof of Proposition 4.2. $\square$

Our next goal is to prove a reciprocity law giving an explicit description of those primes $v \in \mathcal{P}_k \setminus \{w, v_\infty\}$ which split completely, respectively stay inert in the cyclic extension $k(w)_n/k$, for a fixed $w \in \mathcal{P}_{k,nd}$. Let $\widehat{v}$ denote the class of v in $\mathrm{Pic}(A_\infty)$, for $v \in \mathcal{P}_k \setminus \{w, v_\infty\}$. Obviously, we have $\widehat{v}^{h_\infty} = 1$ in $\mathrm{Pic}(A_\infty)$. This means that there exists $x_v \in k^\times$, unique up to multiplication by an element in $\mathbb{F}_q^\times$, such that its divisor $\mathrm{div}(x_v)$ satisfies

$$\mathrm{div}(x_v) = h_\infty \cdot v - (h_\infty d_v / d_{v_\infty}) \cdot v_\infty\,. \tag{8}$$

For each $v \in \mathcal{P}_k \setminus \{w, v_\infty\}$, we fix an $x_v \in k^\times$ with this property.

Proposition 4.3 (a reciprocity law). *Let $w \in \mathcal{P}_{k,nd}$ and $v \in \mathcal{P}_k \setminus \{w, v_\infty\}$. Then, the following statements hold true.*

1. *The prime v splits completely in $k(w)_n/k$ is and only if the prime w splits completely in $k_{nd}(x_v^{1/n})/k$.*
2. *The prime v is inert in $k(w)_n/k$ if and only if the prime w does not split completely in $k_{nd}(x_v^{1/\ell})/k$.*

Proof. We begin with the following remark.

Remark. Since d was chosen so that $n \,|\, (q^d-1)/(q-1)$, we have an inclusion $\mathbb{F}_q^\times \subseteq (\mathbb{F}_{q^{nd}}^\times)^n \subseteq (k_{nd}^\times)^n$. Consequently, the fields $k_{nd}(x_v^{1/n})$ and $k_{nd}(x_v^{1/\ell})$ do not depend on the above choice for x_v and the extensions $k_{nd}(x_v^{1/n})/k$ and $k_{nd}(x_v^{1/\ell})/k$ are Galois. Also, equality (8) combined with the assumption that $\gcd(n, h_\infty) = 1$ shows that the extensions $k_{nd}(x_v^{1/n})/k_{nd}$ and $k_{nd}(x_v^{1/\ell})/k_{nd}$ are cyclic of degrees n and ℓ, respectively, totally ramified at all the primes in k_{nd} sitting above v, and unramified at all primes in k_{nd} sitting above primes other than v and v_∞ (in particular, unramified at w).

Let $\sigma_v \in G(k(w)_n/k)$ denote the Frobenius morphism associated to v in the extension $k(w)_n/k$. Then v splits completely in $k(w)_n/k$ if and only if $\sigma_v = 1$. However, since $\gcd(n, h_\infty) = 1$ and $\sigma_v \in G(k(w)_n/k) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$, we have $\sigma_v = 1$ if and only if $\sigma_v^{h_\infty} = 1$. Equality (8) combined with the definition of Artin's global reciprocity map $\rho_{w,n}$, shows that via the isomorphisms (7) we have

$$\sigma_v^{h_\infty} = \rho_{w,n}(\widetilde{x_v}) = \rho_{w,n} \circ i_{w,n}(\widehat{x_v}^{-1})\,, \tag{9}$$

where $\widetilde{x_v}$ is the class in $J_k/J_k^n k^\times[k_{v_\infty}^\times U_w^{(1)} \times \prod_v' U_v]$ of the idèle whose v–component is equal to x_v and the remaining components are trivial, and $\widehat{x_v}$ is the class of x_v in U_w/U_w^n. This shows that $\sigma_v^{h_\infty} = 1$ if and only if $x_v \in U_w^n$ or, equivalently, if and only if x_v is an n–power modulo w. This happens if and only if the equation $X^n - x_v = 0$ has at least one root modulo w. However, since $nd \,|\, d_w$ and therefore all the order n roots of unity are in $\mathbb{F}_{q^{d_w}}^\times$, the equation above has one root modulo w if and only if the polynomial $X^n - x_v$ splits into n distinct linear factors modulo w. However, since w splits completely in k_{nd}/k, this happens if and only if all the primes sitting above w in k_{nd} split completely in $k_{nd}(x_v^{1/n})/k_{nd}$, which is equivalent to the complete splitting of w in $k_{nd}(x_v^{1/n})/k_{nd}$. This concludes the proof of (1).

