SOLUTIONS

Exam 1, Mathematics 109
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Name:
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Note: There are 3 questions on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (30 points)

(1) Use truth tables to prove the following logical equivalence

$$\overline{P \vee Q} \Longleftrightarrow \overline{P} \wedge \overline{Q},$$

for any two propositions P and Q.

(2) Use part (1) above and the contrapositive proof technique to prove the following statement:
"For all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, if $x^n > 0$ then $x > 0$ or n is even."

Hint for part (2): You may assume that an integer n is odd if and only if it can be written as $n = 2k + 1$, with $k \in \mathbb{Z}$.

1)

P	Q	$\overline{P \vee Q}$	$\overline{P} \wedge \overline{Q}$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

2) For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ $(x^n > 0) \Rightarrow (x > 0) \vee (n \text{ even})$

According to (1) and the contrapositive method of proof, this is equivalent to:

For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, $(x \leq 0) \wedge (n \text{ odd}) \Rightarrow x^n \leq 0$.

Proof. Let $x \in \mathbb{R}$, $x \leq 0$. and $n \in \mathbb{N}$ odd. Write $n = 2k+1$, with $k \in \mathbb{Z}_{\geq 0}$. We have $x^n = x^{2k+1} = (x^2)^k \cdot x$.

However $x^2 \geq 0$, for all $x \in \mathbb{R}$. Therefore $(x^2)^k \geq 0$, for all $x \in \mathbb{R}$. Therefore $(x^2)^k \cdot x \leq 0$, for $x \leq 0$. $\square$

II. (40 points)

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers defined inductively as follows:

$$a_1 = 1, \quad a_2 = 1, \quad a_n = 2a_{n-1} + 3a_{n-2}, \quad \text{for all } n \geq 3.$$

(1) Prove that $a_n > 3^{n-2}$, for all $n \geq 3$.

(2) Prove that $a_n < 2 \cdot 3^{n-2}$, for all $n \geq 3$.

(3) Prove that

$$a_n = \frac{1}{6} \cdot 3^n - \frac{1}{2} \cdot (-1)^n, \quad \text{for all } n \geq 1.$$

Note that $a_3 = 2\cdot 1 + 3 \cdot 1 = 5$, $a_4 = 2 \cdot 5 + 3 \cdot 1 = 13$.

(1) We use strong mathematical induction to prove $P(n): a_n > 3^{n-2}$, for all $n \geq 3$.

Base case $n = 3$ $P(3)$: $5 > 3$ true.

$n = 4$ $P(4)$: $13 > 9$ true.

Inductive step Let $n \geq 4$ and assume that $P(3), \dots, P(n)$ are all true. We will prove $P(n+1)$.

$P(n+1)$: $a_{n+1} > 3^{n-1}$.

However $a_{n+1} = 2 \cdot a_n + 3a_{n-1} \overset{(P(n),\, P(n-1))}{>} 2 \cdot 3^{n-2} + 3 \cdot 3^{n-3} = (2+1)3^{n-2} = 3^{n-1}$ $\square$.

(2) Similar to (1)

(3) Use strong math. induction to prove: $P(n): a_n = \frac{1}{6} 3^n - \frac{1}{2}(-1)^n$ for all $n \geq 1$.

Base case ($n=1$) $1 = \frac{1}{6} \cdot 3 - \frac{1}{2}(-1) \iff 1 = \frac{1}{2} + \frac{1}{2} \iff 1 = 1$ true.

($n=2$) $1 = \frac{1}{6} \cdot 9 - \frac{1}{2}(-1)^2 \iff 1 = \frac{9}{6} - \frac{1}{2} \iff 1 = \frac{6}{6}$ true.

Inductive step. Let $n \geq 2$; assume $P(1), \dots, P(n)$ are all true.

We will prove that $P(n+1): a_{n+1} = \frac{1}{6}3^{n+1} - \frac{1}{2}(-1)^{n+1}$ holds true.

$$a_{n+1} = 2a_n + 3a_{n-1} = 2\left(\frac{1}{6}3^n - \frac{1}{2}(-1)^n\right) + 3\left(\frac{1}{6}3^{n-1} - \frac{1}{2}(-1)^{n-1}\right) =$$

$$= \frac{1}{6}\left(2\cdot 3^n + 3\cdot 3^{n-1}\right) - \frac{1}{2}\left(2\cdot(-1)^n + 3\cdot(-1)^{n-1}\right)$$

$$= \frac{1}{6}3^n\cdot(2+1) - \frac{1}{2}(-1)^{n+1}\left[2\cdot(-1) + 3\right] =$$

$$= \frac{1}{6}3^{n+1} - \frac{1}{2}(-1)^{n+1}. \qquad \square$$

III. (30 points)

(1) Prove that $3 \mid (n^3 - n)$, for all $n \in \mathbb{N}$.

(2) Prove or disprove the following statement:

$$4 \mid (n^4 - n), \quad \text{for all } n \in \mathbb{N}.$$

Definition

$a, b \in \mathbb{Z}$.

$a \mid b \iff$ there exists $k \in \mathbb{Z}$ such that $b = k \cdot a$.

(1) Prove $\left(P(n) : 3 \mid n^3 - n\right)$, for all $n \in \mathbb{N}$ by induction.

Base case $n = 1$ $P(1) : 3 \mid 0$ which is true because $0 = 3 \cdot 0$.

Inductive step Let $n \in \mathbb{N}$. Assume that $3 \mid n^3 - n$.

~~We~~ We will prove that $3 \mid (n+1)^3 - (n+1)$.

$$(n+1)^3 - (n+1) = \left(n^3 + 3n^2 + 3n + 1\right) - (n+1) =$$

$$= \left(n^3 - n\right) + 3\left(n^2 + n\right) =$$

$$\underset{P(n)}{=} 3 \cdot k + 3\left(n^2 + n\right), \text{ for some } k \in \mathbb{Z}$$

Therefore $(n+1)^3 - (n+1) = 3\left(k + n^2 + n\right)$.

Since $k + n^2 + n \in \mathbb{Z}$, $3 \mid (n+1)^3 - (n+1)$ $\square$.

(2) The statement is false because $2 \in \mathbb{N}$ yet

$$4 \nmid 2^4 - 2.$$

Indeed, $2^4 - 2 = 14$ and if $14 = 4 \cdot k$, then

$$k = \frac{14}{4} = \frac{7}{2} \notin \mathbb{Z} \quad \square.$$