BASE CHANGE FOR STARK–TYPE CONJECTURES "OVER Z"

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Abstract. Building upon ideas of Rubin, we formulate a new version "over **Z**" of Stark's Conjecture for abelian L–functions of arbitrary order of vanishing at $s = 0$. We prove a base change property for this statement, generalizing results of Hayes and Sands on base change for the Brumer–Stark Conjecture. We prove a theorem of comparison between the new statement and Rubin's own version "over **Z**" of Stark's Conjecture.

INTRODUCTION

In the 1970s and early 1980s Stark [15] developed a conjecture regarding the values of Artin L–functions at $s = 0$. In the final paper of [St], Stark formulated a refined conjecture ("over **Z**") for *abelian* L–functions of order of vanishing $r = 1$ at $s = 0$. The Brumer–Stark conjecture can be viewed as a particular case of this refined statement (see [18], Chapitre IV). Answering a question posed by Tate in [17], Hayes proved in [8] a base change property for the Brumer–Stark Conjecture.

Our work began as an attempt to prove the same type of property for a more general Stark–type conjecture "over **Z**", formulated by Rubin in [13], for abelian L–functions of arbitrary order of vanishing at $s = 0$. In the process, we have formulated a new generalization of Stark's original conjecture "over **Z**" (see conjecture $\mathrm{C}(K/k,\, S,\, r)$ in §2.1). This new statement, although seemingly slightly weaker than Rubin's (see §5.5 for the comparison theorem), behaves very naturally with respect to base change and it has the advantage of being formulated in terms of a single set of primes S of the base field, very much like Stark's original statement.

After stating the new conjecture in §2, we prove in §3 that it satisfies the desired base change property (see Theorem 3.1.1). In §4 we recover, as a corollary, Hayes' base change theorem for the Brumer–Stark Conjecture (see Theorem 4.1.1). In the process, we generalize the base change multipliers of Hayes and Sands, and interpret them as determinants in a new way. The new interpretation allows us to give very short and straightforward proofs of two integrality theorems of Hayes and Sands (see Theorems 4.2.3 and 4.2.4). In §5 we prove the comparison theorem between conjecture $\mathrm{C}(K/k,\, S,\, r)$ and Rubin's Conjecture. In §6 we list the cases in which conjecture $\mathrm{C}(K/k,\, S,\, r)$ is presently known to hold true.

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1. PRELIMINARY CONSIDERATIONS AND NOTATIONS

Let K/k be a finite, abelian extension of global fields, of Galois group $G = G(K/k)$. Let $\widehat{G}$ be the set of complex valued, irreducible characters of G. For every $\chi \in \widehat{G}$, let $e_\chi = 1/|G| \sum_{\sigma\in G} \chi(\sigma)\cdot\sigma^{-1}$ be the corresponding idempotent in the group–ring $\mathbf{C}[G]$.

If M is a $\mathbf{Z}[G]$–module and R is a commutative ring, then $RM := R\otimes_{\mathbf{Z}} M$, $M^* := \mathrm{Hom}_{\mathbf{Z}[G]}(M, \mathbf{Z}[G])$. For every $\chi \in \widehat{G}$, $(\mathbf{C}M)^\chi$ will denote the χ–component of the $\mathbf{C}[G]$–module $\mathbf{C}M$.

If w is a prime of K, we write K_w for the completion of K at w, and $|\ |_w : K_w \longrightarrow \mathbf{R}^+ \cup \{0\}$ for the w–absolute value, normalized so that

$$|x|_w = \begin{cases} \pm x \text{ (the usual absolute value)}, & \text{if } K_w = \mathbf{R} \\ x\bar{x}, & \text{if } K_w = \mathbf{C} \\ (\mathbf{N}w)^{-\mathrm{ord}_w(x)}, & \text{if } K_w \text{ is nonarchimedian,} \end{cases}$$

where $\mathbf{N}w$ is the cardinality of the residue field $K(w)$ at w. For a prime v in k, unramified in K/k, σ_v denotes its Frobenius automorphism in G.

Let μ_K be the group of roots of unity in K. Let w_K and $\mathcal{A}(K/k)$ denote respectively the order and the annihilator of μ_K as a $\mathbf{Z}[G]$–module. If one considers a function $\mathrm{N} : G \longrightarrow \mathbf{Z}$, such that

$$\zeta^\sigma = \zeta^{\mathrm{N}\sigma},\ \forall\zeta \in \mu_K\,, \forall\sigma \in G\,,$$

then $\mathcal{A}(K/k)$ is generated as a $\mathbf{Z}$–module by the set $\{\sigma - \mathrm{N}\sigma \mid \sigma \in G\} \cup \{w_K\}$.

For a finite, nonempty set S of primes in k, containing all primes which ramify in K/k and all the infinite primes (in the number field case), U_S will denote the group of S–units in K. If T is an auxiliary, finite set of finite primes in k, $U_{S,T}$ will denote the subgroup of U_S consisting of elements congruent to 1 modulo every prime in K dividing a prime in T. For any $u \in U_S$, $\tilde{u}$ will denote its image in $\mathbf{Q}U_S$ via the natural map $U_S \longrightarrow \mathbf{Q}U_S$. If M is a subgroup of U_S, then $\widetilde{M} = \{\tilde{m} \mid m \in M\}$ is the image of M in $\mathbf{Q}U_S$. In what follows, we will identify U_S/μ_K and $\widetilde{U_S}$ via the natural map $U_S/\mu_K \longrightarrow \widetilde{U_S}$, which is obviously a $\mathbf{Z}[G]$–isomorphism.

Definition 1.1. For K/k and S as above, let

$$U^{\mathrm{ab}}_{K/k,S} = \{u \in U_S \mid K(u^{1/w_K})/k \text{ abelian}\}.$$

Let us remark that $\widetilde{U^{\mathrm{ab}}_{K/k,S}}$ is a lattice of maximal rank in $\mathbf{Q}U_S$. For the following characterization theorem, the reader should consult Proposition 1.2, Chapitre IV in [18]:

Proposition 1.2. *Let $\varepsilon \in \mathbf{Q}U_S$. Then $\varepsilon \in \frac{1}{w_K}\widetilde{U^{\mathrm{ab}}_{K/k,S}}$ if and only if either* **(A)** *or* **(B)** *below are satisfied:*

(A) *There exist $u \in U_S$ and $\{\varepsilon_\alpha \mid \alpha \in \mathcal{A}(K/k)\} \subseteq U_S$ satisfying the following:*

(1) $\varepsilon^{w_K} = \tilde{u}$ *(in $\mathbf{Q}U_S$).*
(2) $u^\alpha = \varepsilon_\alpha^{w_K}$, $\forall\alpha \in \mathcal{A}(K/k)$.
(3) $\varepsilon_\alpha^\gamma = \varepsilon_\gamma^\alpha$, $\forall\alpha\,, \gamma \in \mathcal{A}(K/k)$.

(B) *Let $\mathcal{T}$ be a set of primes in k, disjoint from S, such that $\{\sigma_v - \mathbf{N}v \mid v \in \mathcal{T}\}$ generates $\mathcal{A}(K/k)$ as a $\mathbf{Z}$–module. Then, for every $v \in \mathcal{T}$, there exists $\varepsilon_v \in U_{S,\{v\}}$ such that $\varepsilon^{\sigma_v - \mathbf{N}v} = \widetilde{\varepsilon_v}$.*

Remark 1. In particular, the proposition above shows that, if $\varepsilon \in \frac{1}{w_K}\widetilde{U^{\text{ab}}_{K/k,S}}$ and $\alpha \in \mathcal{A}(K/k)$, then $\varepsilon^\alpha \in \widetilde{U_S}$.

For every $\chi \in \widehat{G}$, let $L_{K/k,S}(s,\chi)$ be the Artin L–function associated to χ, with Euler factors at primes in S removed. If $\chi \neq \mathbf{1}_G$, $L_{K/k,S}(s,\chi)$ is holomorphic at every $s \in \mathbf{C}$, while for $\chi = \mathbf{1}_G$, $L_{K/k,S}(s,\chi)$ is holomorphic everywhere except for $s = 1$, where it has a pole of order 1. For $\mathrm{Re}(s) > 1$, one has a uniformly convergent product expansion $L_{K/k,S}(s,\chi) = \prod_{v \notin S}(1 - \mathbf{N}v^{-s} \cdot \chi(\sigma_v))^{-1}$.

For fixed K/k and S, and for every $\chi \in \widehat{G}$, let $r_{\chi,S} = \mathrm{ord}_{s=0} L_{K/k,S}(s,\chi)$ be the order of vanishing of $L_{K/k,S}(s,\chi)$ at $s = 0$. As Tate shows in Chapitre 0 of [18],

$$r_{\chi,S} = \dim_{\mathbf{C}} (\mathbf{C}U_S)^\chi = \mathrm{card}\{v \in S \,:\, \chi|_{G_v} = \mathbf{1}_{G_v}\}\,, \tag{1}$$

where G_v is the decomposition group of v relative to K/k.

Let us now consider the Stickelberger function

$$\Theta_{K/k,S}(s) = \sum_{\chi \in \widehat{G}} L_{K/k,S}(s,\chi) \cdot e_{\chi^{-1}}\,.$$

We think of $\Theta_{K/k,S}$ as a complex meromorphic function with values in $\mathbf{C}\,[G]$, holomorphic everywhere, except for $s = 1$, where it has a pole of order 1. The value $\Theta_{K/k,S}(0)$ satisfies the following most remarkable integrality properties, proved by Deligne and Ribet in [3]:

Theorem 1.3 (Deligne–Ribet). *If $\alpha \in \mathcal{A}(K/k)$, then $\alpha \cdot \Theta_{K/k,S}(0) \in \mathbf{Z}\,[G]$.*

2. THE CONJECTURE

2.1. The statement of the conjecture

Let K/k and S be as in §1, and let $r \geq 1$ be an integer. Let us assume that the set of data $(K/k,\, S,\, r)$ satisfies the following extended set of hypotheses:

$$\text{(H)} \qquad \begin{cases} S \text{ contains all the infinite primes.} \\ S \text{ contains all the primes which ramify in } K/k. \\ S \text{ contains at least } r \text{ primes which split completely in } K/k. \\ |S| \geq r+1. \end{cases}$$

Hypotheses (H) imply that, for any $\chi \in \widehat{G}$, we have $r_{\chi,S} \geq r$ (see (1)), and therefore $\Theta^{(r)}_{K/k,S}(0) := \lim_{s \longrightarrow 0} s^{-r}\Theta_{K/k,S}(s)$ makes sense in $\mathbf{C}\,[G]$.

If R is a subring of $\mathbf{C}$ and M an $R\,[G]$–module without R–torsion, let

$$M_{r,S} = \{x \in M \mid e_\chi \cdot x = 0 \text{ in } \mathbf{C}M, \forall \chi \in \widehat{G} \text{ such that } r_{\chi,S} > r\}\,.$$

Let us choose an r-tuple $V=(v_1,\dots,v_r)$ of r distinct primes in S which split completely in K/k, and let us fix $W=(w_1,\dots,w_r)$, where w_i is a prime in K lying above v_i, for any $1\le i\le r$. For every $\mathbf{Z}[G]$–module M, let $\wedge_G^r M$ denote the r–th exterior power over $\mathbf{Z}[G]$. As in [13], one can define a regulator map:

$$\mathbf{C}\wedge_G^r U_S \xrightarrow{R_W} \mathbf{C}[G]\ ,$$

by letting $R_W(u_1\wedge\cdots\wedge u_r)=\det\limits_{1\le i,j\le r}\left(-\sum\limits_{\sigma\in G}\log|u_j|_{w_i^\sigma}\cdot\sigma\right)$, for $u_1,\dots,u_r\in U_S$, and then extending to $\mathbf{C}\wedge_G^r U_S$ by $\mathbf{C}$–linearity.

Remark 1. As pointed out in Remark 2, §1.6 of [10], R_W is a $\mathbf{C}[G]$–morphism, which induces an isomorphism

$$R_W|_{(\mathbf{C}\wedge_G^r U_S)_{r,S}}:(\mathbf{C}\wedge_G^r U_S)_{r,S}\xrightarrow{\sim}\mathbf{C}[G]_{r,S}\,.$$

For every $(r-1)$–tuple $(\phi_1,\dots,\phi_{r-1})\in\{U_S^*\}^{r-1}$, one can view each ϕ_i as an element of $\mathrm{Hom}_{\mathbf{C}[G]}(\mathbf{C}U_S,\mathbf{C}[G])=\mathbf{C}U_S^*$, and define a $\mathbf{C}[G]$–morphism

$$\mathbf{C}\wedge_G^r U_S\xrightarrow{\phi_1\wedge\cdots\wedge\phi_{r-1}}\mathbf{C}U_S\ ,$$

by $\phi_1\wedge\cdots\wedge\phi_{r-1}(u_1\wedge\cdots\wedge u_r)=\sum\limits_{1\le k\le r}(-1)^{k+1}\det\limits_{\substack{1\le i\le r-1\\1\le j\le r\\ j\ne k}}(\phi_i(u_j))\cdot u_k$, for every $u_1,\dots,u_r\in\mathbf{C}U_S$. (In order to simplify notations, here and often throughout this paper, we write the internal operation on $\mathbf{C}U_S$ additively rather than multiplicatively.)

