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Gras-type conjectures for function fields. (English summary)
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FEATURED REVIEW.

Let $K/\mathbf{Q}$ be finite abelian with group G and admitting a real embedding $\varphi\colon K\to\mathbf{R}$. Let $g=|G|$. It is well known that the index of the cyclotomic units $\mathcal{E}_K$ in the unit group E_K of K equals the class number of K. Let A_K be the class group of K, and let $B_K=E_K/\mathcal{E}_K$. A conjecture of Gras asserts that for any prime $l\nmid g$, the l-parts of A_K and B_K have isomorphic Jordan-Hölder series as $\mathbf{Z}_l[G]$-modules. Equivalently, one can state the conjecture as $|e_\psi A_K|=|e_\psi B_K|$ for all irreducible characters ψ over $\mathbf{Q}_l$, where $e_\psi\in\mathbf{Z}_l[G]$ is the idempotent associated to ψ. As R. Greenberg showed [Nagoya Math. J. **67** (1977), 139–158; MR0444614 (56 #2964)], Gras' conjecture follows from the Main Conjecture of Iwasawa theory. It is therefore no longer a conjecture but rather a corollary of the theorem of Mazur-Wiles.

Fix an extension of φ to the real subfield $\mathcal{K}$ of the abelian closure of $\mathbf{Q}$. For any number field $F\subset\mathcal{K}$, let S_F be the set consisting of the Archimedean place of $\mathbf{Q}$ together with the primes that ramify in $F/\mathbf{Q}$. The restriction of φ to F defines a Stark S_F-unit $\varepsilon_F\in F$ having the property that $F(\sqrt{\varepsilon_F})/\mathbf{Q}$ is abelian. The group $\mathcal{E}_K$ is the intersection with E_K of the subgroup of $K^\times$ generated by the elements $\varepsilon_F^{(\sigma+1)/2}$, $\sigma\in\mathrm{Gal}(F(\sqrt{\varepsilon_F})/\mathbf{Q})$, for $F\subseteq K$. Since the ε_F are defined using the special values at $s=0$ of the imprimitive L-functions of $F/\mathbf{Q}$, they may be aligned into an Euler system [K. Rubin, J. Reine Angew. Math. **425** (1992), 141–154; MR1151317 (93d:11117)], thereby providing an alternative proof (cf. Rubin's appendix to [S. Lang, *Cyclotomic fields I and II*, Combined second edition, Springer, New York, 1990; MR1029028 (91c:11001)]) of Gras' conjecture.

Let K/k be an abelian extension of global fields with group G, and put $e_K=|\mu(K)|$. Given an integer $r\geq 0$, let S be a finite set of k-places containing any Archimedean places, any places that ramify in K/k, and at least r places that split in K/k. Further, assume $|S| \geq r+1$. Let T be any finite set of k-places disjoint from S. For $\chi \in \widehat{G}$, define

$$L_{S,T}(s,\chi) = \prod_{v \notin S} \left(1 - \chi(\sigma_v) \cdot Nv^{-s}\right)^{-1} \cdot \prod_{v \in T} (1 - \chi(\sigma_v) \cdot Nv^{1-s}) \tag{1}$$

[cf. B. H. Gross, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 1, 177–197; MR0931448 (89h:11071)]. Let S_K and T_K be the sets of K-places dividing S and T respectively. When $T = \varnothing$ and $w \in S_K$ is a distinguished split place, H. M. Stark [Adv. in Math. **35** (1980), no. 3, 197–235; MR0563924 (81f:10054)] conjectured the existence of an S_K-unit $\varepsilon = \varepsilon_w \in K$ such that $e_K \cdot L'_{S,T}(0,\chi) = -\sum_{\sigma \in G} \chi(\sigma) \cdot \log|\varepsilon^\sigma|_w$ for all $\chi \in \widehat{G}$. In case $k = \mathbf{Q}$ and w is the Archimedean place defined by an embedding $\varphi\colon K \to \mathbf{R}$, the ε_w exist and generate the group of cyclotomic units in the manner indicated above. Unfortunately, the ε_w will be trivial whenever $r \geq 2$.

