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# HW 3 Solutions - Math 102 Winter 2012

2.1

# 40 a) Find system of 2 linear equations

in  $x_1, x_2, x_3$  w/solution set  $\begin{cases} x_1 = t \\ x_2 = 1+t \\ x_3 = 2-t \end{cases}$

Naively, we see that  $\begin{cases} x_2 - x_1 = 1 \\ x_3 + x_1 = 2 \end{cases}$

this is (one of many) answers - geometrically, we are finding 2 planes in  $\mathbb{R}^3$  ~~which~~ whose intersection is the line  $\ell: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

We check our answer geometrically:

1st plane is  $-x_1 + x_2 + 0x_3 = 1$

the point  $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$  is in the plane, and

the normal vector for the plane  $\vec{n}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

is orthogonal to the direction vector for

the line  $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ , i.e.  $\vec{n}_1 \cdot \vec{d} = 0$

(in general, a plane will contain  $\ell$  if it contains

$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$  and its normal vector is orthogonal to  $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ )

Similarly, for the second plane:  $x_1 + 0x_2 + x_3 = 2$ ,  
 $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$  is in this plane, and  $\vec{n}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  is  
 orthogonal to  $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

b) If  $x_3 = s = 2-t$ , then we get another parametric  
 equation for the same line:  $\begin{cases} x_1 = 2-s \\ x_2 = 3-s \\ x_3 = s \end{cases}$

We can also obtain this by solving the  
 system  $\begin{cases} x_2 - x_1 = 1 \\ x_3 + x_1 = 2 \end{cases}$  by Gaussian elimination:

$$\left[ \begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \end{array} \right] \xrightarrow[\text{add row}]{\text{switch}} \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ -1 & 1 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array} \right], \quad \begin{array}{l} \text{now set } x_3 = s \\ \text{(free variable)} \\ + \text{ solve by back} \\ \text{substitution} \end{array}$$

$$\text{row 1} \Rightarrow x_1 + x_3 = 2 \Rightarrow x_1 = 2-s$$

$$\text{row 2} \Rightarrow x_2 + x_3 = 3 \Rightarrow x_2 = 3-s$$

~~Also~~ We also check geometrically that this is  
 our original line:  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$   
 w/ different parameterization  
 this point lies in both planes!  
 $\uparrow$  this is parallel to  $\vec{d}$

2.2 #39 : Two Methods!

Method 1 : (Algebraic) Suppose ~~ad-bc=0~~  $ad-bc \neq 0$

The system  $\begin{cases} ax+by=r \\ cx+dy=s \end{cases}$  is solved using Gaussian elimination

$$\left[ \begin{array}{cc|c} a & b & r \\ c & d & s \end{array} \right]$$

Case 1:  $a \neq 0$

$$\frac{1}{a} \left[ \begin{array}{cc|c} a & b & r \\ c & d & s \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & b/a & r/a \\ c & d & s \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cc|c} 1 & b/a & r/a \\ 0 & d - \frac{cb}{a} & s - \frac{rc}{a} \end{array} \right]$$

Since  $ad-bc \neq 0$ , we get  $d - \frac{bc}{a} \neq 0$

Solve by back substitution:  
to find unique solution

$$y = \frac{s - \frac{rc}{a}}{d - \frac{bc}{a}}$$

$$x = \frac{r}{a} - \frac{b}{a}(y)$$

(Or just notice that 2 non-zero leading entries  
immediately implies there's a unique  
solution )



Case 2:  $a=0$

$$\text{G} \left[ \begin{matrix} 0 & b \\ c & d \end{matrix} \mid \begin{matrix} r \\ s \end{matrix} \right] \rightarrow \left[ \begin{matrix} c & d \\ 0 & b \end{matrix} \mid \begin{matrix} s \\ r \end{matrix} \right]$$

Since  $a=0$ ,  $ad - bc = -bc \neq 0 \Rightarrow c \neq 0$   
and  $b \neq 0$

Hence, since we have 2 nonzero leading entries, there is a unique solution,

which is  $y = r/b$ ,  $x = s/c - \frac{d}{c}y$

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Method 2: (Geometric)

