Math 103 HW 8 Solutions to Selected Problems

2. **Let**

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 6 & 1 & 7 & 8 & 6 \end{bmatrix} \textbf{ and } \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

Write α, β, and $\alpha\beta$ as

(i) products of disjoint cycles

Solution: Notice that $\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 4, \alpha(4) = 5$, and $\alpha(5) = 1$, so the cycle (12345) appears in α. Similarly, we find that the other cycle is (678). Thus we can write $\alpha = (12345)(678)$ (the order doesn't matter because disjoint cycles commute). Meanwhile, $\beta = (23847)(56)$, and $\alpha\beta = (12485736)$

(ii) products of 2-cycles

Solution: To do this, we just need to write each individual cycle as a product of 2-cycles. There is a standard way to do this: $(a_1a_2 \cdots a_n) = (a_na_{n-1}) \cdots (a_na_2)(a_na_1)$. Thus from part a, $\alpha = (45)(35)(25)(15)(78)(68)$, while $\beta = (47)(87)(37)(27)(56)$. Of course, this means $\alpha\beta = (45)(35)(25)(15)(78)(68)(47)(87)(37)(27)(56)$.

6. **What is th order of each of the following permutations?**

(i) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$

Solution: Call this permutation σ. Since disjoint cycles commute, we know that the order of σ is simply the lcm of the orders of its disjoint cycles, and that an n-cycle has order n. In this case, $\sigma = (12)(356)$, so the order is $lcm(2, 3) = 6$.

(ii) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

Solution: Now call *this* permutation σ. Then we have $\sigma = (1753)(264)$, so by the same reasoning as above $|\sigma| = lcm(4, 3) = 12$.

24. **Suppose that H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n.**

Solution: This looks similar to problem 25 of homework 5, although since H might not be cyclic we cannot use method 2 from the solutions. Instead, we can copy method 1. Suppose H contains an odd permutation σ. Given any other permutation $\tau \in S_n$,

$\sigma\tau$ is even if τ is odd and odd if τ is even. This is easy to see: if $\sigma = \sigma_1 \cdots \sigma_k$ and $\tau = \tau_1 \cdots \tau_m$ with the σ_i, τ_j 2-cycles, the $\sigma\tau = \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_m$, hence can be written as a product of $k + m$ 2-cycles (as we have seen, this means every representation of $\sigma\tau$ as a product of 2-cycles has the same parity as this one). Since τ is odd, k is odd, and hence $m + k \equiv m + 1 \mod 2$; ie, multiiplying by σ changes the parity of the permutation.

This means that the function $f(\tau) = \sigma\tau$ is a function from $H \to H$ (since H is closed under multiplication), that restricts to a bijection (because multiplying by σ^{-1} is the inverse of f) between the subsets of even and odd elements of H. Therefore, the two have the same size, and the sum of their orders, $|H|$ must be even, a contradiction. Thus, H cannot contain an odd element.

32. **Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form.**

Solution: It is easy to take powers of individual cycles, so it will be helpful to write β as a product of *disjoint* cycles. To do this, we just check what β does to each element of $\{1, 2, 3, 4, 5\}$:

$$\begin{aligned}
\beta(1) &= (123)(145)(1) \\
&= (123)(4) \\
&= 4 \\
\beta(2) &= (123)(2) \\
&= 3 \\
\beta(3) &= (123)(3) \\
&= 1 \\
\beta(4) &= (123)(5) \\
&= 5 \\
\beta(5) &= (123)(1) \\
&= 2
\end{aligned}$$

Putting it all together, β is just the 5-cycle (14523). This means that

$$\begin{aligned}
\beta^{99} &= \beta^{100}\beta^{-1} \\
&= (\beta^5)^{20}\beta^{-1} \\
&= e\beta^{-1} \\
&= \beta^{-1}
\end{aligned}$$

However, the inverse of a cycle is also easy to calculate: $\beta^{-1} = (13254)$.This is in disjoint cycle form already, so we are done.

48. **Let α and β belong to S_n. Prove that $\beta\alpha\beta^{-1}$ and α are both even or both odd.**

Solution: Write α as a product of k 2-cycles α_i, and β as a product of r 2-cycles β_j. It is easy to check (it follows from the general form of the inverse of a product, and that 2-cycles are their own inverses) that $\beta^{-1} = \beta_r \cdots \beta_1$, so it also is a product of k 2-cycles. Thus $\beta\alpha\beta^{-1}$ can be written as a product $k + 2r$ 2-cycles. $2r$ is certainly even, so k and $k + 2r$ have the same parity, giving the result.

55. **Show that a permutation with odd order must be an even permutation.**

Solution: Let σ be such a permutation, so in particular $\sigma^r = e$, with r odd. As usual, if we write σ as a product of k 2-cycles. Then σ^r will be a product of kr 2-cycles. But e is an even permutation (for example, $e = (12)(12)$) so kr must be even by the well-definedness of the parity of a permutation. Since 2 divides rk but not r, the only option is if 2 divides k; ie, k is even. Thus σ is even.