Definition 2.1.1. Assuming that $(K/k,\,S,\,r)$ satisfies hypotheses (H), let Λ_S be the $\mathbf{Z}[G]$–submodule of $\mathbf{Q}\wedge_G^r U_S$ defined by

$$\Lambda_S=\{\varepsilon\in(\mathbf{Q}\wedge_G^r U_S)_{r,S}\mid(\phi_1\wedge\cdots\wedge\phi_{r-1})(\varepsilon)\in\tfrac{1}{w_K}\widetilde{U^{\mathrm{ab}}_{K/k,S}},\ \forall\phi_1,\dots,\phi_{r-1}\in U_S^*\}\,.$$

Conjecture $C(K/k,\,S\,,r)$. *Assuming that the set of data $(K/k,\,S,\,r)$ satisfies hypotheses (H) then, for any choice of V and W, there exists a unique $\varepsilon_{S,W}\in\Lambda_S$ such that $R_W(\varepsilon_{S,W})=\Theta^{(r)}_{K/k,S}(0)$.*

Remark 2. Let us note that, under the hypotheses above, there always exists a unique element $\varepsilon_{S,W}$ in $(\mathbf{C}\wedge_G^r U_S)_{r,S}$, satisfying the above regulator condition (see Remark 1 above, and notice that $\Theta^{(r)}_{K/k,S}(0)\in\mathbf{C}[G]_{r,S}$). In particular, the uniqueness in conjecture C is automatic.

Remark 3. Let us observe that, in the case $r=1$, $\Lambda_S=\frac{1}{w_K}\{\widetilde{U^{\mathrm{ab}}_{K/k,S}}\}_{1,S}$, and therefore $C(K/k,\,S\,,1)$ is precisely Stark's own conjecture "over $\mathbf{Z}$", in Tate's equivalent formulation (see [18], Chapitre IV, §2, Conjecture 2.2).

2.2. Changing S and r in conjecture C

The following considerations have perfect analogues of those in [13], §5.

Let us assume that the set of data $(K/k,\,S,\,r)$ satisfies hypotehses (H), let us fix r–tuples $V=(v_1,\dots,v_r)$ and $W=(w_1,\dots w_r)$ as in §2.1, and let $\varepsilon_{S,W}$ be the unique element in $(\mathbf{C}\wedge_G^r U_S)_{r,S}$ satisfying the regulator condition $R_W(\varepsilon_{S,W}) = \Theta_{K/k,S}^{(r)}(0)$ (see Remark 2 in §2.1).

Let us consider an integer $r' > r$ and a set $\{v_{r+1}, \dots, v_{r'}\}$ of $r'-r$ distinct primes in k, which are not in S, and which split completely in K/k. Let us fix primes w_i in K lying above v_i, for every $i = r+1, \dots, r'$, and let $S_{r'} = S \cup \{v_{r+1}, \dots, v_{r'}\}$, $V_{r'} = (v_1, \dots, v_r, v_{r+1}, \dots, v_{r'})$ and $W_{r'} = (w_1, \dots, w_r, w_{r+1}, \dots, w_{r'})$. Obviously, the set of data $(K/k, S_{r'}, r')$ satisfies hypotheses (H) as well. We will denote by $\varepsilon_{S_{r'},W_{r'}}$ the unique element in $\left(\mathbf{C}\wedge_G^{r'} U_{S_{r'}}\right)_{r',S_{r'}}$ satisfying the proper regulator condition.

Since the primes $w_{r+1}, \dots, w_{r'}$ are necessarily finite primes, we can define the $\mathbf{C}[G]$–morphisms

$$\widehat{w_i} : \mathbf{C}U_{S_{r'}} \longrightarrow \mathbf{C}[G]\ ,\ \text{for every } i = r+1, \dots, r'\,,$$

by $\widehat{w_i}(u) = \sum_{\sigma\in G} \mathrm{ord}_{w_i^\sigma}(u)\cdot\sigma$, for every $u \in U_{S_{r'}}$ and then extend by $\mathbf{C}$–linearity.

For every $\mathbf{C}[G]$–module M, and every $\phi \in \mathrm{Hom}_{\mathbf{C}[G]}(M, \mathbf{C}[G])$, and every integer $t \geq 1$, one can define a $\mathbf{C}[G]$–morphism

$$\wedge_{\mathbf{C}[G]}^t M \xrightarrow{\phi^{(t)}} \wedge_{\mathbf{C}[G]}^{t-1} M\,,$$

by $\phi^{(t)}(x_1\wedge\cdots\wedge x_t) = \sum_{1\leq i\leq t} (-1)^{i+1}\phi(x_i)\cdot(x_1\wedge\cdots\wedge x_{i-1}\wedge x_{i+1}\wedge\cdots\wedge x_t)$. With this definition in mind, we obtain a $\mathbf{C}[G]$–morphism

$$\mathbf{C}\wedge_G^{r'} U_{S_{r'}} \xrightarrow{\Psi} \mathbf{C}\wedge_G^r U_{S_{r'}}$$

defined by $\Psi = (\widehat{w_{r'}})^{(r')}\circ\cdots\circ(\widehat{w_{r+1}})^{(r+1)}$.

Proposition 2.2.1. *Under the hypotheses above, if we identify $\mathbf{Q}\wedge_G^r U_S$ with its image in $\mathbf{Q}\wedge_G^{r'} U_{S_{r'}}$, via the inclusion $U_S \subseteq U_{S_{r'}}$, then:*

(1) *Ψ induces a $\mathbf{C}[G]$–isomorphism $\left(\mathbf{C}\wedge_G^{r'} U_{S_{r'}}\right)_{r',S_{r'}} \xrightarrow{\sim} (\mathbf{C}\wedge_G^r U_S)_{r,S}$.*

(2) *$\Psi\left(\Lambda_{S_{r'}}\right) \subseteq \Lambda_S$, where $\Lambda_{S_{r'}}$ is the lattice associated to $(K/k, S_{r'}, r')$.*

(3) *$\Psi(\varepsilon_{S_{r'},W_{r'}}) = \varepsilon_{S,W}$.*

Proof. The proof is almost identical to that of Lemma 5.1 and Proposition 5.2 in [13]. The main facts to be used are that the inclusion $U_S \subseteq U_{S_{r'}}$ induces a surjective group morphism $U_{S_{r'}}^* \twoheadrightarrow U_S^*$, and the equalities:

$$R_{W_{r'}} = \left(\prod_{r+1\leq i\leq r'} \log \mathrm{N}v_i\right)\cdot(R_W\circ\Psi)\ \text{ on }\ \left(\mathbf{C}\wedge_G^{r'} U_{S_{r'}}\right)_{r',S_{r'}}$$

$$\Theta_{S_{r'}}^{(r')}(0) = \left(\prod_{r+1\leq i\leq r'} \log \mathrm{N}v_i\right)\cdot\Theta_S^{(r)}(0)\,.$$

□

An easy but important consequence of this proposition is the following

Corollary 2.2.2. *Under the hypotheses above, assume that* $C(K/k,\, S_{r'},\, r')$ *is true. Then* $C(K/k,\, S,\, r)$ *is true as well.*

3. BASE CHANGE FOR CONJECTURE C

3.1. The formulation of the problem

Let K/k be a finite, abelian extension of global fields, and let us assume that S is a set of primes in k, such that the set of data $(K/k,\, S,\, 1)$ satisfies hypotheses (H). Let k'/k be an intermediate extension ($k \subseteq k' \subseteq K$) of Galois group $\Gamma = G(k'/k)$ and degree r, and let $G' = G(K/k')$.

We will denote by S' the set of primes in k' lying above primes in S. Under the present assumptions, it is obvious that the set of data $(K/k',\, S',\, r)$ also satisfies hypotheses (H). The goal of this section is to prove the following base change property of conjecture C:

Theorem 3.1.1. *Under the assumptions above, if* $C(K/k,\, S,\, 1)$ *is true, then* $C(K/k',\, S',\, r)$ *is also true.*

Before proceeding to the proof of the above theorem, we would like to make several reductions. Let $\pi : \mathbf{Z}[G] \longrightarrow \mathbf{Z}[\Gamma]$ denote the ring homomorphism induced by the natural surjection $G \twoheadrightarrow \Gamma$, and let us fix $\gamma_1, \dots, \gamma_r \in G$, such that

(1) $\gamma_r = 1$
(2) $\{\overline{\gamma_1} = \pi(\gamma_1), \dots, \overline{\gamma_r} = \pi(\gamma_r)\} = \Gamma$.

Let us fix a prime $v \in S$ which splits completely in K/k, and a prime w in K lying above v. Let $w_i = w^{\gamma_i}$, and let v_i be the prime lying below w_i in k', for all $1 \le i \le r$. Let $\varepsilon_S := \varepsilon_{S,W}$ be the unique element of Λ_S whose existence is predicted by $C(K/k,\, S,\, 1)$, with $V = \{v\}$ and $W = \{w\}$. We remind the reader that in this case $\Lambda_S \subseteq \mathbf{Q}U_S$.

According to Remark 2 in §5.1 below, it is enough to show $C(K/k',\, S',\, r)$ for the r–tuples $V' = (v_1, \dots, v_r)$ and $W' = (w_1, \dots, w_r)$. Let $\varepsilon_{S'} = \varepsilon_{S',W'}$ be the unique element in $(\mathbf{C} \wedge^r_{G'} U_S)_{r,S}$ satisfying the regulator condition $R_{W'}(\varepsilon_{S',W'}) = \Theta^{(r)}_{K/k',S'}(0)$ (see Remark 2, §2.1).

Proposition 3.1.2. *Under the above assumptions, we have*

$$\varepsilon_{S'} = \varepsilon_S^{\gamma_1} \wedge \cdots \wedge \varepsilon_S^{\gamma_r}\,, \tag{2}$$

where the exterior powers are considered with respect to $\mathbf{C}[G']$.

Proof. Due to the uniqueness property (see Remark 2, §2.1), all we have to prove is that the right–hand side of the equality above satisfies the corresponding regulator condition, and that it belongs to $(\mathbf{C} \wedge^r_{G'} U_S)_{r,S'}$.

Let us first remark that if one considers the endomorphism of the free $\mathbf{C}[G']$–module $\mathbf{C}[G]$ given by multiplication by $\Theta'_{K/k,S}(0)$, then one has the equality:

$$\Theta^{(r)}_{K/k',S'}(0) = \det_{\mathbf{C}[G']}\left(\mathbf{C}[G] \xrightarrow{\Theta'_{K/k,S}(0)} \mathbf{C}[G]\right). \tag{3}$$

Indeed, this follows from the functorial properties of the Artin L–functions:

$$
(4)\qquad
\begin{aligned}
L_{K/k',S'}(s,\chi') &= \prod_{\substack{\chi\in\widehat{G}\\ \chi|_{G'}=\chi'}} L_{K/k,S}(s,\chi)\,,\\
L^{(r)}_{K/k',S'}(0,\chi') &= \prod_{\substack{\chi\in\widehat{G}\\ \chi|_{G'}=\chi'}} L'_{K/k,S}(0,\chi)\,,\ \text{for all } \chi'\in\widehat{G}'\,.
\end{aligned}
$$

For any prime $\wp$ of K, let us consider the $\mathbf{C}[G]$–morphism $\lambda_\wp$ and the $\mathbf{C}[G']$–morphism $\widetilde{\lambda_\wp}$,

$$
\mathbf{C}U_S \xrightarrow{\lambda_\wp,\widetilde{\lambda_\wp}} \mathbf{C}U_S\,,
$$

defined on U_S by $\lambda_\wp(\varepsilon) = -\sum_{\sigma\in G}\log|\varepsilon|_{\wp^\sigma}\cdot\sigma$ and $\widetilde{\lambda_\wp}(\varepsilon) = -\sum_{\sigma'\in G'}\log|\varepsilon|_{\wp^{\sigma'}}\cdot\sigma'$ respectively, and then extended to $\mathbf{C}U_S$ by $\mathbf{C}$–linearity. Obviously, we then have

$$
(5)\qquad \lambda_\wp(\varepsilon) = \sum_{1\le j\le r}\widetilde{\lambda_{\wp^{\gamma_j}}}(\varepsilon)\cdot\gamma_j\,,\ \text{for all } \varepsilon\in\mathbf{C}U_S\,.
$$

By first noting that the W–regulator condition for ε_S can now be writen as $\Theta'_{K/k,S}(0) = \lambda_w(\varepsilon_S)$, let us compute the determinant in (3) with respect to the $\mathbf{C}[G']$–basis $\{\gamma_1,\dots,\gamma_r\}$ of $\mathbf{C}[G]$. For every $i=1,\dots,r$ we have

$$
\begin{aligned}
\Theta'_{K/k,S}(0)\cdot\gamma_i &= \lambda_w(\varepsilon_S)\cdot\gamma_i = \\
&= \lambda_w(\varepsilon_S^{\gamma_i}) = \sum_{1\le j\le r}\widetilde{\lambda_{w^{\gamma_j}}}(\varepsilon_S^{\gamma_i})\cdot\gamma_j\,.
\end{aligned}
$$

This gives $\Theta^{(r)}_{K/k',S'}(0) = \det_{1\le i,j\le r}\left(\widetilde{\lambda_{w^{\gamma_j}}}(\varepsilon_S^{\gamma_i})\right) = R_{W'}(\varepsilon_S^{\gamma_1}\wedge\dots\wedge\varepsilon_S^{\gamma_r})$, which shows that the proper regulator condition is satisfied.