The solution to  $\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$

is the intersection of two lines in  $\mathbb{R}^2$

Since  $ad - bc \neq 0 \Rightarrow$  the vectors  $\begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} c \\ d \end{bmatrix}$

are not parallel

$$\left( \text{if } \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} c \\ d \end{bmatrix} \Rightarrow \begin{array}{l} a = \lambda c \\ b = \lambda d \end{array} \Rightarrow \lambda = \frac{a}{c} = \frac{b}{d} \Rightarrow ad - bc = 0 \right)$$

If  $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$  which are normal vectors for

the lines are not parallel, then the lines

themselves are not parallel, hence must intersect in a unique point

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Q.2 #45 Find the line of intersection  
of the given planes: P.  $3x + 2y + z = -1$   
 $P_2 \quad 2x - y + 4z = 5$

$$\frac{1}{3} \left[ \begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 2 & -1 & 4 & 5 \end{array} \right]$$

$$\left[ \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{7}{3} & \frac{10}{3} & \frac{17}{3} \end{array} \right] \begin{matrix} ② \\ ① \end{matrix}$$

$z$  is a free variable  
we set  $z = t$ , our parameter

① gives:  $-\frac{7}{3}y + \frac{10}{3}t = \frac{17}{3}$

$$y = -\frac{3}{7} \left( \frac{17}{3} - \frac{10}{3}t \right)$$

$$= -\frac{17}{7} + \frac{10}{7}t$$

② gives:  $x + \frac{2}{3}y + \frac{1}{3}t = -\frac{1}{3}$

$$x = -\frac{2}{3} \left( -\frac{17}{7} + \frac{10}{7}t \right) - \frac{1}{3}t - \frac{1}{3}$$

$$= \frac{34}{21} - \frac{20}{21}t - \frac{7}{21}t - \frac{7}{21}$$

$$= \frac{27}{21} - \frac{27}{21}t = \frac{9}{7} - \frac{9}{7}t$$

So our solution is the line

$$l: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ -\frac{17}{7} \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{9}{7} \\ \frac{10}{7} \\ 1 \end{bmatrix}$$

Note that our answer is different from the book's solution, but we can check our work by ~~check~~ plugging into the 2 planes: ⑥

Plug  $\ell$  into  $P_1$ :  $3x + 2y + z$

$$= 3\left(\frac{9}{7} - \frac{9}{7}t\right) + 2\left(-\frac{17}{7} + \frac{10}{7}t\right) + t$$

$$= \frac{27}{7} - \frac{27}{7}t - \frac{34}{7} + \frac{20}{7}t + t$$

$$= -\frac{7}{7} - \frac{27}{7}t + \frac{27}{7}t = -1 \quad \checkmark$$

Plug  $\ell$  into  $P_2$ :  $2x - y + 4z$

$$= 2\left(\frac{9}{7} - \frac{9}{7}t\right) - \left(-\frac{17}{7} + \frac{10}{7}t\right) + 4t$$

$$= \frac{18}{7} - \frac{18}{7}t + \frac{17}{7} - \frac{10}{7}t + 4t$$

$$= \frac{35}{7} - \frac{28}{7}t + 4t = 5 \quad \checkmark$$

Hence, since the line  $\ell$  is in both planes which are nonparallel to each other,  
 $\ell$  is the intersection of the two planes.

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Determine whether  $\vec{x} = \vec{p} + s\vec{u}$  and  $\vec{x} = \vec{q} + t\vec{v}$  intersect, i.e. ~~is there~~ there are  $s, t$  scalars so that  $\vec{p} + s\vec{u} = \vec{q} + t\vec{v} \Leftrightarrow s\vec{u} - t\vec{v} = \vec{q} - \vec{p}$

$$\Leftrightarrow s \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

This is a system of 3 equations ~~with~~ in two unknowns,  $s$  and  $t$ :

$$\left\{ \begin{array}{l} s+t=3 \\ 2s-t=0 \\ -s=-1 \end{array} \right. \Rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{array} \right] \text{ } \cancel{\text{non}}$$

We can immediately see from row 3 that  $s=1$ , and both row 1 and 2 give  $t=2$

so this is the unique point of intersection

(equivalently, row reduce the matrix to

obtain  $\left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$ , a rank 2 matrix  
since  $\text{rank} = 2 = n$ ,  
unique solution )

