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3.1 #36. The critical fact is that  $B^8 = I_2$ .

One way to see this is by computing

$$B^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B^4 = (B^2)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B^8 = (B^4)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by dividing  $\frac{2011}{8} = 251 \frac{3}{8}$ , we see that

$$B^{2011} = \underbrace{(B^8)^{251}}_I B^3 = B^3.$$

$$\begin{aligned} \text{Then } B^3 &= BB^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

$$\text{So } B^{2011} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

3.2 #42. A matrix  $B$  is skew-symmetric if  $B^T = -B$ .

Set  $B = A - A^T$ . Then

$$\begin{aligned} B^T &= (A - A^T)^T = A^T - (A^T)^T \quad \text{by Thm 3.4} \\ &= A^T - A \quad \text{by Thm 3.4} \\ &= -(A - A^T) \\ &= -B, \end{aligned}$$

Thus  $A - A^T$  is skew-symmetric.

3.2 #43 First we prove that  $A + A^T$  is symmetric. (i.e.  $(A + A^T)^T = A + A^T$ ). Indeed,

$$\begin{aligned} (A + A^T)^T &= A^T + (A^T)^T \quad \text{by Thm 3.4} \\ &= A^T + A \quad \text{by Thm 3.4} \end{aligned}$$

Then we can write

$$A = \underbrace{\frac{A + A^T}{2}}_{\text{symmetric by}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew-symmetric}}$$

above calculation.  $\quad$  by 3.2 #42

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3.2 #47 Either method is OK.

Method 1:  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ . Then

$$\begin{aligned} AB - BA &= \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} - \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix} \\ &= \begin{bmatrix} bg - cf & af + bh - be - df \\ ce + dg - ag - ch & cf - bg \end{bmatrix} \end{aligned}$$

Notice that  $cf - bg = -(bg - cf)$ , so the  $(2, 2)$ -entry is negative the  $(1, 1)$ -entry. Thus both entries cannot be 1, and  $AB - BA \neq I_2$ .

$$\begin{aligned} \text{Method 2: } \operatorname{tr}(AB - BA) &= \operatorname{tr}(AB) + \operatorname{tr}(-BA) \\ &= \operatorname{tr}(AB) - \operatorname{tr}(BA) \\ &= \operatorname{tr}(AB) - \operatorname{tr}(AB) \quad - \text{By 3.2 #44} \\ &= 0 \quad \text{By 3.2 #45} \end{aligned}$$

But  $\operatorname{tr}(I_2) = 2$ . Since their traces are different,

$$I_2 \neq AB - BA.$$