Now let us consider $\chi'\in\widehat{G}'$ such that $r_{\chi',S'}>r$. The definitions and equalities (4) show that there exists at least one $\chi\in\widehat{G}$, such that $r_{\chi,S}>1$, and therefore $e_\chi\cdot\varepsilon_S=0$ in $\mathbf{C}U_S$. Let $\{\chi_1,\dots,\chi_r\}=\{\chi\in\widehat{G}\,:\,\chi|_{G'}=\chi'\}$. If we take into account that $e_{\chi'}=e_{\chi_1}+\dots+e_{\chi_r}$ in $\mathbf{C}[G]$, we obtain

$$
e_{\chi'}\cdot(\varepsilon_S^{\gamma_1}\wedge\dots\wedge\varepsilon_S^{\gamma_r}) = \left[\sum_{\sigma\in\Sigma_r}\epsilon(\sigma)\chi_{\sigma(1)}(\gamma_1)\cdot\ldots\cdot\chi_{\sigma(r)}(\gamma_r)\right]\cdot(e_{\chi_1}\varepsilon_S\wedge\dots\wedge e_{\chi_r}\varepsilon_S)=0\,.
$$

Here Σ_r is the permutation group in r letters and $\epsilon(\sigma)$ is the signature of $\sigma\in\Sigma_r$.□

Remark 1. Let us now note that Proposition 3.1.2 reduces Theorem 3.1.1 to the following statement in $\mathbf{Q}U_S$ (see Remark 3, §2.1 as well):

$$
\varepsilon_S\in\frac{1}{w_K}\{\widetilde{U^{\mathrm{ab}}_{K/k,S}}\}_{1,S}\implies \varepsilon_{S'}=\varepsilon_S^{\gamma_1}\wedge\dots\wedge\varepsilon_S^{\gamma_r}\in\Lambda_{S'}\,.
$$

3.2. The $\beta_{S,\Phi}$–multipliers

With the same notations used in the previous section, let us consider an $(r-1)$–tuple $\Phi = (\phi_1, \dots, \phi_{r-1}) \in \mathrm{Hom}_{\mathbf{Z}[G']}(U_S, \mathbf{Z}[G'])^{r-1}$. Let us fix, once and for all, a function $\mathrm{N} : G \longrightarrow \mathbf{Z}$, such that $\zeta^\sigma = \zeta^{\mathrm{N}\sigma}$, $\forall \zeta \in \mu_K$, $\forall \sigma \in G$. We then have

$$(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon_{S'}) = \pm \beta_{S,\Phi} \cdot \varepsilon_S \,, \tag{6}$$

where

$$\begin{aligned}\beta_{S,\Phi} &= \begin{vmatrix} \phi_1(\varepsilon_S^{\gamma_1}) & \dots & \phi_1(\varepsilon_S^{\gamma_r}) \\ \vdots & & \vdots \\ \phi_{r-1}(\varepsilon_S^{\gamma_1}) & \dots & \phi_{r-1}(\varepsilon_S^{\gamma_r}) \\ \gamma_1 & \dots & \gamma_r \end{vmatrix} = \\ &= \begin{vmatrix} \phi_1(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \dots & \phi_1(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \phi_1(\varepsilon_S) \\ \vdots & & \vdots & \vdots \\ \phi_{r-1}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \dots & \phi_{r-1}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \phi_{r-1}(\varepsilon_S) \\ \gamma_1 - \mathrm{N}\gamma_1 & \dots & \gamma_{r-1} - \mathrm{N}\gamma_{r-1} & 1 \end{vmatrix}.\end{aligned} \tag{7}$$

The above determinants are viewed over $\mathbf{Q}[G]$. Equality (6) and the first equality in (7) follow directly from the definition of $(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon_{S'})$. The second equality in (7) is obtained by multiplying the last column of the determinant in the middle by $\mathrm{N}\gamma_1, \dots, \mathrm{N}\gamma_{r-1}$, and then subtracting it from columns $1, \dots, r-1$ respectively. (We remind the reader that $\gamma_r = 1$.)

Remark 1. Remark 1 in §3.1, combined with equalities (6) and (7) above, reduces Theorem 3.1.1 to the following statement:

$$\varepsilon_S \in \frac{1}{w_K} \widetilde{U_{K/k,S}^{\mathrm{ab}}} \Longrightarrow \beta_{S,\Phi} \cdot \varepsilon_S \in \frac{1}{w_K} \widetilde{U_{K/k,S}^{\mathrm{ab}}}, \ \forall \Phi \in \mathrm{Hom}_{\mathbf{Z}[G']}(U_S, \mathbf{Z}[G'])^{r-1} \,,$$

where $\beta_{S,\Phi}$ is either one of the determinants in (7).

It is now clear that in order to prove Theorem 3.1.1, our attention should first focus on studying the "multipliers" $\beta_{S,\Phi}$. The following proposition establishes a most remarkable property of these objects.

Proposition 3.2.1. *Under the hypotheses of Theorem 3.1.1, let $\alpha \in \mathcal{A}(K/k')$ and $\Phi \in Hom_{\mathbf{Z}[G']}(U_S, \mathbf{Z}[G'])^{r-1}$. Then*

$$\alpha \cdot \beta_{S,\Phi} \in \mathcal{A}(K/k) \,.$$

Proof. Since $\phi_1, \dots, \phi_{r-1}$ are $\mathbf{Z}[G']$–linear, if we multiply the last column of the second determinant in (7) by α, we get the equality

$$\alpha \cdot \beta_{S,\Phi} = \begin{vmatrix} \phi_1(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \dots & \phi_1(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \phi_1(\varepsilon_S^{\alpha}) \\ \vdots & & \vdots & \vdots \\ \phi_{r-1}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \dots & \phi_{r-1}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \phi_{r-1}(\varepsilon_S^{\alpha}) \\ \gamma_1 - \mathrm{N}\gamma_1 & \dots & \gamma_{r-1} - \mathrm{N}\gamma_{r-1} & \alpha \end{vmatrix}. \tag{8}$$

Now let us recall that since $\varepsilon_S \in \frac{1}{w_K}\widetilde{U^{\mathrm{ab}}_{K/k,S}}$, Remark 1 in §1 shows that $\varepsilon_S^\theta \in \widetilde{U_S}$, for any $\theta \in \mathcal{A}(K/k)$, and in particular for $\theta = \alpha, \gamma_1 - \mathrm{N}\gamma_1, \ldots, \gamma_{r-1} - \mathrm{N}\gamma_{r-1}$. This shows that the determinant above has entries in $\mathbf{Z}[G]$. Since its last row has entries in $\mathcal{A}(K/k)$, we can conclude that $\alpha \cdot \beta_{S,\Phi} \in \mathcal{A}(K/k)$. □

In §4.2 we will show that Hayes' and Sands' base–change multiplier (see [8] and [14]) is a particular case of $\beta_{S,\Phi}$. Based on equality (8), we recover integrality theorems of Hayes and Sands (see Theorem 1 in [8]).

3.3. The proof of Theorem 3.1.1

Let $u \in U_S$ and $\{\varepsilon_\alpha \,|\, \alpha \in \mathcal{A}(K/k)\} \subseteq U_S$ satisfying properties **(A)**(1)–(3) in Proposition 1.2, for $\varepsilon = \varepsilon_S$. Let $\lambda = u^{1/w_K}$ be a w_K–root of u in the algebraic closure of k. Proposition 1.2 **(A)**(1) then implies that the extension $K(\lambda)/k$ is abelian. Let Σ be its Galois group and let $s : G \longrightarrow \Sigma$ be a section of the natural projection $\Sigma \twoheadrightarrow G$. We extend s by $\mathbf{Z}$–linearity to an additive map $s : \mathbf{Z}[G] \longrightarrow \mathbf{Z}[\Sigma]$. In general, s cannot be chosen to be multiplicative.

Lemma 3.3.1.

(1) *For every $\alpha \in \mathcal{A}(K/k)$, $\lambda^{s(\alpha)} \in U_S$. In particular $\lambda^{s(\alpha\cdot\beta_{S,\Phi})} \in U_S$, for any $\alpha \in \mathcal{A}(K/k')$, and any $\Phi \in \mathrm{Hom}_{\mathbf{Z}[G']}(U_S, \mathbf{Z}[G'])^{r-1}$.*

(2) *If $\alpha \in \mathcal{A}(K/k)$ and $a, b \in \mathbf{Z}[G]$, then $\lambda^{s(\alpha)s(ab)} = \lambda^{s(\alpha)s(a)s(b)}$.*

Proof. (1) Let $\alpha \in \mathcal{A}(K/k)$, and let $\tau \in G(K(\lambda)/k)$. Since Σ is abelian, we have $\{\lambda^{s(\alpha)}\}^{\tau-1} = \{\lambda^{\tau-1}\}^{s(\alpha)}$. However, $\lambda^{\tau-1} = \zeta\cdot\lambda$, for a certain $\zeta \in \mu_K$, and therefore $\{\lambda^{\tau-1}\}^{s(\alpha)} = \zeta^{s(\alpha)} = \zeta^\alpha = 1$. This shows that $\lambda^{s(\alpha)}$ is fixed by $G(K(\lambda)/K)$, and therefore $\lambda^{s(\alpha)} \in K^\times$. Since $u \in U_S$, $\lambda \in U_{K(\lambda),S}$, and therefore $\lambda^{s(\alpha)} \in U_S$. The second statement in (1) is now a direct consequence of Proposition 3.2.1.

(2) Since (1) implies that $\lambda^{s(\alpha)} \in K^\times$, we obviously have $\{\lambda^{s(\alpha)}\}^{s(ab)-s(a)s(b)} = \{\lambda^{s(\alpha)}\}^{ab-ab} = 1$. □

Now we would like to conclude that $\varepsilon_{S,\Phi} \overset{\mathrm{def}}{=} \beta_{S,\Phi}\cdot\varepsilon_S \in \frac{1}{w_K}\widetilde{U^{\mathrm{ab}}_{K/k',S'}}$, for every $\Phi = (\phi_1, \ldots \phi_{r-1})$ as in Proposition 3.2.1. According to Remark 1, §3.2, this would end the proof of Theorem 3.1.1. Let us fix such a Φ. We are planning on running the test provided by Proposition 1.2 **(A)**, for $\varepsilon = \varepsilon_{S,\phi}$, and $k = k'$.

Definition 3.3.2. Let $u_\Phi = \lambda^{s(w_K\cdot\beta_{S,\Phi})}$ and $\varepsilon_{\alpha,\Phi} = \lambda^{s(\alpha\cdot\beta_{S,\Phi})}$, for every $\alpha \in \mathcal{A}(K/k')$.

Proposition 3.3.3. *The quantities u_Φ and $\varepsilon_{\alpha,\Phi}$ defined above satisfy the following properties:*

(1) $u_\Phi \in U_S$; $\varepsilon_{\alpha,\Phi} \in U_S$, $\forall\alpha \in \mathcal{A}(K/k')$.
(2) $w_K \cdot \varepsilon_{S,\Phi} = \widetilde{u}_\Phi$ *(in $\mathbf{Q}U_S$).*
(3) $u_\Phi^\alpha = \varepsilon_{\alpha,\Phi}^{w_K}$, $\forall\alpha \in \mathcal{A}(K/k')$.
(4) $\varepsilon_{\alpha,\Phi}^\gamma = \varepsilon_{\gamma,\Phi}^\alpha$, $\forall\alpha\,,\,\gamma \in \mathcal{A}(K/k')$.

