

Math 102 Homework 8 Solutions

6.3

#16) $B = \{x, 1+x, 1-x+x^2\}$, $C = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

then $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C} = [\vec{v}_1]_B \ [\vec{v}_2]_B \ [\vec{v}_3]_B$

Compute $(P_{C \leftarrow B})^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 & 2 \end{array} \right]$$

so $(P_{C \leftarrow B})^{-1} = P_{B \leftarrow C} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

$$\begin{aligned} \vec{v}_1 &= 1(x) - 1(1+x) + 2(1-x+x^2) \\ &= 1 - 2x + 2x^2 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 &= 0(x) + 1(1+x) - 1(1-x+x^2) \\ &= 2x - x^2 \end{aligned}$$

$$\begin{aligned} \vec{v}_3 &= 0(x) - 1(1+x) + 2(1-x+x^2) \\ &= 1 - 3x + 2x^2 \end{aligned}$$

so $C = \{1 - 2x + 2x^2, 2x - x^2, 1 - 3x + 2x^2\}$

21) Prove that $P_{D \leftarrow C} \cdot P_{C \leftarrow B} = P_{D \leftarrow B}$

Pf: Suppose \vec{x} is in V

Then $(P_{D \leftarrow C} \cdot P_{C \leftarrow B}) [\vec{x}]_B = P_{D \leftarrow C} (P_{C \leftarrow B} [\vec{x}]_B)$

$$= P_{D \leftarrow C} ([\vec{x}]_C) = [\vec{x}]_D$$

Hence $P_{D \leftarrow C} \cdot P_{C \leftarrow B}$ is a matrix that sends $[\vec{x}]_B$ to $[\vec{x}]_D$ for all \vec{x} in V

By 6.12 (b), (uniqueness)

$$P_{D \leftarrow C} \cdot P_{C \leftarrow B} = P_{D \leftarrow B} //$$

6.4

to the contrary

#20) Suppose $T: \mathbb{R}^3 \rightarrow P_2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = 1+x, \quad T\left(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right) = 2-x+x^2,$$

$$T\left(\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix}\right) = -2+2x^2$$

$$\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

Since T is linear transf,

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix}\right) &= T\left(6 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right) \\ &= 6 \cdot T\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) - 4 \cdot T\left(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right) \\ &= 6(1+x) - 4(2-x+x^2) \\ &= -2 + 10x - 4x^2 \end{aligned}$$

But $T\left(\begin{bmatrix} 0 \\ 6 \\ -8 \end{bmatrix}\right) = -2+2x^2$ by definition

This is a contradiction.

Hence, T cannot be a linear transf.



#32) a) \Rightarrow Suppose $\{\vec{v}, T(\vec{v})\}$ linearly dep.

Then $T(\vec{v}) = \lambda \vec{v}$ for some scalar

Applying T to both sides gives

$$(T \circ T)\vec{v} = T(T(\vec{v})) = T(\lambda \vec{v}) = \lambda T(\vec{v}) = \lambda^2 \vec{v}$$

$$\stackrel{\vec{v}}{\Rightarrow} \text{Hence } \vec{v} = \lambda^2 \vec{v} \quad \cancel{\text{so } \lambda^2 = 1}$$

$$\Rightarrow \lambda^2 = 1$$

$$\text{so } \lambda = 1 \text{ or } \lambda = -1$$

$$\Rightarrow T(\vec{v}) = \vec{v} \text{ or } T(\vec{v}) = -\vec{v} \quad \cancel{\text{so } \lambda^2 = 1}$$

\Leftarrow) Suppose $T(\vec{v}) = \pm \vec{v}$

If $T(\vec{v}) = \vec{v}$, then $\{\vec{v}, T(\vec{v})\} = \{\vec{v}, \vec{v}\}$

is clearly a lin. dependent set

Similarly, if $T(\vec{v}) = -\vec{v}$, $\{\vec{v}, -\vec{v}\}$ also
linearly dependent ✓

b) $V = \mathbb{R}^2$, let T = identity = I

i.e. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \end{bmatrix}$ in \mathbb{R}^2

Then clearly $T \circ T = I$