LOCAL / GLOBAL RECIPROCITY MAPS

$$
\begin{tikzcd}
1 \arrow[r] & D_K \cap K_v^\times \arrow[rr] \arrow[dr, hook] & & K_v^\times \arrow[rr, "\rho_{K_v}"] \arrow[dr, hook] \arrow[dd] & & G_{K_v}^{ab} \arrow[dr, hook] \arrow[dd, dashed] & \\
 & 1 \arrow[r] & D_K \arrow[rr] & & C_K \arrow[rr, "\rho_K"] \arrow[dd] & & G_K^{ab} \arrow[dd, "\mathrm{res}_{K^{ab}/L}"] \\
 & & & K_v^\times / N_{L_w/K_v}(L_w^\times) \arrow[rr, dashed, "\rho_{L_w/K_v}", "\sim"'] \arrow[dr, hook] & & G(L_w/K_v) \arrow[dr, dashed, hook] & \\
 & & & & C_K / N_{L/K}(C_L) \arrow[rr, "\sim", "\rho_{L/K}"'] & & G(L/K)
\end{tikzcd}
$$

$$
\begin{tikzcd}
K_v^{ab} & K^{ab} \arrow[l] \\
L_w \arrow[u, dash] & L \arrow[l] \arrow[u, dash] \\
K_v \arrow[u, dash] & K \arrow[l] \arrow[u, dash, "\text{finite}"'] \\
\text{local} & \text{global}
\end{tikzcd}
$$

1. • $\mathrm{Im}(\rho_K)$ is dense in G_K^{ab}. If $\mathrm{char}(K)=0$, then $\mathrm{Im}(\rho_K)=G_K^{ab}$. If $\mathrm{char}(K)=p>0$, then $\mathrm{Im}(\rho_K)=\{\sigma\in G_K^{ab} \mid \sigma(\zeta)=\zeta^{q_K^n}, \text{ for some } n\in\mathbb{Z}, \ \forall \zeta\in\overline{\mathbb{F}_p}^\times\}$.
   • $\ker(\rho_K)=D_K=\bigcap_{\substack{L/K \subseteq K^{ab}\\ \text{finite}}} N_{L/K}(C_L)$ is the connected component of 1 in C_K. If $\mathrm{char}(K)=0$, then $D_K\neq\{1\}$ (structure thm. due to Weil). If $\mathrm{char}(K)=p>0$, then $D_K=\{1\}$.

2. • $\mathrm{Im}(\rho_{K_v})$ is dense in $G_{K_v}^{ab}$. If $K_v=\mathbb{R}$ or $\mathbb{C}$, then $\mathrm{Im}(\rho_{K_v})=G_{K_v}^{ab}$. If v is non-archimedean, then $\mathrm{Im}(\rho_{K_v})=\{\sigma\in G_{K_v}^{ab} \mid \sigma|_{K_v^{unr}}=\sigma_v^n, \text{ for some } n\in\mathbb{Z}\}$.
   • $\ker(\rho_{K_v})=D_K\cap K_v^\times=$ the connected comp. of 1 in $K_v^\times = \begin{cases} 1, & \text{if } v \text{ is non-archimedean} \\ \mathbb{C}^\times, & \text{if } K_v=\mathbb{C} \\ \mathbb{R}^+, & \text{if } K_v=\mathbb{R}. \end{cases}$