Proof. (1) follows imediately from Lemma 3.3.1 (1).

(2) Since $\mathbf{Q}U_S$ injects in $\mathbf{Q}U_{K(\lambda),S}$, one can prove this equality in $\mathbf{Q}U_{K(\lambda),S}$, as an easy exercise.

(3)&(4) In order to prove (3) and (4), it is enough to show that

$$\lambda^{s(\alpha\cdot\beta_{S,\Phi})\cdot s(\gamma)} = \lambda^{s(\gamma\cdot\beta_{S,\Phi})\cdot s(\alpha)}, \ \forall \alpha, \gamma \in \mathcal{A}(K/k')\,. \tag{9}$$

Let $\alpha, \gamma \in \mathcal{A}(K/k')$. Then $\alpha \cdot \beta_{S,\Phi}$ and $\gamma \cdot \beta_{S,\Phi}$ can be written as determinants with entries in $\mathbf{Z}[G]$, as described by formula (8). These two determinants differ only at the entries of their r–th columns. If we expand them with respect to their r–th columns, we obtain respectively:

$$\begin{aligned} \alpha \cdot \beta_{S,\Phi} &= \mu_1 \cdot \phi_1(\varepsilon_S^{\alpha}) + \cdots + \mu_{r-1} \cdot \phi_{r-1}(\varepsilon_S^{\alpha}) + \mu \cdot \alpha \\ \gamma \cdot \beta_{S,\Phi} &= \mu_1 \cdot \phi_1(\varepsilon_S^{\gamma}) + \cdots + \mu_{r-1} \cdot \phi_{r-1}(\varepsilon_S^{\gamma}) + \mu \cdot \gamma\,, \end{aligned}$$

where $\mu_1, \ldots, \mu_{r-1} \in \mathcal{A}(K/k)$, $\mu \in \mathbf{Z}[G]$, and $\mu_1, \ldots, \mu_{r-1}, \mu$ are independent of α or γ. Therefore, we obtain:

$$\begin{aligned} \lambda^{s(\alpha\cdot\beta_{S,\Phi})\cdot s(\gamma) - s(\gamma\cdot\beta_{S,\Phi})\cdot s(\alpha)} = & \left(\prod_{1\leq i\leq r-1} \lambda^{s(\mu_i\cdot\phi_i(\varepsilon_S^{\alpha}))\cdot s(\gamma) - s(\mu_i\cdot\phi_i(\varepsilon_S^{\gamma}))\cdot s(\alpha)} \right) \cdot \\ & \cdot \lambda^{s(\mu\alpha)\cdot s(\gamma) - s(\mu\gamma)\cdot s(\alpha)}\,. \end{aligned}$$

Now, if we apply Lemma 3.3.1(2) to each factor of the product above, and take into account that $\alpha, \gamma \in \mathcal{A}(K/k')$, $\mu_i \in \mathcal{A}(K/k)$, $\forall i = 1, \ldots, r-1$, and that the ϕ_i's are all $\mathbf{Z}[G']$–linear, we obtain:

$$\begin{aligned} \lambda^{s(\mu_i\cdot\phi_i(\varepsilon_S^{\alpha}))\cdot s(\gamma) - s(\mu_i\cdot\phi_i(\varepsilon_S^{\gamma}))\cdot s(\alpha)} &= \lambda^{s(\mu_i)s(\phi_i(\varepsilon_S^{\alpha}))s(\gamma) - s(\mu_i)s(\phi_i(\varepsilon_S^{\gamma}))s(\alpha)} = \\ &= \lambda^{s(\mu_i)s(\phi_i(\varepsilon_S^{\alpha})\cdot\gamma) - s(\mu_i)s(\phi_i(\varepsilon_S^{\gamma})\cdot\alpha)} = \\ &= \lambda^{s(\mu_i)s(\phi_i(\varepsilon_S^{\alpha\gamma})) - s(\mu_i)s(\phi_i(\varepsilon_S^{\gamma\alpha}))} = 1\,, \end{aligned}$$

$$\lambda^{s(\mu\alpha)\cdot s(\gamma) - s(\mu\gamma)\cdot s(\alpha)} = \lambda^{s(\mu)\cdot s(\alpha)\cdot s(\gamma) - s(\mu)\cdot s(\gamma)\cdot s(\alpha)} = 1\,.$$

This concludes the proofs of (9), Proposition 3.3.1 and Theorem 3.1.1. □

4. ANOTHER LOOK AT BASE CHANGE FOR THE BRUMER–STARK CONJECTURE

4.1. The conjecture and its base change property

Throughout §4 we will asume that K/k is a finite, abelian extension of *number fields*, and that S_0 is a finite set of primes in k, containing the infinite primes and the primes which ramify in K/k. We keep the same notations as in the previous sections. The Deligne–Ribet theorem (Theorem 1.3 above) shows that, under the present hypotheses, $w_K \Theta_{K/k,S_0}(0) \in \mathbf{Z}[G]$.

Let v be a prime in k, such that $v \notin S_0$ and v splits completely in K/k. If $S_v = S_0 \cup \{v\}$, then the set of data $(K/k,\, S_v,\, 1)$ satisfies hypotheses (H). One of the equivalent forms of the Brumer–Stark conjecture is the following:

Conjecture BS$(K/k,\, S_0)$. *For every prime v in k which splits in K/k, and every prime w in K dividing v, there exists $u_w \in \{U^{\rm ab}_{K/k,S_v}\}_{1,S_v}$, such that*

$$w^{w_K \Theta_{K/k,S_0}(0)} = (u_w)\,,$$

where (u_w) is the K–fractional ideal generated by u_w.

It is not difficult to show (see Remark 3 in §2.1) that BS$(K/k,\, S_0)$ is equivalent to the following statement:

Conjecture C$(K/k,\, S_0)$. *Let $\mathcal{S}$ be a finite set of primes in K, not dividing primes in S_0, whose classes generate the ideal class group of K, and which divide only completely split primes in K/k. Then C$(K/k,\, S_0 \cup \{v\},\, 1)$ is true for any prime v in k lying below a prime in $\mathcal{S}$.*

Keeping the same notations as in §3.1, let S_0' be the set of primes in k', dividing primes in S_0. The following base change property was proved by Hayes (see the Main Theorem in [8]):

Theorem 4.1.1 (Hayes). *If BS$(K/k,\, S_0)$ is true, then BS$(K/k',\, S_0')$ is true as well.*

In the remaining part of this section, we give a short proof of Theorem 4.1.1, based on our Theorem 3.1.1 above.

Proof. Let us assume that BS$(K/k,\, S_0)$ is true. Let $\mathcal{S}$ be a set of primes as in the statement of C$(K/k,\, S_0)$. Let $w \in \mathcal{S}$, and let v_1 and v_1' be the primes in k' and respectively k lying below w. Since BS$(K/k,\, S_0)$ is true, C$(K/k,\, S_0 \cup \{v_1\},\, 1)$ is also true. Let $\{v_1', \dots, v_r'\}$ be the set of primes in k' dividing v_1. Theorem 3.1.1 implies that C$(K/k',\, S_0' \cup \{v_1' \dots v_r'\},\, r)$ is true. Now, Corollary 2.2.2 implies that C$(K/k',\, S_0' \cup \{v_1'\},\, 1)$ is true. Since this happens for every $w \in \mathcal{S}$, the equivalence between BS$(K/k',\, S_0')$ and C$(K/k',\, S_0')$ shows that BS$(K/k',\, S_0')$ is true. □

4.2. The base change multiplier of Hayes and Sands

Let $(K/k,\, S_0)$ be as in §4.1. We consider an intermediate extension k'/k ($k \subseteq k' \subseteq K$), and bring forth all the notations and conventions of §3.1. For simplicity, let $\Theta \overset{\rm def}{=} \Theta_{K/k,S_0}(0)$ and $\Theta' \overset{\rm def}{=} \Theta_{K/k',S_0'}(0)$. For a ring R, one can take the direct sum decomposition $R\,[G] = \bigoplus_{i=1}^{r} R[G'] \cdot \gamma_i$. For any $x \in R\,[G]$, $x(\overline{\gamma_i})$ will denote the projection of x on the factor $R[G'] \cdot \gamma_i$. The projection depends only on the image $\overline{\gamma_i}$ of γ_i in Γ.

For every $\chi \in \widehat{G}$, let $t_\chi : \mathbf{C}\,[G] \longrightarrow \mathbf{C}\,[G]$ be the $\mathbf{C}$–linear map satisfying $t_\chi(\sigma) = \chi(\sigma) \cdot \sigma$, for every $\sigma \in G$. Based on equalities (4), it is easy to prove that

$$\Theta' = \prod_{\chi \in \widehat{\Gamma}} t_\chi(\Theta)\,.$$

Then, we obviously have $\Theta' = \beta \cdot \Theta$, where

$$\beta = \prod_{\substack{\chi \in \widehat{\Gamma} \\ \chi \neq \mathbf{1}_\Gamma}} t_\chi(\Theta)\,.$$

The quantity β defined above is what Hayes in [8] and Sands in [14] call the *base change multiplier*. Our first goal is to show that $\beta = \beta_{S,\Phi}$, for an appropriate choice of S and $\Phi \in \mathrm{Hom}_{\mathbf{Z}[G']}(U_S, \mathbf{Z}[G'])^{r-1}$.

For a finite prime $\wp$ in K and any finite nonempty set of primes S, let $\lambda_\wp$ and $\widetilde{\lambda_\wp}$ be as defined in §3.1, and let $\widehat{\wp}$ and $\widetilde{\widehat{\wp}}$ be the $\mathbf{C}\,[G]$–linear and respectively $\mathbf{C}[G']$–linear maps

$$\mathbf{C}U_S \xrightarrow{\widehat{\wp},\, \widetilde{\widehat{\wp}}} \mathbf{C}\,[G]\ ,$$

defined as $\widehat{\wp}(u) = \sum\limits_{\sigma \in G} \mathrm{ord}_{\wp^\sigma}(u) \cdot \sigma$, and $\widetilde{\widehat{\wp}}(u) = \sum\limits_{\sigma' \in G'} \mathrm{ord}_{\wp^{\sigma'}}(u) \cdot \sigma'$, for all $u \in U_S$. We obviously have equalities $\lambda_\wp = \log|\mathrm{N}\wp| \cdot \widehat{\wp}$, $\widetilde{\lambda_\wp} = \log|\mathrm{N}\wp| \cdot \widetilde{\widehat{\wp}}$, and also

$$\widehat{\wp}(u) = \sum_{1 \le i \le r} \widetilde{\widehat{\wp^{\gamma_i}}}(u) \cdot \gamma_i\,, \forall u \in \mathbf{C}U_S\,. \tag{10}$$

Let v be a prime in k, $v \notin S_0$, which splits completely in K/k, let w be a prime in K dividing v, and let $S = S_0 \cup \{v\}$. According to Remark 2, §2.1, there exists a unique $\varepsilon_S \in (\mathbf{C}U_S)_{1,S}$, such that $\lambda_w(\varepsilon_S) = \Theta'_{K/k,S}(0)$. If we take into account that $\Theta'_{K/k,S}(0) = \log|\mathrm{N}w| \cdot \Theta$, the equality above can be written as

$$\widehat{w}(\varepsilon_S) = \Theta\,, \text{ or } \quad \widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S) \cdot \gamma_i = \Theta(\overline{\gamma_i}), \, \forall i = 1, \ldots, r\,. \tag{11}$$

Proposition 4.2.1. *With notations as above, we have the equality*

$$\beta = \beta_{S,\Phi}\,,$$

where $\Phi = (\widetilde{\widehat{w^{\gamma_1}}}, \ldots, \widetilde{\widehat{w^{\gamma_{r-1}}}})$.

