Exam 1, Mathematics 103B
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Name:
Student ID:

Note: There are 3 problems on this exam. You will not receive credit unless you show all your work. No books, calculators, notes or tables are permitted.

I. (35 points)

Let R be a ring with $0_R \neq 1_R$ and such that $x^2 = x$, for all $x \in R$.

(1) Show that $x + x = 0$, for all $x \in R$.
(2) Show that $\mathrm{char}(R) = 2$.
(3) Show that R is commutative.
(4) Give an example of a ring R with more than two elements satisfying the above properties.

II. (30 points)

Let R be the subset of the field $\mathbb{Q}$ of rational numbers consisting of all fractions $\frac{m}{n}$, with $m, n \in \mathbb{Z}$, such that $2 \nmid n$.

(1) Show that R is a subring of $\mathbb{Q}$, with the usual addition and multiplication of rational numbers.
(2) Show that $\frac{m}{n} \in R^{\times}$ if and only if $2 \nmid m$ and $2 \nmid n$.
(3) Is R a field ? Justify your answer.
(4) What is the characteristic of R ? Justify your answer.

III. (35 points)

Let p and q be two prime numbers, such that $p < q$. Let $m = p^2 \cdot q$.

(1) List all the nilpotent elements in the ring $\mathbb{Z}_m$. Justify your answer.
(2) List all the zero–divisors in the ring $\mathbb{Z}_m$. Justify your answer.
(3) Show that $\widehat{p-1}$ is a unit in $\mathbb{Z}_m$.
(4) Is $\widehat{q-1}$ necessarily a unit in $\mathbb{Z}_m$? Justify your answer.