Let $v_1, v_2, \cdots, v_r \in S$ split in K/k, and assume $T \neq \varnothing$. Refining ideas of Stark [Advances in Math. **7** (1971), 301–343 (1971); MR0289429 (44 #6620); Advances in Math. **17** (1975), no. 1, 60–92; MR0382194 (52 #3082); Advances in Math. **22** (1976), no. 1, 64–84; MR0437501 (55 #10427)], Rubin [Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 33–62; MR1385509 (97d:11174)] formulated a conjecture in number fields that enables one to construct special units in $U_{S,T}$, the S-units ε in K such that $\varepsilon \equiv 1 \pmod{w}$ for all $w \in T_K$. Rubin showed that these conjectural units can be aligned into Euler systems over k when k is totally real and $r = [k\colon \mathbf{Q}]$. The Rubin conjecture has an exact analog in function fields, and Popescu [Compositio Math. **116** (1999), no. 3, 321–367; MR1691163 (2000m:11115); see the preceding review] recently proved the conjecture in that context up to primes dividing $g = |G|$. After defining a regulator map $R\colon \mathbf{C} \otimes \bigwedge^r_{\mathbf{Z}[G]} U_{S,T} \to \mathbf{C}[G]$ using the r split places v_i, Popescu showed that there is a unique element

$$\varepsilon_{S,T} \in \mathrm{Fitt}_{\mathbf{Z}[G]} A_{S,T} \cdot \mathbf{Z}[1/g] \otimes \bigwedge{}^r_{\mathbf{Z}[G]} U_{S,T} \tag{2}$$

such that $\chi(R(\varepsilon_{S,T})) = \lim_{s \to 0} s^{-r} L_{S,T}(s,\chi)$ for all $\chi \in \widehat{G}$. Here, $A_{S,T}$ is the S-class group of K trivialized along T. As Rubin [op. cit., 1996] observed, when $k = \mathbf{Q}$ and $r = 1$, (2) is a form of Gras' conjecture.

Given $\varphi_1, \varphi_2, \cdots, \varphi_{r-1} \in \mathrm{Hom}_{\mathbf{Z}[G]}(U_{S,T}, \mathbf{Z}[G])$, put $\Phi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_{r-1}$. In the paper under review, Popescu defines $\Phi\colon \mathbf{Z}[1/g] \otimes \bigwedge^r_{\mathbf{Z}[G]} U_{S,T} \to \mathbf{Z}[1/g] \otimes U_{S,T}$ by

$$\Phi(u_1 \wedge \cdots \wedge u_r) = \sum_{1 \leq i \leq r} (-1)^i \det_{\substack{k,j \\ j \neq i}} (\varphi_k(u_j)) \cdot u_i.$$

The elements $\Phi(\varepsilon_{S,T})$ as Φ varies generate a subgroup of $\mathbf{Z}[1/g] \otimes U_{S,T}$. The elements in the intersection of this subgroup with $U_{S,T}$ are analogues of cyclotomic units. Similar constructs in sub-extensions of K/k provide a full group $\mathcal{E}_{S,T}$ of special units in $U_{S,T}$. Then $B_{S,T} := U_{S,T}/\mathcal{E}_{S,T}$ is finite and for primes $l \nmid g$,

$$|e_\psi A_{S,T}|^{r_\psi} = |e_\psi B_{S,T}| \tag{3}$$

for all irreducible characters ψ of G over $\mathbf{Q}_l$, where r_ψ is the number of places $v \in S$ where ψ

is locally trivial, except that $r_\psi = \text{Card}(S) - 1$ when $\psi = 1_G$. The exponent r_ψ appears in (3) because the prime ideals in S_K do not contribute to the class group $A_{S,T}$. The proofs do not invoke Euler systems, as that technique fails when l equals the characteristic of k.

The construction of special units in an arbitrary abelian extension of global function fields via the data encoded in the leading terms of its L-functions (1) at $s = 0$ and the demonstration of a Gras-type connection to the class group provide powerful evidence supporting Stark's ideas and Rubin's conjecture.