Proof. In [14], Sands shows that one has an equality

$$\beta = \begin{vmatrix} \Theta(\overline{\gamma_1} \cdot \overline{\gamma_1}^{-1}) & \ldots & \Theta(\overline{\gamma_1} \cdot \overline{\gamma_r}^{-1}) \\ \vdots & & \vdots \\ \Theta(\overline{\gamma_{r-1}} \cdot \overline{\gamma_1}^{-1}) & \ldots & \Theta(\overline{\gamma_{r-1}} \cdot \overline{\gamma_r}^{-1}) \\ 1 & \ldots & 1 \end{vmatrix}\,. \tag{12}$$

On the other hand, equalities (11) easily imply that $\Theta(\overline{\gamma_i} \cdot \overline{\gamma_j}^{-1}) = \widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\gamma_j})\gamma_i\gamma_j^{-1}$, for all $i, j = 1, \ldots, r$. If we substitute this in the determinant above and multiply column j by γ_j and row i by γ_i^{-1}, for any $i, j = 1, \ldots, r$, we obtain:

$$\begin{aligned} \beta &= \begin{vmatrix} \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_r}) \\ \vdots & & \vdots \\ \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_r}) \\ \gamma_1 & \ldots & \gamma_r \end{vmatrix} = \\ &= \begin{vmatrix} \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S) \\ \vdots & & \vdots & \vdots \\ \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S) \\ \gamma_1 - \mathrm{N}\gamma_1 & \ldots & \gamma_{r-1} - \mathrm{N}\gamma_{r-1} & 1 \end{vmatrix} = \beta_{S,\Phi}\,. \end{aligned} \tag{13}$$

The reader should refer to the technical trick explained in §3.2 in order to get the equality between the two determinants in (13). □

Now, that we have realized β as a particular case of $\beta_{S,\Phi}$, we will apply the techniques developed in §3 in order to recover two integrality theorems due to Hayes and Sands respectively (see Theorem 1 of [8]). Before proceeding, we need the following:

Lemma 4.2.2. *If $\alpha \in \mathbf{Z}[G]$ and $\alpha \cdot \Theta \in \mathbf{Z}[G]$, then*

$$\widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\alpha}) \in \mathbf{Z}[G']\,, \text{ for any } i = 1, \ldots, r\,.$$

Proof. Indeed, since $\widehat{w}$ is $\mathbf{C}[G]$–linear, equalities (10) and (11) show that

$$\alpha \cdot \Theta = \alpha \cdot \widehat{w}(\varepsilon_S) = \widehat{w}(\varepsilon_S^{\alpha}) = \sum_{1 \le i \le r} \widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\alpha}) \cdot \gamma_i\,.$$

This shows that $\widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\alpha}) \cdot \gamma_i \in \mathbf{Z}[G]$, and therefore $\widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\alpha}) \in \mathbf{Z}[G']$, for all i. □

Theorem 4.2.3 (Hayes). *If $\alpha \in \mathcal{A}(K/k')$ then $\alpha \cdot \beta \in \mathcal{A}(K/k)$.*

Proof. Let us first remark that this statement was proved in §3.2 for any $\beta_{S,\Phi}$, under the assumption that $\mathrm{C}(S,\, K/k,\, 1)$ is true (see Proposition 3.2.1). It turns out that, for the particular case of β, this assumption is not needed. Indeed, if we multiply the last column of the second determinant in (13) by α, and keep in mind that the maps $\widetilde{\widehat{w^{\gamma_i}}}$ are $\mathbf{C}[G']$–linear, we obtain

$$\alpha \cdot \beta = \begin{vmatrix} \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \widetilde{\widehat{w^{\gamma_1}}}(\varepsilon_S^{\alpha}) \\ \vdots & & \vdots & \vdots \\ \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_1 - \mathrm{N}\gamma_1}) & \ldots & \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\gamma_{r-1} - \mathrm{N}\gamma_{r-1}}) & \widetilde{\widehat{w^{\gamma_{r-1}}}}(\varepsilon_S^{\alpha}) \\ \gamma_1 - \mathrm{N}\gamma_1 & \ldots & \gamma_{r-1} - \mathrm{N}\gamma_{r-1} & \alpha \end{vmatrix}.$$

Now, by applying Theorem 1.3, we know that $(\gamma_i - \mathrm{N}\gamma_i) \cdot \Theta \in \mathbf{Z}[G]$, for all i, and that $\alpha \cdot \Theta \in \mathbf{Z}[G]$. Therefore, Lemma 4.2.2 shows that the determinant above has entries in $\mathbf{Z}[G]$, and since its last row has entries in $\mathcal{A}(K/k)$, this implies the desired result. □

Theorem 4.2.4 (Sands). *Let $\alpha \in \mathbf{Z}[G']$, such that $\alpha \cdot \Theta \in \mathbf{Z}[G]$. Then $\alpha \cdot \beta \in \mathbf{Z}[G]$ and $\alpha \cdot \Theta' \in \mathbf{Z}[G']$.*

Proof. The proof of $\alpha \cdot \beta \in \mathbf{Z}[G]$ is identical to that of Theorem 4.2.3 above. The same technique works for the second statement if we take into account that

$$\Theta' = \det_{1 \le i,j \le r} \left(\Theta(\overline{\gamma_i} \cdot \overline{\gamma_j}^{-1})\right) = \det_{1 \le i,j \le r} \left(\widetilde{\widehat{w^{\gamma_i}}}(\varepsilon_S^{\gamma_j})\right).$$

The first equality above can be found in [14], and it is a direct consequence of (3), for example. For the second equality, see the proof of Proposition 4.2.1 above. □

Remark. While the base change property for the Brumer–Stark conjecture is non-trivial and meaningful only in the case $k \subseteq k' \subseteq K$, with k and k' totally real and K totally imaginary number fields, the base change property for conjecture C is non-trivial in the most general of cases. This phenomenon is due to the fact that, when passing from L–functions for K/k to L–functions for K/k', higher order zeroes at $s = 0$ can occur. Precisely these give the "raison d'être" of conjecture C.

5. LINKS WITH A CONJECTURE OF RUBIN

The main goal of this section is the proof of a theorem of comparison between conjecture C and Rubin's version "over $\mathbf{Z}$" of Stark's Conjecture for abelian L–functions of arbitrary order of vanishing at $s = 0$. After a few preparations, this will be achieved in §5.5.

5.1. Rubin's Conjecture

Let K/k, S and r be as in §2, and let T be an auxiliary, finite set of primes in k. Let us assume that the set of data $(K/k, S, T, r)$ satisfies the following group of hypotheses:

$$(\mathrm{H}_T) \qquad \begin{cases} \text{hypotheses (H)} \\ T \neq \emptyset,\ S \cap T = \emptyset \\ U_{S,T} \text{ has no } \mathbf{Z}\text{–torsion.} \end{cases}$$

Let us remark that since the index $[U_S : U_{S,T}]$ is finite, we have an equality of $\mathbf{Q}$–vector spaces $\mathbf{Q} \wedge_G^r U_S = \mathbf{Q} \wedge_G^r U_{S,T}$.

Definition 5.1.1. Assuming that $(K/k, S, T, r)$ satisfies hypotheses (H_T), let $\Lambda_{S,T}$ be the $\mathbf{Z}[G]$–submodule of $\mathbf{Q} \wedge_G^r U_S$ defined by

$$\Lambda_{S,T} = \{\varepsilon \in (\mathbf{Q} \wedge_G^r U_S)_{r,S} \mid (\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon) \in U_{S,T},\ \forall \phi_1, \dots, \phi_{r-1} \in U_{S,T}^*\}.$$

Based on our constructions in §2.1 above, for $\phi_1, \dots, \phi_r \in U_{S,T}^*$, let us define

$$\mathbf{C} \wedge_G^r U_S \xrightarrow{\phi_1 \wedge \cdots \wedge \phi_r} \mathbf{C}[G],$$

by setting $\phi_1 \wedge \cdots \wedge \phi_r(\varepsilon) := \phi_r((\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon))$, for all $\varepsilon \in \mathbf{C} \wedge_G^r U_S$. Corollary 1.3 of [13] (see (16) below as well) implies that Definition 5.1.1 above is equivalent to the following

Definition 5.1.1'. Under the hypotheses and notations of Definition 5.1.1.,

$$\Lambda_{S,T} = \{\varepsilon \in (\mathbf{Q} \wedge_G^r U_S)_{r,S} \mid (\phi_1 \wedge \cdots \wedge \phi_r)(\varepsilon) \in \mathbf{Z}[G],\ \forall \phi_1, \dots, \phi_r \in U_{S,T}^*\}.$$

Let $\Theta_{K/k,S,T}$ be the T–modified Stickelberger function, defined by

$$\Theta_{K/k,S,T}(s) = \prod_{v \in T} (1 - \mathbf{N}v^{1-s} \cdot \sigma_v^{-1}) \Theta_{K/k,S}(s), \text{ for all } s \in \mathbf{C}.$$

Obviously, the extra Euler factors at primes in T do not alter the holomorphicity or order of vanishing of $\Theta_{K/k,S}$ at $s = 0$, and we have the equality

$$\Theta_{K/k,S,T}^{(r)}(0) = \prod_{v \in T} (1 - \mathbf{N}v \cdot \sigma_v^{-1}) \cdot \Theta_{K/k,S}^{(r)}(0). \tag{14}$$

Let V and W and R_W be as in §2.1. In [13], Rubin conjectures the following (see [10] for the function field case):

Conjecture $B(K/k, S, T, r)$**.** *Assuming that the set of data $(K/k, S, T, r)$ satisfies hypotheses (H_T) then, for any choice of V and W, there exists a unique $\varepsilon_{S,T,W} \in \Lambda_{S,T}$ such that $R_W(\varepsilon_{S,T,W}) = \Theta_{K/k,S,T}^{(r)}(0)$.*

Remark 1. Let S and $T \subseteq T'$ be finite sets of primes in k such that (H_T) and $(\mathrm{H}_{T'})$ are satisfied. Let $\varepsilon_{S,W}$, $\varepsilon_{S,T,W}$ and $\varepsilon_{S,T',W}$ be the unique elements in $(\mathbf{C} \wedge_G^r U_S)_{r,S}$ satisfying the regulator relations in conjectures $\mathrm{C}(K/k,\, S,\, r)$, $\mathrm{B}(K/k,\, S,\, T,\, r)$ and $\mathrm{B}(K/k,\, S,\, T',\, r)$ respectively. Then, as a direct consequence of (14), we have

$$(15)\quad \varepsilon_{S,T,W} = \prod_{v\in T}(1-\mathbf{N}v\cdot\sigma_v^{-1})\cdot\varepsilon_{S,W}\,, \qquad \varepsilon_{S,T',W} = \prod_{v\in T'\setminus T}(1-\mathbf{N}v\cdot\sigma_v^{-1})\cdot\varepsilon_{S,W,T}$$

Remark 2. It is not dificult to show that given $(K/k,\, S,\, T,\, r)$, if conjectures $\mathrm{B}(K/k,\, S,\, T,\, r)$ and respectively $\mathrm{C}(K/k,\, S,\, r)$ are true for a choice of V and W then they are true for any other choice. In fact, one can show that if one has "too much freedom" in choosing V (in the sense that S contains at least $r+1$ primes which split completely in K/k), then $\mathrm{B}(K/k,\, S,\, T,\, r)$ and $\mathrm{C}(K/k,\, S,\, r)$ are trivially true in virtue of the S–class number formula. (See [13], §3.1 for a proof of conjecture B in this case. The same arguments work for conjecture C.) Also, once V is chosen, the elements $\varepsilon_{S,T,W}$ and $\varepsilon_{S,W}$ depend in a very simple way of the choice of W. For these reasons we will suppress V and W from our future notations.

Remark 3. For $r = 1$, Definition 5.1.1 gives $\Lambda_{S,T} = U_{S,T}$.

5.2. A few points of homological and commutative algebra

In this section we will prove a homological algebra lemma and make a few general commutative algebra remarks. These will play an important role in our future considerations. The main references for this section are [9], §IV.7 and [4], §11.3.

Lemma 5.2.1. *Let D be a Dedekind domain, G a finite, abelian group, and M a finitely generated $D\,[G]$–module. Then*

1. $\mathrm{Ext}^n_{D[G]}(M\,, D\,[G]) = 0$*, for all* $n \geq 2$.
2. *If M has no D–torsion* $\mathrm{Ext}^n_{D[G]}(M\,, D\,[G]) = 0$*, for all* $n \geq 1$.

Proof. We will first show statement (2) in the Lemma. It is not hard to show (see [1], Proposition VI.3.4) that, for any $D\,[G]$–module X which has no D–torsion, one has an isomorphism of D–modules

$$(16)\qquad \mathrm{Hom}_{D[G]}(X\,, D\,[G]) \xrightarrow[\pi]{\sim} \mathrm{Hom}_D(X\,, D)\,,$$

whose inverse is given by $\pi^{-1}(\phi) = \psi$, with $\psi(x) = \sum_{g\in G}\phi(g^{-1}x)\cdot g$, for all $x \in X$ and $\phi \in \mathrm{Hom}_D(X\,, D)$. Let us consider a $D\,[G]$–projective resolution of M

$$\cdots \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0\,.$$

Under the assumption of (2), M is a projective D-module. The isomorphism (16) thus shows that, if one applies the functor $\mathrm{Hom}_{D[G]}(*\,, D\,[G])$ to the long exact sequence above, one gets a long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{D[G]}(M\,, D\,[G]) \xrightarrow{\varepsilon^*} \mathrm{Hom}_{D[G]}(P_0\,, D\,[G]) \xrightarrow{\delta_0^*} \cdots\,.$$

Since $\mathrm{Ext}^n_{D[G]}(*\,,D\,[G])$ are the right derived functors of $\mathrm{Hom}_{D[G]}(*\,,D\,[G])$, the exactness of the last sequence shows that indeed $\mathrm{Ext}^n_{D[G]}(M\,,D\,[G])=0$, for all $n\geq 1$.