In order to prove (2), let us observe that the arguments above show that v is inert in $k(w)_n/k$ if and only if $\sigma_v^{h_\infty}$ is not an ℓ–power in $G(k(w)_n/k)$. Equalities (9) show that this happens if and only if x_v is not an ℓ–power modulo w. This happens if and only if the polynomial $X^\ell - x_v$ is irreducible modulo w. As above, this happens if and only if all the primes sitting above w in k_{nd} are inert in $k_{nd}(x_v^{1/\ell})/k_{nd}$. Since w splits completely in k_{nd}/k and the extension $k_{nd}(x_v^{1/\ell})/k_{nd}$ is of prime degree ℓ (see the Remark above), this happens if and only if w does not split completely in $k_{nd}(x_v^{1/n})/k$. $\square$

Proof of Theorem 4.1. As remarked at the beginning of §4.1, it suffices to construct an extension L/k as in the statement of Theorem 4.1, for $n = \ell^m$, with ℓ prime, $\gcd(\ell, ph_\infty) = 1$. The extension L/k will be constructed as a compositum of countably many linearly disjoint Galois extensions $\{L_i/k\}_{i\geq 0}$, with cyclic Galois groups $G(L_i/k)$ of degree n. The construction of L_i/k is done inductively and described in detail below. In what follows, we refer to the ordering of $\mathcal{P}_k$ fixed above the statement of Proposition 4.2.

Step 0. We let $L_0 := k_n$.

Step 1. We let $L_1 := k(v_{j_1})_n$ where v_{j_1} is a fixed prime in $\mathcal{P}_{k,nd}$.

Step 2. We let $L_2 := k(v_{j_2})_n$, where $j_2 > j_1$, and $v_{j_2} \in \mathcal{P}_{k,nd}$, with the following properties.

1. The prime v_{j_1} splits completely in $k(v_{j_2})_n/k$.
2. The primes $v_i \in \mathcal{P}_k$, with $i < j_1$ are inert in $k(v_{j_2})_n/k$.

Of course, before declaring **Step 2.** complete, we would need to prove the existence of a prime $v_{j_2} \in \mathcal{P}_{k,nd}$ with properties (1) and (2) above. Proposition 4.2 implies that a prime $v_{j_2} \in \mathcal{P}_k$ satisfies properties (1) and (2) above if, firstly, it splits completely in k_{nd}/k (which is equivalent to $v_{j_2} \in \mathcal{P}_{k,nd}$, according to Lemma 1.2), and secondly, it splits completely in $k_{nd}(x_{v_{j_1}}^{1/n})/k$ and it does not split completely in $k_{nd}(x_{v_i}^{1/\ell})/k$, for all $i < j_1$. Let $\mathcal{L}_2$ the extension of k_{nd} generated by $\{x_{v_{j_1}}^{1/n}\} \cup \{x_{v_i}^{1/\ell} \mid 1 \leq i < j_1\}$. Then, the Remark made in the proof of Proposition 4.2 shows that $\mathcal{L}_2/k$ is a Galois extension and, for ramification reasons (see the Remark above) one has an isomorphism of groups.

$$G(\mathcal{L}_2/k_{nd}) \xrightarrow{\sim} G(k_{nd}(x_{v_{j_1}}^{1/n})/k_{nd}) \times \prod_{i=1}^{j_1-1} G(k_{nd}(x_{v_i}^{1/\ell})/k_{nd})\,.$$

Chebotarev's Density Theorem applied to the finite Galois extension $\mathcal{L}_2/k$ shows that there are infinitely many primes $v \in \mathcal{P}_k$ which are completely split in k_{nd}/k and whose Frobenius morphism $\sigma_v \in G(\mathcal{L}_2/k_{nd})$ is sent under the isomorphism above to $(1, \sigma_1, \sigma_2, \dots, \sigma_{j_1-1})$, where σ_i is a generator of (the cyclic group, see the Remark above) $G(k_{nd}(x_{v_i}^{1/\ell})/k_{nd})$ for all i, $1 \leq i < j_1 - 1$. Any such prime v belongs to $\mathcal{P}_{k,nd}$ (since it is completely split in k_{nd}/k) and satisfies properties (1) and (2). In order to complete **Step 2.**, we fix $j_2 > j_1$, so that v_{j_2} is such a prime.

Now, assuming that we have constructed $L_0 := k_{nd}, L_1 := k(v_{j_1})_n, \dots, L_s := k(v_{j_s})_n$, for $s \geq 2$, we describe the construction of L_{s+1}.

Step (s+1). We let $L_{s+1} := k(v_{j_{s+1}})_n$, where $v_{j_{s+1}} \in \mathcal{P}_{k,nd}$, with $j_{s+1} > j_s$, satisfying the following properties.

1. The primes $v_{j_1}, \dots, v_{j_s}$ split completely in $k(v_{j_{s+1}})_n/k$.
2. The primes $v_i \in \mathcal{P}_k$, with $j_{s-1} < i < j_s$ are inert in $k(v_{j_{s+1}})_n/k$.

