

6.5

#3] $T: P_2 \rightarrow \mathbb{R}^2$

$$T(a+bx+cx^2) = \begin{bmatrix} a-b \\ b+c \end{bmatrix}.$$

(a) (i) $T(1+x) = \begin{bmatrix} 1-1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $(1+x)$ is not in $\ker(T)$.

(ii) $T(x-x^2) = \begin{bmatrix} 0-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so $(x-x^2)$ is not in $\ker(T)$

(iii) $T(1+x-x^2) = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $1+x-x^2$ is in $\ker(T)$.

(b) All three are in the range, as

(i) $T(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (ii) $T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (iii) $T(-x^2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) $a+bx+cx^2$ is in $\ker(T)$, if $T(a+bx+cx^2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\text{i.e. } \begin{bmatrix} a-b \\ b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{i.e. } \begin{array}{l} a-b=0 \\ b+c=0 \end{array} \quad \text{i.e. } a=b=c=0.$$

Hence $\ker(T)$ is a one-dimensional subspace with basis

$$\mathcal{B} = \{(1+x-x^2)\}$$

By the Rank Theorem,

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

or $\text{rank}(T) + 1 = 3 \Rightarrow \text{rank}(T) = 2$. Since \mathbb{R}^2 is 2-dimensional, this means that $\text{im}(T) = \mathbb{R}^2$, i.e. T is onto.

#9] $T: M_{22} \rightarrow \mathbb{R}^2$

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix}.$$

Note that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $\ker(T)$ iff $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

i.e. if $a=b$ and $c=d$. Thus $\ker(T) = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

is 2-dimensional. By the Rank Theorem,

$$\begin{aligned} \text{rank}(T) &= \dim V - \text{nullity}(T) \\ &= 4 - 2 = 2. \end{aligned}$$

#22] $V = S_3$ (symmetric 3×3), $W = U_3$ (upper triangular 3×3).

These are isomorphic via the map below:

$$T: V \rightarrow W \text{ by } T \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

T is linear:

$$\begin{aligned} (1) \quad T \left(\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} + \begin{pmatrix} a' & b' & c' \\ b' & d' & e' \\ c' & e' & f' \end{pmatrix} \right) &= T \begin{pmatrix} a+a' & b+b' & c+c' \\ b+b' & d+d' & e+e' \\ c+c' & e+e' & f+f' \end{pmatrix} \\ &= \begin{pmatrix} a+a' & b+b' & c+c' \\ 0 & d+d' & e+e' \\ 0 & 0 & f+f' \end{pmatrix} \end{aligned}$$

#22 cont'd

$$= \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} + \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} = T \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} + T \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix}.$$

(2) $T \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = T \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ \tilde{b} & \tilde{d} & \tilde{e} \\ \tilde{c} & \tilde{e} & \tilde{f} \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} & \tilde{c} \\ 0 & \tilde{d} & \tilde{e} \\ 0 & 0 & \tilde{f} \end{pmatrix} = \tilde{a} T \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$

T is one-to-one:

If $T \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow a = b = c = d = e = f = 0$$

\Rightarrow Only 3 maps to 3 (i.e. T is one-to-one).

T is onto:

If $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ is an arbitrary upper-triangular matrix,

then $T \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

$\Rightarrow T$ is onto.

$\therefore T$ is an isomorphism.

#35] (b) $T: V \rightarrow W$. Prove if $\dim V > \dim W$, then T cannot be one-to-one.

Pf:

Method #1: By the Rank Theorem,

$$\text{nullity}(T) = \dim(V) - \text{rank}(T).$$

As $\text{rank}(T) \leq \dim W < \dim V$,

$$\text{nullity}(T) \geq 1$$

so T is not one-to-one.

Method #2 Suppose for contradiction T was one-to-one.

By Theorem 6.22, if $B = \{b_1, \dots, b_n\}$ is a basis of V with $n = \dim V$, then $T(b_1), \dots, T(b_n)$ are linearly independent in W . But W ~~can~~ can have at most $\dim W < n$ vectors, \Rightarrow contradiction.