Statement (1) in the Lemma is an easy consequence of (2). Indeed, let us take an exact sequence of $D\,[G]$–modules

$$0\longrightarrow N\longrightarrow D\,[G]^s\longrightarrow M\longrightarrow 0\,.$$

Since N and $D\,[G]^s$ are both D–torsion free, statement (2) implies that the above exact sequence gives the following long exact Ext–sequence in the first variable

$$0\longrightarrow \mathrm{Hom}_{D[G]}(M\,,D\,[G])\longrightarrow \mathrm{Hom}_{D[G]}(D\,[G]^s\,,D\,[G])\longrightarrow \mathrm{Hom}_{D[G]}(N\,,D\,[G])\longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_{D[G]}(M\,,D\,[G])\longrightarrow 0\longrightarrow\cdots\longrightarrow 0\longrightarrow \mathrm{Ext}^n_{D[G]}(M\,,D\,[G])\longrightarrow 0\longrightarrow\cdots$$

This shows that $\mathrm{Ext}^n_{D[G]}(M\,,D\,[G])=0$, for all $n\geq 2$. □

Let R be a commutative noetherian ring and $\mathcal{K}(R)$ be its total field of fractions. Let I be a fractional ideal of R (i.e. a finitely generated R–submodule of $\mathcal{K}(R)$). We will denote by I^{-1} the R–fractional given by

$$I^{-1}=\{x\in\mathcal{K}(R)\mid xI\subseteq R\}\,,$$

and by I^* the R–dual of I, $I^*=\mathrm{Hom}_R(I\,,R)$.

Remark 1. It is a well known fact that if an R–fractional ideal I is projective of rank 1 as an R–module, then so is I^{-1} and we have the following isomorphisms respectively equalities of R–modules

$$I^{-1}\xrightarrow{\sim} I^* \qquad I^{-1}\cdot I=R \qquad \left(I^{-1}\right)^{-1}=I\,.$$

The isomorphism above is given by $a\longrightarrow\phi_a$, where $\phi_a(t)=at$, for all $a\in I^{-1}$ and $t\in I$. (See [4], §11.3 for all these facts.)

From now on, we will refer to projective, rank 1 R–fractional ideals as invertible R–fractional ideals.

Remark 2. Let us now assume that the ring R is a finite direct sum of local rings $R=\overset{s}{\underset{i=1}{\oplus}}R_i$. Let I be an invertible R–fractional ideal. Then obviously $I=\overset{s}{\underset{i=1}{\oplus}}I_i$, with I_i principal R_i–fractional ideals $I_i=R_i\cdot f_i$ generated by nonzerodivisors $f_i\in\mathcal{K}(R)$. If we further assume that I contains R, then we have an isomorpfism of R–modules

$$I/R\xrightarrow[\cdot f^{-1}]{\sim} R/I^{-1}\,,$$

given by multiplication with the nonzerodivisor $f^{-1}=(f_1^{-1},\ldots,f_s^{-1})\in\mathcal{K}(R)$.

We will apply the remarks above to the case $R=\mathbf{Z}_\ell\,[G]$. The group ring $\mathbf{Z}_\ell\,[G]$ can be written as a direct sum of local rings as follows. Let L be the ℓ–Sylow subgroup of G and Δ its complement ($G=L\times\Delta$). Then one has the following ring isomorphism

$$\mathbf{Z}_\ell\,[G]\xrightarrow{\sim}\underset{\chi}{\oplus}\,\mathbf{Z}_\ell\,[\chi]\,[L]\,.$$

Here χ runs over all the $G(\overline{\mathbf{Q}_\ell}/\mathbf{Q}_\ell)$–conjugacy classes of irreducible $\overline{\mathbf{Q}_\ell}$–valued characters of Δ and $\mathbf{Z}_\ell[\chi]$ is the finite extension of $\mathbf{Z}_\ell$ obtained by adjoining the values of χ. Each of the rings $\mathbf{Z}_\ell[\chi][L]$ in the direct sum above is a local ring of maximal ideal generated by ℓ and the augmentation ideal I_L of L. The following proposition will enable us to apply Remarks 1 and 2 above to $\mathbf{Z}_\ell[G]$–fractional ideals.

Proposition 5.2.2. *If D is a principal ideal domain, G is a finite, abelian group and M is a finitely generated $D[G]$–module, the following are equivalent:*

(1) *M is $D[G]$–projective.*
(2) *M has no D–torsion and is G–cohomologically trivial (i.e. $\widehat{\mathrm{H}}^i(H, M) = 0$, for all integers i and all subgroups H of G.)*

Proof. See [2], IV.9, for the case $D = \mathbf{Z}$. The general case follows similarly. □

5.3. Dependence on T in Rubin's Conjecture

Definition 5.3.2. For a finite prime v of k, unramified in K/k, let

$$\Delta_v = \bigoplus_{w|v} K(w)^\times, \quad \delta_v = 1 - \sigma_v^{-1}\mathrm{N}v,$$

where the direct sum is taken with respect to all the primes w in K dividing v, and $K(w)$ is the residue field of w.

Remark 1. Δ_v is endowed with a natural $\mathbf{Z}[G]$-module structure. Let G_v denote the decomposition group of v associated to K/k. The cyclicity of $K(w)^\times$ as a $\mathbf{Z}[G_v]$–module immediately gives the following $\mathbf{Z}[G]$–isomorphisms

$$\Delta_v \xrightarrow{\sim} K(w)^\times \underset{\mathbf{Z}[G_v]}{\otimes} \mathbf{Z}[G] \xrightarrow{\sim} \mathbf{Z}[G_v]/(\delta_v) \underset{\mathbf{Z}[G_v]}{\otimes} \mathbf{Z}[G] \xrightarrow{\sim} \mathbf{Z}[G]/(\delta_v),$$

for all $w|v$.

For any prime v as above, we will denote by res_v the $\mathbf{Z}[G]$–morphism

$$\{x \in K^* \mid \mathrm{ord}_v(x) = 0\} \xrightarrow{\mathrm{res}_v} \Delta_v,$$

sending every x into into the element of $\Delta_v = \oplus K(w)^\times$ whose w–component is the residue class of x modulo w, for all $w|v$.

Proposition 5.3.1. *Let $(K/k, S, T, r)$ and $(K/k, S, T', r)$ two sets of data satisfying hypotheses (H_T) and $(H_{T'})$ respectively. Assume that $T \subseteq T'$. Then*

$$B(K/k, S, T, r) \Longrightarrow B(K/k, S, T', r).$$

Proof. It obviously suffices to prove the implication above under the assumption $T' = T \cup \{v\}$, for any prime v of k which is not in $T \cup S$. One has an exact sequence of $\mathbf{Z}[G]$–modules

$$0 \longrightarrow U_{S,T'} \longrightarrow U_{S,T} \xrightarrow{\mathrm{res}_v} \Delta_v.$$

Let us denote by $\mathcal{I}$ and $\mathcal{C}$ the image and respectively the cokernel of res_v. We obtain the following short exact sequences of $\mathbf{Z}[G]$–modules

$$0 \longrightarrow U_{S,T'} \longrightarrow U_{S,T} \xrightarrow{\mathrm{res}_v} \mathcal{I} \longrightarrow 0 \tag{17}$$

$$0 \longrightarrow \mathcal{I} \longrightarrow \Delta_v \longrightarrow \mathcal{C} \longrightarrow 0\,. \tag{18}$$

Let us notice that since Δ_v is finite and therefore a torsion $\mathbf{Z}[G]$–module, we have $\mathcal{I}^* = \Delta_v^* = \mathcal{C}^* = 0$. Since $U_{S,T}$ and $U_{S,T'}$ have no $\mathbf{Z}$–torsion, Lemma 5.2.1 shows that the short exact sequences (17) and (18) give the following long exact Ext–sequences

(19)

$$0 \longrightarrow U_{S,T}^* \longrightarrow U_{S,T'}^* \xrightarrow{\mathrm{res}_v^1} \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathcal{I}\,, \mathbf{Z}[G]) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$
$$0 \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathcal{C}\,, \mathbf{Z}[G]) \longrightarrow \mathbf{Z}[G]/(\delta_v) \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}[G]}(\mathcal{I}\,, \mathbf{Z}[G]) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\,.$$

The above exact sequences show that $U_{S,T'}^*/U_{S,T}^*$ is a cyclic $\mathbf{Z}[G]$–module annihilated by δ_v. Let ϕ_0 be an element of $U_{S,T'}^*$ such that its image in $U_{S,T'}^*/U_{S,T}^*$ generates $U_{S,T'}^*/U_{S,T}^*$ as a $\mathbf{Z}[G]$–module. Then obviously $\delta_v\phi_0$ belongs to $U_{S,T}^*$.

Let us now assume that $\mathrm{B}(K/k,\, S,\, T,\, r)$ is true. Let us denote by $\varepsilon_{S,T}$ the unique element in the lattice $\Lambda_{S,T}$ satisfying the regulator condition in the statement of $\mathrm{B}(K/k,\, S,\, T,\, r)$. Remark 1, §5.1, shows that

$$\varepsilon_{S,T'} \stackrel{\text{def}}{=} \delta_v \cdot \varepsilon_{S,T}$$

is the unique element of $(\mathbf{Q} \wedge_G^r U_S)_{r,S}$ satisfying the regulator condition in the statement of $\mathrm{B}(K/k,\, S,\, T',\, r)$. Therefore, in order to prove that $\mathrm{B}(K/k,\, S,\, T',\, r)$ is true, we must show that in fact $\delta_v \cdot \varepsilon_{S,T}$ belongs to $\Lambda_{S,T'}$.

Let $\phi_1', \ldots, \phi_r'$ be elements of $U_{S,T'}^*$. The choice of ϕ_0 implies that there exist elements α_i in $\mathbf{Z}[G]$ and ϕ_i in $U_{S,T}^*$ such that

$$\phi_i' = \phi_i + \alpha_i\phi_0\,, \text{ for all } i = 1, \ldots, r\,.$$

These equalities imply that

$$\begin{aligned}(\phi_1' \wedge \cdots \wedge \phi_r')\,(\varepsilon_{S,T'}) &= \delta_v \cdot (\phi_1 \wedge \cdots \wedge \phi_r)\,(\varepsilon_{S,T}) + \\ &+ \sum_{i=1}^r (-1)^{i+1}\alpha_i \cdot (\phi_1 \wedge \cdots \wedge \overset{i}{\overbrace{(\delta_v\phi_0)}} \wedge \cdots \wedge \phi_r)\,(\varepsilon_{S,T})\,.\end{aligned} \tag{20}$$

Since $\varepsilon_{S,T}$ belongs to $\Lambda_{S,T}$ and $\delta_v\phi_0$ belongs to $U_{S,T}^*$ we have

$$(\phi_1 \wedge \cdots \wedge \phi_r)\,(\varepsilon_{S,T}) \in \mathbf{Z}[G]$$

$$(\phi_1 \wedge \cdots \wedge \overset{i}{\overbrace{(\delta_v\phi_0)}} \wedge \cdots \wedge \phi_r)\,(\varepsilon_{S,T}) \in \mathbf{Z}[G]\,, \text{ for all } i = 1, \ldots, r\,.$$

These observations show that the left hand side of (20) belongs to $\mathbf{Z}[G]$ for any $\phi_1', \ldots, \phi_r'$ in $U_{S,T'}^*$, and therefore $\mathrm{B}(K/k,\, S,\, T',\, r)$ is indeed true. $\square$

5.4. A few remarks on roots of unity

Let K/k, $G = G(K/k)$ be as in the previous sections. As in Definition 5.3.2, let v be a finite prime of k, unramified in K/k. Let us assume that there are no elements in μ_K congruent to 1 modulo w for all primes w in K dividing v. This is another way of saying that the residue $\mathbf{Z}[G]$–morphism

$$\mu_K \xrightarrow{\mathrm{res}_v} \Delta_v$$

is injective. Therefore, the $\mathbf{Z}[G]$–isomorphism $\Delta_v \xrightarrow{\sim} \mathbf{Z}[G]/(\delta_v)$ (see Remark 1, §5.3) implies that $\delta_v \in \mathcal{A}(K/k)$. Let us also remark that for any prime v as above, $\delta_v = 1 - \sigma_v^{-1}\mathrm{N}v$ is a nonzerodivisor in $\mathbf{Z}[G]$.