The proof of the existence of a prime $v_{j_{s+1}}$ with the above properties is almost identical to our proof of the existence of v_{j_2} (see **Step 2.** above.) As above, the Chebotarev's Density Theorem applied to the extension $\mathcal{L}_s/k$, where $\mathcal{L}_s$ is the field generated over k_{nd} by the set $\{x_{v_{j_1}}^{1/n}, \dots x_{v_{j_s}}^{1/n}\} \cup \{x_{v_i}^{1/\ell} \mid j_{s-1} \leq i < j_s\}$, implies the existence of infinitely many primes $v \in \mathcal{P}_k$ which are completely split in k_{nd}/k and whose Frobenius morphism $\sigma_v \in G(\mathcal{L}_s/k_{nd})$ is sent via the group isomorphism

$$G(\mathcal{L}_2/k_{nd}) \xrightarrow{\sim} \prod_{t=1}^{s} G(k_{nd}(x_{v_{j_t}}^{1/n})/k_{nd}) \times \prod_{i=j_{s-1}+1}^{j_s-1} G(k_{nd}(x_{v_i}^{1/\ell})/k_{nd})$$

into $(1, \dots, 1, \sigma_{j_{s-1}+1}, \dots \sigma_{j_s-1})$, where σ_i is a generator of $G(k_{nd}(x_{v_i}^{1/\ell})/k_{nd})$, for all i. We fix $j_{s+1} > j_s$, so that $v_{j_{s+1}}$ is such a prime.

We let the extension L/k be the compositum of all the extensions $\{L_i/k\}_{i\geq 0}$ defined above. We claim that L/k satisfies the property $[L:k]_v = n$, for all $v \in \mathcal{P}_k$. First of all, let us note that Proposition 4.2 implies that L_i/k are cyclic of degree n and linearly disjoint. Therefore, we have group isomorphisms

$$G(L/k) \xrightarrow{\sim} \prod_{i\geq 0} G(L_i/k) \xrightarrow{\sim} \prod_{i\geq 0} \mathbb{Z}/n\mathbb{Z}\,. \tag{10}$$

Let $v \in \mathcal{P}_k$. There are three distinct cases.

Case 0. Assume that $v = v_\infty$. Then, since v splits completely in L_i/k, for all $i \geq 1$ (see Proposition 4.2(3)) and v is inert in L_0/k, we have equalities $[L:k]_v = [L_0:k]_v = [L_0:k] = n$.

Case 1. Assume that $v = v_i \in \mathcal{P}_k \setminus \{v_\infty, v_{j_1}, v_{j_2}, \dots\}$. Then L_i/k is unramified at v for all $i \geq 0$ (see Proposition 4.2(2) and Lemma 1.2) and therefore L/k is unramified at v. Consequently, $[L:k]_v$ equals the order $\mathrm{ord}(\sigma_v)$ of the Frobenius morphism σ_v of v in L/k. The isomorphisms (10) above show that $\mathrm{ord}(\sigma_v) \leq n$. On the other hand, we claim that $[L:k]_v \geq n$. Indeed, we either have $i < j_1$, or there exists (a unique) $s \geq 2$, such that $j_{s-1} < i < j_s$. In the former case, **Step 2** above shows that $[L:k]_v \geq [L_2:k]_v = n$, and in the later case **Step (s+1)** above shows that $[L:k]_v \geq [L_{s+1}:k]_v = n$. Therefore $[L:k]_v = n$.

Case 2. Assume that $v = v_{j_s}$, for some $s \geq 1$. If $s = 1$, since v splits completely in L_0/k (see Lemma 1.2 and recall that $v \in \mathcal{P}_{k,nd}$ in this case), v splits completely in L_i/k, for all $i \geq 2$ (see **Step s**, for $s \geq 2$), and v is totally ramified in L_1/k (see Proposition 4.2), we have

$$[L:k]_v = [L_1:k]_v = [L_1:k] = n\,.$$

If $s \geq 2$, then v splits completely in L_i/k, for all $i < s$ (see Lemma 1.2 for $i = 0$ and **Step s** for $0 < i < s$), v splits completely in L_i/k, for all $i > s$ (see **Step i**, for $i > s$), and v is totally ramified in L_s/k (Proposition 4.2(2)), we have

$$[L:k]_v = [L_s:k]_v = [L_s:k] = n\,.$$

This concludes the proof of Theorem 4.1. □

5. Conclusions

We conclude with the following.

Theorem 5.1. *Let k be a global function field of characteristic p and $n \in \mathbb{Z}_{\geq 1}$, such that n is coprime to $h_k/p^{\mathrm{ord}_p(h_k)}$. Then, there exists an abelian Galois extension L/k, such that $[L:k]_v = n$, for all $v \in \mathcal{P}_k$.*

Proof. Apply Lemma 2.1, Theorem 3.1 and Theorem 4.1. □

Corollary 5.2. *For k and n as in Theorem 5.1, the group $\mathrm{Br}_n(k)$ is equal to $\mathrm{Br}(L/k)$, for an abelian extension L/k. In particular, $\mathrm{Br}_n(k)$ is algebraic.*

Proof. Combine Theorem 5.1 with Proposition 1.5. □

Remark. Finally, let us note that the hypotheses in Theorem 5.1 and Corollary 5.2 are met for all n if h_k is a p–power. In particular, they are met if k is a rational function field $\mathbb{F}_q(T)$.

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