Math 103 HW 1 Solutions to Selected Problems

6. **Suppose a and b are integers that divide the integer c. If a and b are relatively prime, show that ab divides c. Show, by example, that if a and b are not relatively prime, then ab need not divide c.**

   Solution: Since a and b are relatively prime, we can find integers x and y such that $ax + by = 1$. Multiplying by c, this means that

   $$cax + cby = c$$

   Both of the summands on the right hand side are divisible by ab, since a and b both divide c, and therefore so is their sum, c.

   Now let $a = b = c = 2$. Then a and b certainly divide c, but of course $ab = 4$ does not divide 2. This shows that the relatively prime assumption is necessary, in general.

12. **Show that $5n + 3$ and $7n + 4$ are relatively prime for all n.**

   Solution: Let $x = 7n + 4$, $y = 5n + 3$. One way to show that x and y are relatively prime is to show $ax + by = 1$ for some $a, b \in \mathbb{Z}$. To start, notice that $x - y = 2n + 1$, and $3y - 2x = n + 1$. Subtracting, we get that

   $$\begin{aligned} 3x - 4y &= 2n + 1 - (n + 1) \\ &= n \end{aligned}$$

   which means that

   $$\begin{aligned} 1 &= n + 1 - n \\ &= (3y - 2x) - (3x - 4y) \\ &= 7y - 5x \end{aligned}$$

   so they are indeed relatively prime.

   Note: If $n \geq 0$, this is exactly the result we would get from running the Euclidean algorithm on x and y, then tracing backwards.

30. **(Generalized Euclid's Lemma) If p is a prime and p divides $a_1a_2\cdots a_n$, prove that p divides a_i for some i.**

Solution: If $n = 1$, then p divides a_1 certainly implies p divides a_1. The case when $n = 2$ is given by the usual Euclid's Lemma. The rest we can take care of by induction: suppose we know for some $n \geq 2$ that the statement is true for any product of n integers, and that p divides $a_1a_2\cdots a_{n+1}$. By Euclid's Lemma applied to the product $(a_1a_2\cdots a_n)\cdot(a_{i+1})$, either p divides a_{n+1}, or p divides $a_1a_2\cdots a_n$. In this case, p divides some a_i (for $1 \leq i \leq n$) by assumption, so either way we're done.

34. **The Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$. In general, the Fibonacci numbers are defined by $f_1 = 1, f_2 = 2$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Prove the nth Fibonacci number f_n satisfies $f_n < 2^n$.**

Solution: To begin with, at least $f_1 = 1 < 2^1$ and $F_2 = 1 < 2^2$. For the rest, suppose that we know that $f_k < 2^k$ for all $1 \leq k \leq n$ ($n \geq 2$). We want to show that $f_{n+1} < 2^{n+1}$. But

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \\ &< 2^n + 2^{n-1} \\ &< 2^n + 2^n \\ &= 2^{n+1} \end{aligned}$$

where the first inequality follows from our inductive hypothesis. By induction, the inquality holds for all positive integers n.

62. **Prove that $3, 5$, and 7 are the only three consecutive integers that are prime.**

Solution: Strictly speaking, this is false if we consider $-7, -5, -3$ to also be primes, but we will rule out any other examples. Any triple of consecutive odd integers is of the form $n, n+2, n+4$ for some integer n. Looking at some examples, we can guess that any such triple has one of its members divisible by 3. To prove this, we use the division algorithm to write $n = 3q + r$ for $q, r \in \mathbb{Z}$, $0 \leq r < 3$. There are three cases:

(i) $n = 3q$. This proves our claim in this case.

(ii) $n = 3q + 1$. Then $n + 2 = 3q + 3$, which is divisible by 3.

(iii) $n = 3q + 2$. Then $n + 4 = 3q + 6$, which is also divisible by 3.

In any case, we've shown that one of the integers (call it m) must be divisible by 3. For almost all m, this forces m to be composite. The only exceptions are when $m = \pm 3$, but we can check these cases by hand. The triples involving 3 or -3 are $(-7, -5, -3), (-5, -3, -1), (-3, -1, 1), (-1, 1, 3), (1, 3, 5)$, and $(3, 5, 7)$. Of these, the only ones whose members are all prime are $3, 5, 7$ and $-7, -5, -3$.