Lemma 5.4.1. *Let $\mathrm{N} : G \longrightarrow \mathbf{Z}$ be a map such that $\zeta^\sigma = \zeta^{\mathrm{N}(\sigma)}$, for all $\sigma \in G$ and $\zeta \in \mu_K$. Then the inverse of $\mathcal{A}(K/k)$ viewed as a $\mathbf{Z}[G]$–fractional ideal inside $\mathcal{K}(\mathbf{Z}[G]) = \mathbf{Q}[G]$ is given by*

$$\mathcal{A}(K/k)^{-1} = \mathbf{Z}\frac{1}{w_K}\xi_N + \mathbf{Z}[G] \ ,$$

where $\xi_N = \sum_{\sigma\in G} \mathrm{N}(\sigma^{-1})\sigma$.

Proof. This is an easy calculation based on the fact that $\mathcal{A}(K/k)$ is generated as a $\mathbf{Z}$–module by the set $\{\sigma - \mathrm{N}(\sigma) \mid \sigma \in G\} \cup \{w_K\}$, for any map N as above. The details are left to the reader. □

Lemma 5.4.2. *Let v be a finite prime in k, unramified in K/k, such that the map res_v is injective on μ_K. Let $\Delta_v[\mathcal{A}(K/k)]$ be the maximal $\mathbf{Z}[G]$–submodule of Δ_v annihilated by $\mathcal{A}(K/k)$. Then:*

(1) $\mathrm{Im}\left(\mu_K \xrightarrow{\mathrm{res}_v} \Delta_v\right) = \Delta_v[\mathcal{A}(K/k)]$.

(2) *Under the isomorphism $\Delta_v \xrightarrow{\sim} \mathbf{Z}[G]/(\delta_v)$, we have*

$$\Delta_v[\mathcal{A}(K/k)] \xrightarrow{\sim} (\delta_v)\cdot\mathcal{A}(K/k)^{-1}/(\delta_v) \ .$$

Proof. Statement (2) follows immediately from the fact that δ_v is a nonzerodivisor of $\mathbf{Z}[G]$.

Since clearly $\mathrm{Im}(res_v) \subseteq \Delta_v[\mathcal{A}(K/k)]$ and res_v is injective, in order to prove (1) it sufffices to show that the cardinality of $\Delta_v[\mathcal{A}(K/k)]$ is w_K. On the other hand, statement (2) above and Lemma 5.4.1 give the following isomorphisms of abelian groups:

$$\Delta_v[\mathcal{A}(K/k)] \xrightarrow{\sim} (\delta_v)\cdot\mathcal{A}(K/k)^{-1}/(\delta_v) \xrightarrow[\cdot\delta_v^{-1}]{\sim} \mathcal{A}(K/k)^{-1}/\mathbf{Z}[G] \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \mathbf{Z}\frac{1}{w_K}\xi_N/\mathbf{Z}\xi_N \xrightarrow{\sim} \mathbf{Z}/w_K\mathbf{Z}.$$

This concludes the proof of (1). □

We conclude this section with a criterion for G–cohomological triviality of the ℓ–Sylow subgroup of μ_K, for a rational prime ℓ. In order to simplify notations, for any global field L and any rational prime ℓ, let $\mu_L^{(\ell)} := \mu_L \otimes \mathbf{Z}_\ell$.

Definition 5.4.3. If ℓ is a rational prime, we call the extension K/k *ℓ–admissible* if $\mu_K^{(\ell)}$ is $G(K/k)$–cohomologically trivial. We call K/k *admissible* if it is ℓ–admissible for all ℓ.

Lemma 5.4.4.

(1) *If ℓ is odd or* $\mathrm{char}(k) \neq 0$*, then K/k is ℓ–admissible if and only if*

$$\ell \nmid w_K \ \text{ or } \ \ell \nmid [K : k(\mu_K^{(\ell)})] \,.$$

(2) *If* $\mathrm{char}(k) = 0$*, then K/k is 2–admissible if and only if*

$$2 \nmid [K : k(\mu_K^{(2)})] \ \text{ and } \ \left\{ \begin{array}{l} k \cap \mathbf{Q}(\mu_K^{(2)}) \text{ is not a (totally) real field} \\ \text{in the case } \mu_K^{(2)} \neq \mu_k^{(2)} \end{array} \right\} .$$

Proof. This is an easy exercise on norms of roots of unity in cyclotomic extensions and group cohomology. The details can be found in [11]. □

5.5. The comparison theorem

We are now ready to state and prove the theorem of comparison between conjectures B and C. If R is a subring of $\mathbf{C}$, we denote by $R\mathrm{B}(K/k,\, S,\, T,\, r)$ and $R\mathrm{C}(K/k,\, S,\, r)$ the statements obtained from conjectures B and C by replacing lattices Λ_S and $\Lambda_{S,T}$ by $R \otimes_{\mathbf{Z}} \Lambda_{S,T}$ and $R \otimes_{\mathbf{Z}} \Lambda_S$ respectively. In particular, if ℓ is a rational prime, we view the ring of ℓ–adic integers $\mathbf{Z}_\ell$ and its field of fractions $\mathbf{Q}_\ell$ as subrings of $\mathbf{C}$ by fixing an arbitrary embedding of $\mathbf{Q}_\ell$ into $\mathbf{C}$. Obviously, proving conjecture $\mathrm{B}(K/k,\, S,\, T,\, r)$ is equivalent to proving $\mathbf{Q}\mathrm{B}(K/k,\, S,\, T,\, r)$ and $\mathbf{Z}_\ell\mathrm{B}(K/k,\, S,\, T,\, r)$ for all ℓ, and similarly for conjecture C.

Theorem 5.5.1. *Assume that $(K/k,\, S,\, r)$ satisfies hypothesis (H). Let $\mathcal{T}$ be a set of primes in k, disjoint from S, such that $\{\sigma_v - \mathrm{N}v \,|\, v \in \mathcal{T}\}$ generates $\mathcal{A}(K/k)$ as a $\mathbf{Z}$–module. Let ℓ be a rational prime. Then the following hold true:*

(1) *Let T be a set of primes in k such that $(K/k,\, S,\, T,\, r)$ satisfies hypothesis (H_T). Then*

$$\mathbf{Q}B(K/k,\, S,\, T,\, r) \Longleftrightarrow \mathbf{Q}C(K/k,\, S,\, r) \,.$$

(2) $\{\mathbf{Z}_\ell B(K/k,\, S,\, \{v\},\, r)\ ,\ \text{for all } v \text{ in } \mathcal{T}\ \} \Longrightarrow \mathbf{Z}_\ell C(K/k,\, S,\, r)$.

(3) *If K/k is ℓ–admissible, then*

$$\mathbf{Z}_\ell C(K/k,\, S,\, r) \Longleftrightarrow \left\{ \begin{array}{l} \mathbf{Z}_\ell B(K/k,\, S,\, T,\, r)\ , \\ \text{for all } T \text{ such that } (K/k,\, S,\, T,\, r) \text{ satisfies } (H_T) \end{array} \right\} .$$

(4) *If $r = 1$, then the equivalence in (3) is true without assuming ℓ–admissibility.*

Proof. Throughout this proof we will be working with $\mathbf{Z}_\ell\,[G]$–modules. All the module–dual and fractional ideal operations will be viewed over $\mathbf{Z}_\ell\,[G]$. In order to simplify notations, let $\mathcal{A}_\ell := \mathcal{A}(K/k) \otimes \mathbf{Z}_\ell$.

(1) This is a straightforward consequence of the fact that $\mathbf{Q}\Lambda_S = \mathbf{Q}\Lambda_{S,T}$, of Remark 1, §5.1, and the fact that $\prod_{v\in T}\left(1-\sigma_v^{-1}\mathrm{N}v\right)$ is invertible in $\mathbf{Q}[G]$.

(2) Let $v \in \mathcal{T}$ and let $\varepsilon_{S,\{v\}}$ be the unique element in $\mathbf{Z}_\ell\Lambda_{S,\{v\}}$ satisfying the regulator condition in conjecture $\mathbf{Z}_\ell\mathrm{B}(K/k,\, S,\, \{v\},\, r)$. Since δ_v is invertible in $\mathbf{Q}_\ell[G]$, (15) implies that $\varepsilon_S := \delta_v^{-1}\varepsilon_{S,\{v\}}$ is the unique element in $\mathbf{Q}_\ell\Lambda_S$ satisfying the regulator condition in $\mathbf{Z}_\ell\mathrm{C}(K/k,\, S,\, r)$. Since $\widetilde{U_S}^{\,*} \subseteq U_{S,T}^*$, for all T disjoint from S, the hypothesis in (1) shows that, if $\phi_1, \ldots, \phi_{r-1} \in \widetilde{U_S}^{\,*}$, we have:

$$\delta_v \cdot \{(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon_S)\} \in \mathbf{Z}_\ell U_{S,\{v\}}\,, \text{ for all } v \in \mathcal{T}\,.$$

Proposition 1.2(B) implies that

$$(\phi_1 \wedge \cdots \wedge \phi_{r-1})(\varepsilon_S) \in \mathbf{Z}_\ell\left\{\frac{1}{w_K}\widetilde{U^{\mathrm{ab}}_{K/k,S}}\right\}\,,$$

which concludes the proof of (2).

(3) The right–to–left implication follows from (2). Assume that $\mathbf{Z}_\ell\mathrm{C}(K/k,\, S,\, r)$ is true and let ε_S be the unique element in $\mathbf{Z}_\ell\Lambda_S$ satisfying the corresponding regulator condition. According to Proposition 5.3.1, (3) will follow if we prove $\mathbf{Z}_\ell\mathrm{B}(K/k,\, S,\, T,\, r)$ for sets T consisting of a single prime. Let us fix such a set $T = \{v\}$. Then (15) implies that the unique element inside $\mathbf{Q}_\ell\Lambda_{S,T}$ satisfying the regulator condition in conjecture $\mathrm{B}(K/k,\, S,\, T,\, r)$ is $\varepsilon_{S,T} := \delta_v\varepsilon_S$. We will have to show that $\varepsilon_{S,T} \in \mathbf{Z}_\ell\Lambda_{S,T}$. The argument is very similar to the one displayed in the proof of Proposition 5.3.1. Slight difficulties set in due to the presence of roots of unity.

We have an exact sequence of $\mathbf{Z}[G]$–modules

$$0 \longrightarrow U_{S,T} \longrightarrow U_S \xrightarrow{\mathrm{res}_v} \Delta_v\,.$$

Since $U_{S,T}$ contains no roots of unity, this leads to an exact sequence

$$0 \longrightarrow U_{S,T} \longrightarrow \widetilde{U_S} \xrightarrow{\widetilde{\mathrm{res}_v}} \Delta_v/\mathrm{res}_v(\mu_K)\,.$$

We will now tensor the last sequence with $\mathbf{Z}_\ell$, and denote by $\mathcal{I}_\ell$ and $\mathcal{C}_\ell$ the image and respectively the cokernel of $\widetilde{\mathrm{res}_{v,l}} := \widetilde{\mathrm{res}_v} \otimes \mathbf{1}_{\mathbf{Z}_\ell}$. According to Lemma 5.4.2, we obtain the following exact sequences of $\mathbf{Z}_\ell[G]$–modules:

$$0 \longrightarrow U_{S,T} \otimes \mathbf{Z}_\ell \longrightarrow \widetilde{U_S} \otimes \mathbf{Z}_\ell \xrightarrow{\widetilde{\mathrm{res}_{v,\ell}}} \mathcal{I}_\ell \longrightarrow 0 \tag{21}$$

$$0 \longrightarrow \mathcal{I}_\ell \longrightarrow \mathbf{Z}_\ell[G]/\delta_v\mathcal{A}_\ell^{-1} \longrightarrow \mathcal{C}_\ell \longrightarrow 0\,. \tag{22}$$

We will now compute the long $\mathrm{Ext}_{\mathbf{Z}_\ell[G]}(*\,,\mathbf{Z}_\ell[G])$–exact sequence associated to

$$0 \longrightarrow \delta_v\mathcal{A}_\ell^{-1} \longrightarrow \mathbf{Z}_\ell[G] \longrightarrow \mathbf{Z}_\ell[G]/\delta_v\mathcal{A}_\ell^{-1} \longrightarrow 0\,.$$

If we apply Lemma 5.2.1, and take into account that $\mathbf{Z}_\ell\,[G]\,/\delta_v\mathcal{A}_\ell^{-1}$ is a finite module, we obtain

$$0 \longrightarrow (\mathbf{Z}_\ell\,[G])^* \longrightarrow \left(\delta_v\mathcal{A}_\ell^{-1}\right)^* \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}_\ell[G]}\left(\mathbf{Z}_\ell\,[G]\,/\delta_v\mathcal{A}_\ell^{-1}\,,\mathbf{Z}_\ell\,[G]\right) \longrightarrow 0 \longrightarrow \cdots\,. \tag{23}$$