Popescu also uses his techniques to give a new derivation of a theorem of S. Bae [Math. Ann. **285** (1989), no. 3, 417–445; MR1019711 (91g:11130)] that established Chinburg's Ω_3-conjecture for cyclic extensions of function fields of prime degree.

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[References]

1. Bae, S.: On the conjecture of Lichtenbaum and of Chinburg over function fields, *Math. Ann.* **285** (1989), 417–445. MR1019711 (91g:11130)
2. Cassou-Noguès, Ph., Chinburg, T., Fröhlich, A. and Taylor, M. J.: L-functions and Galois modules, L-functions and arithmetic. In: J. Coates and M. J. Taylor (eds), *Proc. Durham Sympos.*, July 1989, London Math. Soc. Lecture Notes Series, 153 Cambridge Univ. Press, 1989. MR1110391 (92d:11124)
3. Chinburg, T.: On the Galois structure of algebraic integers and S-units, *Invent. Math* **74** (1983), 321–349. MR0724009 (86c:11096)
4. Chinburg, T.: Exact sequences and Galois module structure, *Ann. of Math.* **121** (1985), 351–376. MR0786352 (86j:11115)
5. Chinburg, T.: Galois structure invariants of global fields, Working paper from a talk at Durham, July 1989.
6. Fröhlich, A.: Some problems of Galois module structure for wild extensions, *Proc. London Math. Soc.* **27** (1978), 193–212. MR0507603 (80a:12013)
7. Goss, D. and Sinnott, W.: Class-groups of function fields, *Duke Math. J.* **52** (1985), 507–516. MR0792185 (87b:11118)
8. Greenberg, R.: On p-adic L-functions and cyclotomic fields II, *Nagoya Math. J.* **67** (1977), 139–158. MR0444614 (56 #2964)
9. Gras, G.: Classes d'ideaux des corps abeliens et nombres de Bernoulli generalises, *Ann. Inst. Fourier* **27** (1977), 1–66. MR0450238 (56 #8534)
10. Hayes, D. R.: Stickelberger elements in function fields, *Compositio Math.* **55** (1985), 209–239. MR0795715 (87d:11091)
11. Lang, S.: *Cyclotomic Fields I and II* (comb. 2nd edn), Springer-Verlag, New York, 1990. MR1029028 (91c:11001)
12. Mazur, B. and Wiles, A.: Class fields of abelian extensions of Q, *Invent. Math.* **76** (1984), 179–330. MR0742853 (85m:11069)

13. Moreno, C.: *Algebraic Curves over Finite Fields*, Cambridge University Press, 1991. MR1101140 (92d:11066)
14. Popescu, C. D.: *On a Refined Stark Conjecture for Function Fields*, PhD Thesis, Ohio State University, 1996.
15. Popescu, C. D.: On a refined Stark conjecture for function fields, to appear in *Compositio Math.* MR1691163 (2000m:11115)
16. Rubin, K.: Stark units and Kolyvagin-Euler systems, *J. Reine Angew. Math.* **425** (1992), 141–154. MR1151317 (93d:11117)
17. Rubin, K.: A Stark conjecture 'over **Z**' for abelian L-functions with multiple zeros, *Ann. Inst. Fourier* **46** (1996). MR1385509 (97d:11174)
18. Swan, R. G.: Induced representations and projective modules, *Ann. of Math.* **71**(3) (1960), 552–578. MR0138688 (25 #2131)
19. Swan, R. G. and Evans, E. G.: *K-theory of Finite Groups and Orders*, Lecture Notes in Math. 149, Springer Verlag, New York, 1970. MR0308195 (46 #7310)
20. Tate, J.: The cohomology groups of tory in finite Galois extensions of number fields, *Nagoya Math. J.* **27** (1966), 709–719. MR0207680 (34 #7495)
21. Tate, J.: *Les conjectures de stark sur les fonctions L d'Artin en $s = 0$*, Progr. in Math. 47, Birkhaüser, Boston, 1984. MR0782485 (86e:11112)