Math 103 HW 3 Solutions to Selected Problems

19. **Prove that the set of all 2×2 matrics with entries from $\mathbb{R}$ and determinant $+1$ is a group under matrix multiplication.**

Solution: Let G be this (putative) group. We first show that G is clsoed under multiplication. This is easy if we remember a fact from linear algebra: given matrices A and B, $\det(AB) = \det(A)\det(B)$. This shows that if A and B are $\in G$, we must have $\det(AB) = 1$ as well, meaning $AB \in G$. We can also check that the identity element in G is just

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has determinant $1 \cdot 1 - 0 \cdot 0 = 1$.

Next we show that G has inverses. From linear algebra, we have the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which works whenever $ad - bc$ (the determinant) is nonzero. This is certainly the case for $A \in G$, so we at least know that the matrix A^{-1} exists. It remains to check that A^{-1} is actually in G. This can be checked by computing the determinant, but we can also notice that since

$$\begin{aligned} AA^{-1} &= I, \\ \det(A)\det(A^{-1}) &= \det(I) \\ &= 1 \end{aligned}$$

Thus if $\det(A) = 1$, the above shows that $\det(A^{-1}) = 1$ as well, which means it is in G.

The only thing left is the tedious task of showing that matrix multiplication is actually associative. To this end, let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, and $Z = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$. We calculate:

$$\begin{aligned} A(BC) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} es + fu & et + fv \\ gs + hu & gt + hv \end{pmatrix} \\ &= \begin{pmatrix} a(es + fu) + b(gs + hu) & a(et + fv) + b(gt + hv) \\ c(es + fu) + d(gs + hu) & c(et + fv) + d(gt + hv) \end{pmatrix} \end{aligned}$$

On the other hand,

$$(AB)C = \begin{pmatrix} ae+bg & af+gh \\ ce+bg & cf+dh \end{pmatrix} \cdot \begin{pmatrix} s & t \\ u & v \end{pmatrix}$$
$$= \begin{pmatrix} (ae+bg)s+(af+gh)u & (ae+bg)t+(af+gh)v \\ (ce+bg)s+(cf+dh)u & (ce+bg)t+(cf+dh)v \end{pmatrix}$$

and by distributing and using commutativity (and associativity) of multiplication of real numbers, we see that each of the four entries is the same. Therefore the two are equal, and G is a group.

22. **Let G be a group with the property that for any x, y, z in the group, $xy = zx$ implies $y = z$. Prove that G is Abelian ("Left-right cancellation" implies commutativity.)**

    Solution: Given $a, b \in G$, we must show that $ab = ba$. Multiplying the LHS by b on the left and the RHS by b on the right, we certainly have $bab = bab$, so letting $x = b, y = ab, z = ba$, the assumption on G forces $ab = ba$, as desired.

26. **Prove that in a group $(a^{-1})^{-1} = a$ for all a.**

    Solution: $(a^{-1})^{-1}$ is the unique element in G such that $(a^{-1})^{-1}a^{-1} = a^{-1}(a^{-1})^{-1} = e$. But since a^{-1} is a's inverse, $aa^{-1} = a^{-1}a = e$, so $a = (a^{-1})^{-1}$.

34. **Prove that in a group, $(ab)^2 = a^2b^2$ if and only if $ab = ba$.**
    Prove that in a group, $(ab)^{-2} = b^{-2}a^{-2}$ if and only if $ab = ba$.

    Solution: The "if" parts follow from problem 23, so assume that $(ab)(ab) = a^2b^2$. Multiplying on the left by a^{-1}, we see that $bab = ab^2$. But then we can multiply on the right by b^{-1}, which yields $ba = ab$.

    Now suppose that $(ab)^{-1}(ab)^{-1} = b^{-2}a^{-2}$. Since $(ab)^{-1} = b^{-1}a^{-1}$, we have that $(b^{-1}a^{-1})^2 = (b^{-1})^2(a^{-1})^2$. But then the previous result (with b^{-1} in place of a and a^{-1} in place of b), shows that $a^{-1}b^{-1} = b^{-1}a^{-1}$. But then the inverses of both sides are the same, so $ba = ab$.

36. **Let a and b belong to a group G. Find an x in G such that $xabx^{-1} = ba$.**

    Solution: It is equivalent to show $xabx^{-1}a^{-1} = b$, since we have b's in the center of both expressions, one easy way to make them equal is to let $xa = e$, or $x = a^{-1}$.