On the other hand, the ideal $\mathcal{A}_\ell$ fits into the following exact sequence

$$0 \longrightarrow \mathcal{A}_\ell \longrightarrow \mathbf{Z}_\ell\,[G] \longrightarrow \mu_K \otimes \mathbf{Z}_\ell \longrightarrow 0\,.$$

Since by hypothesis $\mu_K \otimes \mathbf{Z}_\ell$ is G–cohomologically trivial, $\mathcal{A}_\ell$ is G–cohomologically trivial as well. Therefore, Proposition 5.2.2 combined with Remark 1, §5.2 show that $\mathcal{A}_\ell$ is an invertible $\mathbf{Z}_\ell\,[G]$–fractional ideal and so is $\delta_v\mathcal{A}_\ell^{-1}$. (Recall that δ_v is a nonzerodivisor in $\mathbf{Z}_\ell\,[G]$.) If one applies Remark 1, §5.2, to $I = \delta_v\mathcal{A}_\ell^{-1}$ and $I = \mathbf{Z}_\ell\,[G]$ respectively, one obtains the following $\mathbf{Z}_\ell\,[G]$–isomorphisms:

$$\left(\delta_v\mathcal{A}_\ell^{-1}\right)^* \xrightarrow{\sim} \delta_v^{-1}\mathcal{A}_\ell\,, \qquad (\mathbf{Z}_\ell\,[G])^* \xrightarrow{\sim} \mathbf{Z}_\ell\,[G]\,.$$

These isomorphisms combined with exact sequence (23) and Remark 2, §5.2 imply

$$\mathrm{Ext}^1_{\mathbf{Z}_\ell[G]}\left(\mathbf{Z}_\ell\,[G]\,/\delta_v\mathcal{A}_\ell^{-1}\,,\mathbf{Z}_\ell\,[G]\right) \xrightarrow{\sim} \mathbf{Z}_\ell\,[G]\,/\delta_v\mathcal{A}_\ell^{-1}\,.$$

We will now combine Lemma 5.2.1 and the isomorphism above to get the following long $\mathrm{Ext}_{\mathbf{Z}_\ell[G]}\,(*\,,\mathbf{Z}_\ell\,[G])$–exact sequences associated to (21) and (22) respectively :

$$0 \longrightarrow \widetilde{U_S}^{\,*} \otimes \mathbf{Z}_\ell \longrightarrow U^*_{S,T} \otimes \mathbf{Z}_\ell \xrightarrow{\widetilde{\mathrm{res}_{v,\ell}}^{1}} \mathrm{Ext}^1_{\mathbf{Z}_\ell[G]}\left(\mathcal{I}_\ell\,,\mathbf{Z}_\ell\,[G]\right) \longrightarrow 0 \longrightarrow \cdots$$

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}_\ell[G]}\left(\mathcal{C}_\ell\,,\mathbf{Z}_\ell\,[G]\right) \longrightarrow \mathbf{Z}_\ell\,[G]\,/\delta_v\mathcal{A}_\ell^{-1} \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}_\ell[G]}\left(\mathcal{I}_\ell\,,\mathbf{Z}_\ell\,[G]\right) \longrightarrow 0 \longrightarrow \cdots\,.$$

These sequences show that $U^*_{S,T}/\widetilde{U_S}^{\,*} \otimes \mathbf{Z}_\ell$ is a cyclic $\mathbf{Z}_\ell\,[G]$–module annihilated by $\delta_v\mathcal{A}_\ell^{-1}$. Let $\phi_0 \in U^*_{S,T}$ such that its class in $U^*_{S,T}/\widetilde{U_S}^{\,*}$ generates $U^*_{S,T}/\widetilde{U_S}^{\,*} \otimes \mathbf{Z}_\ell$ as a $\mathbf{Z}_\ell\,[G]$–module. We obviously have

$$\delta_v\mathcal{A}_\ell^{-1} \cdot \phi_0 \subseteq \widetilde{U_S}^{\,*} \otimes \mathbf{Z}_\ell\,. \tag{24}$$

Take $\phi_1\,,\cdots,\phi_r \in U^*_{S,T}$. Then there exist $\phi'_1\,,\cdots,\phi'_r \in \widetilde{U_S}^{\,*}$ and $\alpha_1\,,\cdots,\alpha_r \in \mathbf{Z}_\ell\,[G]$ such that, inside $U_{S,T} \otimes \mathbf{Z}_\ell$, we have

$$\phi_i = \phi'_i + \alpha_i\phi_0\,, \ \text{for all } i = 1,\cdots,r\,.$$

These equalities show that, inside $\mathbf{Q}_\ell\,[G]$, we have

$$\begin{aligned}(\phi_1 \wedge \cdots \wedge \phi_r)\,(\varepsilon_{S,T}) &= \delta_v \cdot (\phi'_1 \wedge \cdots \wedge \phi'_r)\,(\varepsilon_S) + \\ &\quad + \sum_{i=1}^{r}(-1)^{i+1}\alpha_i\delta_v \cdot (\phi'_1 \wedge \cdots \wedge \overset{i}{\widehat{\phi_0}} \wedge \cdots \wedge \phi'_r)\,(\varepsilon_S)\,.\end{aligned} \tag{25}$$

On the other hand, the definition of Λ_S combined with Proposition 1.2(A) imply that

$$(\psi_1 \wedge \cdots \wedge \psi_r)(\varepsilon_S) \in \mathcal{A}_\ell^{-1}\,, \text{ for all } \psi_1, \ldots, \psi_r \text{ in } \widetilde{U_S}^{\,*}\,. \tag{26}$$

This implies right away that

$$\delta_v \cdot (\phi_1' \wedge \cdots \wedge \phi_r')\,(\varepsilon_S) \in \delta_v \mathcal{A}_\ell^{-1} \subseteq \mathbf{Z}_\ell\,[G]\,. \tag{27}$$

We will not rely one more time on the fact that $\mathcal{A}_\ell$ is an invertible $\mathbf{Z}_\ell\,[G]$–fractional ideal. This implies that $\delta_v \mathcal{A}_\ell^{-1}\mathcal{A}_\ell = \mathbf{Z}_\ell\,[G] \cdot \delta_v$. Therefore, we can write $\delta_v = \sum_j \xi_j \cdot a_j$, with $\xi_j \in \delta_v \mathcal{A}_\ell^{-1}$ and $a_j \in \mathcal{A}_\ell$. Since $\xi_j \phi_0 \in \widetilde{U_S}^{\,*}$ for all j, (24) and (26) imply that:

(28)

$$\delta_v \cdot (\phi_1' \wedge \cdots \wedge \overset{i}{\widehat{\phi_0}} \wedge \cdots \wedge \phi_r')\,(\varepsilon_S) = \sum_j a_j \cdot (\phi_1' \wedge \cdots \wedge \overset{i}{\widehat{(\xi_j \phi_0)}} \wedge \cdots \wedge \phi_r')\,(\varepsilon_S) \in$$
$$\in \mathcal{A}_\ell \mathcal{A}_\ell^{-1} = \mathbf{Z}_\ell\,[G]\,, \text{ for all } i = 1, \ldots, r\,.$$

We now combine (27) and (28) to conclude that the left hand side of (25) is in $\mathbf{Z}_\ell\,[G]$. This concludes the proof of (3) in the comparison theorem.

(4) is a direct consequence of Remark 3, §2.1, Remark 3, §5.1, Proposition 1.2(B), and Proposition 5.3.1. □

6. EVIDENCE FOR CONJECTURE C

In this section we will list the cases in which we can presently prove Conjecture C. In what follows g denotes the order of the Galois group $G = G(K/k)$.

6.1. Global fields of characteristic $p > 0$

If K/k is a finite, abelian extension of function fields of characteristic $p > 0$, then the following statements are presently known to hold true:

- $\mathrm{C}(K/k,\, S,\, 1)$. In [18], Chapitre V, and [7] Deligne and Hayes give independent proofs of the Brumer-Stark Conjecture for finite, abelian extensions K/k of function fields. As remarked in §4.1, this is equivalent to $\mathrm{C}(K/k,\, S,\, 1)$, for all S such that $(K/k,\, S,\, 1)$ satisfies (H).
- $\mathbf{Z}\,[1/g]\,\mathrm{C}\,(K/k,\, S,\, r)$. In [10] we prove $\mathbf{Z}\,[1/g]\mathrm{B}\,(K/k,\, S,\, T,\, r)$ in this case. Since K/k is ℓ–admissible for all $\ell \nmid g$, Theorem 5.5.1(3) shows that this result is equivalent to $\mathbf{Z}\,[1/g]\mathrm{C}(K/k,\, S,\, r)$.
- $\mathbf{Z}_p\mathrm{C}(K/k,\, S,\, r)$. In [11], we show that a result obtained by Tan in [16] on a conjecture of Gross implies $\mathbf{Z}_p\mathrm{B}\,(K/k,\, S,\, T,\, r)$, for all $(K/k,\, S,\, T,\, r)$ satisfying hypothesis (H_T). On the other hand, $\mu_K^{(p)} = \{1\}$ for fields K of characteristic p. Therefore, Theorem 5.5.1(3) shows that this result is equivalent to $\mathbf{Z}_p\mathrm{C}(K/k,\, S,\, r)$.
- $\mathrm{C}(K/k,\, S,\, r)$, for $K \subseteq k_\infty \cdot k^{(p)}$. In [11] we show $\mathrm{B}(K/k,\, S,\, T,\, r)$ for any K contained in the compositum of the maximal constant field extension k_∞ and the maximal abelian pro–p extension $k^{(p)}$ of k. As an application of the criterion provided by Lemma 5.4.4, one can show that K/k is admissible for any such K (see [11]). Therefore, Theorem 5.1.1 implies that this result is equivalent to $\mathrm{C}(K/k,\, S,\, r)$.

6.2. Global fields of characteristic 0

Assume now that K/k is a finite, abelian extension of number fields. Then the following results are known to hold true:

• C(K/k, S, 1) for $k = \mathbf{Q}$ or k quadratic imaginary. In [15] Stark proves his conjecture "over $\mathbf{Z}$" for L–functions of order of vanishing 1 at $s = 0$ in these cases (see [18] as well). Remark 3, §2.1, and Theorem 5.5.1(4) show that this is equivalent to C(K/k, S, 1) and respectively B(K/k, S, T, 1), for all T such that (K/k, S, T, 1) satisfies (H_T).

Let us now make the additional assumption that K is a CM field and k is totally real. Under these hypotheses, Greither proves in [6] a strong form of Brumer's Conjecture up to a power of 2 for a large class of extensions K/k. Greither calls these extensions "nice". We refer the reader to [6] for the definition of a "nice extension." Here we will limit ourselves to saying that if K/k has a prime power conductor, then K/k tends to be "nice". In particular, if K is an imaginary subfield of the cyclotomic field $\mathbf{Q}(\zeta_{\ell^n})$, for ℓ prime and n natural, then $K/\mathbf{Q}$ is always a "nice extension". An essential property of Greither's "nice extensions" is that they are always ℓ–admissible, for all odd primes ℓ. Based on Greither's results, we show in [12] that the following statements are true:

• $\mathbf{Z}_\ell$C(K/k, S, r) for all nice extensions K/k and all odd primes ℓ. Based on the above observations, Theorem 5.5.1(3) implies that this result is equivalent to $\mathbf{Z}_\ell$B(K/k, S, T, r), for all T such that hypothesis (H_T) is satisfied.

• C(K/k, S, r), for $k = \mathbf{Q}$ and K an imaginary subfield of $\mathbf{Q}(\zeta_{\ell^n})$, with ℓ odd prime and n natural. We prove this in [12], based on results obtained by Greither in [5] and [6]. Please note that Lemma 5.4.4 implies that, under the present hypotheses, K/k is q–admissible for any odd prime q, but not 2–admissible. Therefore, we cannot conclude based on Theorem 5.5.1(3) that conjecture $\mathbf{Z}_2$B is also true in this case.

6.3. Evidence arising from base change

As observed in §§6.1 and 6.2 above, if (K/k, S, 1) satisfies (H), then C(K/k, S, 1) is true if either k is a function field of characteristic $p > 0$, or $k = \mathbf{Q}$, or k is an imaginary quadratic field. Theorem 3.1.1 therefore implies that, under these hypotheses, C(K/k', S', r) is also true for any intermediate field $k \subseteq k' \subseteq K$ of relative degree $[k' : k] = r$. (As in §3.1, here S' denotes the set of primes in k' lying above primes in S.)

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