Math 103 HW 7 Solutions to Selected Problems

Chapter 3

50. **Consider the elements** $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **and** $B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ **from** $SL(2,\mathbb{R})$. **Find** $|A|, |B|$, **and** $|AB|$. **Does your answer surprise you?**

    Solution: The answer doesn't surprise us because we found these orders a few weeks ago—see problem 52 of Homework 5.

77. **Let** a **belong to a group and** $|a| = m$. **If** n **is relatively prime to** m**, show that** a **can be written as the** n**th power of some element in the group.**

    Solution: Since n and m are relatively prime, we can find $x, y \in \mathbb{Z}$ such that $nx + my = 1$. Then

    $$\begin{aligned} a &= a^{nx+my} \\ &= (a^x)^n (a^m)^y \\ &= (a^x)^n e^y \\ &= (a^x)^n \end{aligned}$$

    meaning a is the nth power of a^x.

78. **Let** G **be a finite group with more than one element. Show that** G **has an element of prime order.**

    Solution: Since G has more than one element, we can take $g \in G$ with $g \neq e$. Then $|g|$ (which must be finite since G is finite) is > 1, so must have some prime divisor p. As usual, $g^{\frac{n}{p}}$ must have order p, so we've found an element of order p in G.

Chapter 4

10. **In** Z_{24}**, list all generators for the subgroup of order 8. Let** $G = < a >$**, and let** $|a| = 24$. **List all generators for the subgroup of order** 8.

Solution: Z_{24} is cyclic, generated by 1, so the fundamental theorem of cyclic groups says that there is a single subgroup of Z_{24} of order 8, generated by $\frac{24}{8} \cdot 1 = 3$. Since $< x > = < gcd(x, 24) >$ (by theorem 4.2, for example), the other generators are precisely those elements of Z_{24} whose gcd with 24 is 3. The only prime factors of 24 are 2 and 3, so these are simply the elements divisible by 3 but not by 2. The generators are thus $3, 9, 15$, and 21. Changing from additive notation to multiplicative and replacing x with a^x, the analogous result holds for any cyclic group $G = < a >$ of order 24.

52. **Suppose that G is a cyclic group and that 6 divides $|G|$. How many elements of order 6 does G have? If 8 divides $|G|$, how many elements of order 8 does G have? If a is one element of order 8, list the other elements of order 8.**

    Solution: Since G is cyclic, for any divisor d of $|G|$ we have a single (necessarily cyclic) subgroup H of order d. The generators of H are therefore the only elements of order d (since any such element will generate a cyclic group of order d), and are of the form h^k, where $gcd(k, d) = 1$ (where k can be taken to be positive and smaller than d) for any generator of h of H. For $d = 6$, the only coprime positive integers smaller than d are 1 and 5, so there are 2 elements of order 6 in this case. Those coprime to and smaller than 8 are 1,3,5, and 7, so there are four elements of 8, which can be written as a, a^3, a^5, and a^7.